Conference in honour of

Magda Peligrad

A relationship between maximal inequalities and SLLN

by
Oleg Klesov

La Sorbonne, Paris

June 22, 2010

A TALK

TO BE DELIVERED BETWEEN

COFFEE BREAK

AND

CONFERENCE DINNER

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A RELATIONSHIP BETWEEN MAXIMAL INEQUALITIES AND SLLN

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June 22, 2010
17:00–17.44

1.
DEFINITIONS

Basic object

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a sequence

S_n, $n \geq 1$,

of random variables.

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a sequence

$$S_n,\ n \geq 1,$$

of random variables.

Remark. We do not assume that S_n is of a special structure.

However, $S_0 = 0$,

$$X_n \stackrel{\text{def}}{=} S_n - S_{n-1} \quad \Longrightarrow \quad S_n = \sum_{k=1}^{n} X_k.$$

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$$X_n \stackrel{\text{def}}{=} S_n - S_{n-1} \quad \Longrightarrow \quad S_n = \sum_{k=1}^{n} X_k.$$

Or, $S_0 = 1$,

$$X_n \stackrel{\text{def}}{=} \frac{S_n}{S_{n-1}} \quad \Longrightarrow \quad S_n = \prod_{k=1}^{n} X_k.$$

Correlation structure

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For the sake of demonstration of general results, we will consider some specific correlation structures in different places.

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For the sake of demonstration of general results, we will consider some specific correlation structures in different places.

Example. Let $X_n = S_n - S_{n-1}$ be the increments of the sequence $\{S_n\}$. We are able to treat the cases where X_n are

- m-dependent
- martingale-difference
- orthogonal
- stationary
- mixing
- positively dependent
- ϕ-subGaussian
- negatively associated
- ... ...

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Let

$$\{b_n, n \geq 1\}$$

be a sequence of real numbers

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such that

(a) $b_n \leq b_{n+1}$
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What do we want to prove

Let

$$\{b_n, n \geq 1\}$$

be a sequence of real numbers

such that

(a) $b_n \leq b_{n+1}$
(b) $b_n \to \infty$

SLLN. We would like to prove that

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad \text{a.s.}$$

2.
HISTORICAL EXAMPLES

Kolmogorov (1930)

Kolmogorov (1930)

Theorem. *If increments*

$$X_n = S_n - S_{n-1}$$

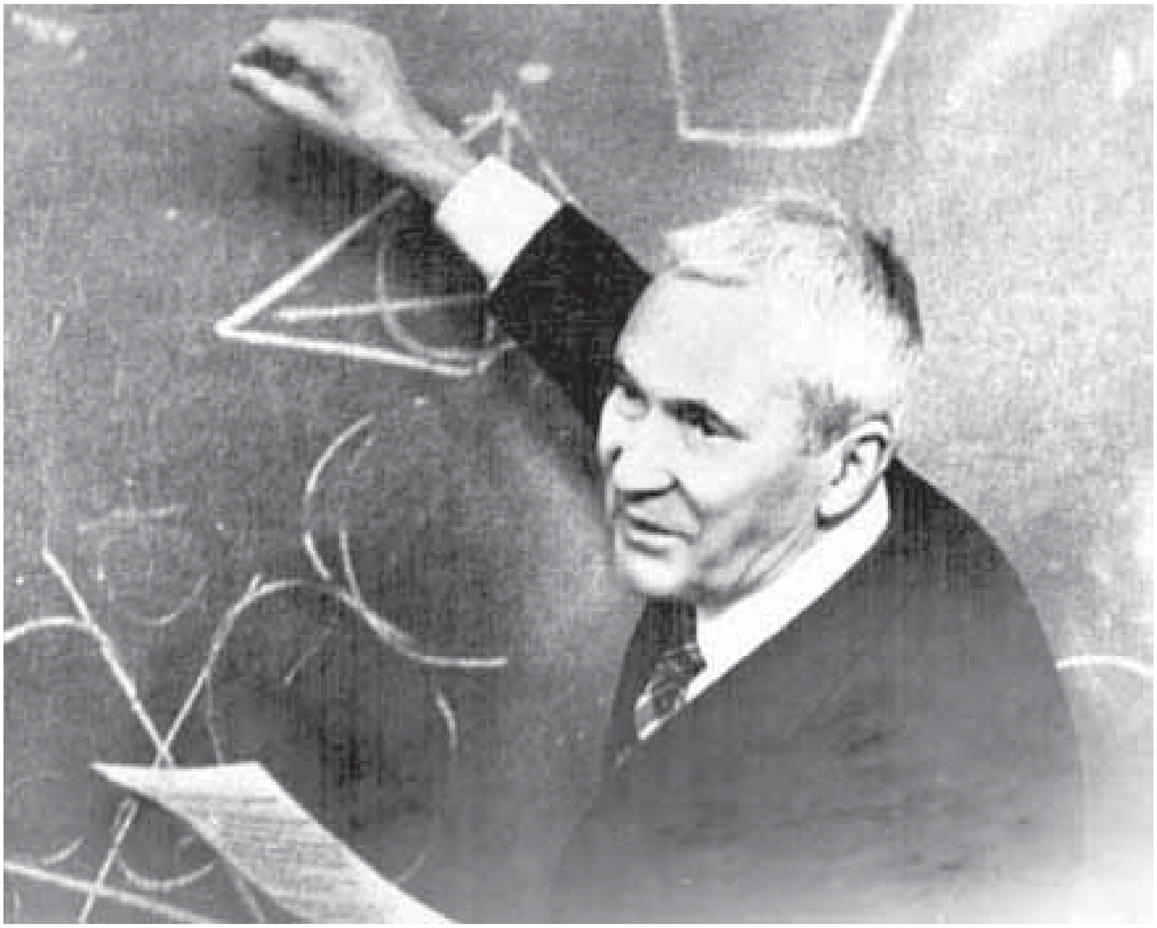
are independent and $\mathsf{E}\, X_n = 0$,

$$\sum_{n=1}^{\infty} \frac{\mathsf{E}\, X_n^2}{b_n^2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad a.s.$$

Андрей Николаевич Колмогоров



(12.04.1903–20.10.1987)

Menschoff–Rademacher ($\approx$ 1922)

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Theorem. *If increments*

$$X_n = S_n - S_{n-1}$$

are orthogonal and $\mathsf{E}\, X_n = 0$,

$$\sum_{n=1}^{\infty} \frac{\mathsf{E}\, X_n^2}{b_n^2} (\log_2 n)^2 < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad a.s.$$

Дмитрий Евгеньевич Меньшов

(18.04.1892–25.11.1988)

Hans Rademacher

(3.04.1892–7.02.1969)

What do we assume

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Let $r > 0$. Assume that

$$\mathsf{E}\,|S_n|^r < \infty.$$

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Let $r > 0$. Assume that

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Majorant sequence

Definition. A sequence of non-negative numbers $\{\lambda_k, k \ge 1\}$ is called a *majorant* for $\{S_n, n \ge 1\}$ if

$$\mathsf{E}\left(\max_{k\le n}|S_k|\right)^r \le \sum_{k=1}^n \lambda_k$$

for all $n \ge 1$.

Existence of a majorant

If $\lambda_1 = \mathsf{E}\,|S_1|^r$ and

$$\lambda_k = \mathsf{E}\left(\max_{i\le k}|S_i|\right)^r - \mathsf{E}\left(\max_{i\le k-1}|S_i|\right)^r$$

for $k > 1$, then

$$\mathsf{E}\left(\max_{k\le n}|S_k|\right)^r = \sum_{k=1}^{n}\lambda_k$$

for all $n \ge 1$.

Historical examples

Doob. *If increments*

$$X_n = S_n - S_{n-1}$$

form a <u>*martingale-difference*</u> *and* $r > 1$, *then*

$$\mathsf{E}\left(\max_{k \le n} |S_k|\right)^r \le \left(\frac{r}{r-1}\right)^r \mathsf{E}\,|S_n|^r,$$

whence

$$\lambda_k = \left(\frac{r}{r-1}\right)^r \left[\,\mathsf{E}\,|S_k|^r - \mathsf{E}\,|S_{k-1}|^r\right].$$

In particular,

In particular,

if $r = 2$ and X_n are independent and centered, then

$$\lambda_k = 4\big[\,\mathsf{E}\,|S_k|^2 - \mathsf{E}\,|S_{k-1}|^2\big]$$

which means

$$\lambda_k = 4\,\mathsf{E}\,X_k^2.$$

Kolmogorov's type inequality

If increments $\{X_k\}$ are independent,

$$\mathsf{E}\left(\max_{k\le n}|S_k|\right)^2 \le 4\sum_{k=1}^n \mathsf{E}\,X_k^2.$$

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$$\mathsf{E}\left(\max_{k\le n}|S_k|\right)^2 \le 4\sum_{k=1}^{n}\mathsf{E}\,X_k^2.$$

Kolmogorov's inequality

For all $\varepsilon > 0$,

$$\mathsf{P}\left(\max_{k\le n}|S_k| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\mathsf{E}\,X_k^2.$$

Historical examples

Menschoff–Rademacher. *If increments*

$$X_n = S_n - S_{n-1}$$

are orthogonal and $r = 2$, *then*

$$\mathsf{E}\left(\max_{k \le n} |S_k|\right)^2 \le (\log_2 4n)^2 \sum_{k=1}^{n} \mathsf{E}\, X_k^2,$$

whence

$$\lambda_k = (\log_2 4n)^2\, \mathsf{E}\, X_k^2, \qquad k \le n.$$

3.
J. HÁJEK AND A. RÉNYI

Hájek–Rényi inequality (1955)

If

$$\{X_n\} \text{ are independent}$$

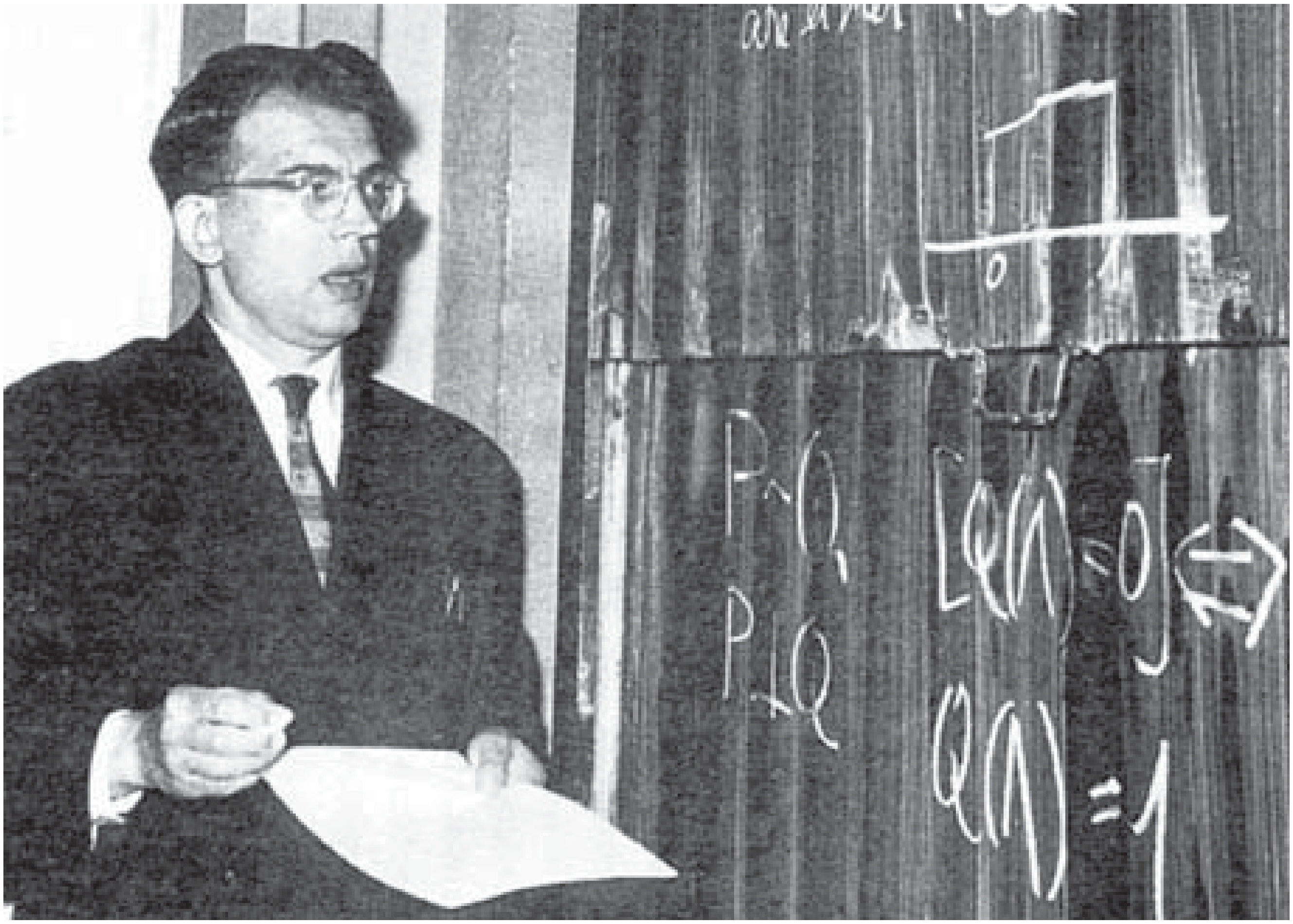
and

$$\{b_n\} \text{ are nondecreasing,}$$

then, for all $\varepsilon > 0$,

$$\mathsf{P}\left(\max_{k \le n}\left|\frac{S_k}{b_k}\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\frac{\mathsf{E}\,X_k^2}{b_k^2}.$$

Jaroslav Hájek



(4.02.1926–10.06.1974)

Alfred Rényi

(20.03.1921–1.02.1970)

4.
HÁJEK–RÉNYI TYPE INEQUALITY

Fazekas, Klesov (1999).

Fazekas, Klesov (1999). *Let a majorant exist for* $\{S_n\}$*:*

$$\mathsf{E}\left(\max_{k\le n}|S_k|\right)^r \le \sum_{k=1}^n \lambda_k.$$

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Then

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|\right)^r \le 4\sum_{k=1}^{n}\frac{\lambda_k}{b_k^r}.$$

Historical examples

Hájek–Rényi. *If increments*

$$X_n = S_n - S_{n-1}$$

are *independent* *and* $r = 2$, *then*

$$\mathsf{P}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\frac{\mathsf{E}\,X_k^2}{b_k^2}$$

for all $\varepsilon > 0$.

Historical examples

Rényi–Zergényi (1956). *If increments*

$$X_n = S_n - S_{n-1}$$

are <u>*orthogonal*</u> *and* $r = 2$, *then*

$$\mathsf{P}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right| \ge \varepsilon\right) \le \frac{(\log_2 4n)^2}{\varepsilon^2}\sum_{k=1}^{n}\frac{\mathsf{E}\,X_k^2}{b_k^2}$$

for all $\varepsilon > 0$.

Rényi–Zergényi (1956). *If increments*

$$X_n = S_n - S_{n-1}$$

are <u>*orthogonal*</u> *and* $r = 2$, *then*

$$\mathsf{P}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right| \ge \varepsilon\right) \le \frac{\left(\log_2 4n\right)^2}{\varepsilon^2}\sum_{k=1}^{n}\frac{\mathsf{E}\,X_k^2}{b_k^2}$$

for all $\varepsilon > 0$.

Remark. Extra assumption

$$\sup_{k\ge 1}\frac{b_{2k}}{b_k} < \infty.$$

Hájek–Rényi maximal inequality

Fazekas, Klesov (1999). *Let a majorant exist*

$$\mathsf{E}\left(\max_{k\le n}|S_k|\right)^r \le \sum_{k=1}^{n}\lambda_k.$$

Then

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|\right)^r \le 4\sum_{k=1}^{n}\frac{\lambda_k}{b_k^r}.$$

Proof

For the sake of simplicity, we consider the case of

$$b_n = n.$$

Let, for $i \geq 0$,

$$A_i = \left\{2^i, 2^i + 1, \ldots, 2^{i+1} - 1\right\},$$

Denoting

$$k_i = 2^{i+1} - 1$$

we obtain

$$A_i = \{k_{i-1} + 1, \ldots, k_i\}.$$

Case of $n = 2^m - 1$

We have

$$[1, n] = \bigcup_{i=0}^{m-1} A_i.$$

Thus

$$\mathsf{E}\left(\max_{k \le n} \left|\frac{S_k}{b_k}\right|^r\right)$$

Case of $n = 2^m - 1$

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$$\begin{aligned}\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|^r\right) \\ = \mathsf{E}\left(\max_{k\in A_0\cup A_1\cup\cdots\cup A_{m-1}}\left|\frac{S_k}{b_k}\right|^r\right)\end{aligned}$$

Case of $n = 2^m - 1$

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Thus

$$\begin{aligned}
&\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|^r\right)\\
&\qquad = \mathsf{E}\left(\max_{k\in A_0\cup A_1\cup\cdots\cup A_{m-1}}\left|\frac{S_k}{b_k}\right|^r\right)\\
&\qquad = \mathsf{E}\left(\max_{i=0,1,\ldots,m-1}\left[\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|\right]^r\right)
\end{aligned}$$

Case of $n = 2^m - 1$

We have

$$[1, n] = \bigcup_{i=0}^{m-1} A_i.$$

Thus

$$\begin{aligned}
&\mathsf{E}\left(\max_{k \le n}\left|\frac{S_k}{b_k}\right|^r\right) \\
&\qquad = \mathsf{E}\left(\max_{k \in A_0 \cup A_1 \cup \cdots \cup A_{m-1}}\left|\frac{S_k}{b_k}\right|^r\right) \\
&\qquad = \mathsf{E}\left(\max_{i=0,1,\ldots,m-1}\left[\max_{k \in A_i}\left|\frac{S_k}{b_k}\right|\right]^r\right) \\
&\qquad \le \sum_{i=0}^{m-1} \mathsf{E}\left(\max_{k \in A_i}\left|\frac{S_k}{b_k}\right|^r\right)
\end{aligned}$$

Recall

$$A_i = \left\{2^i, 2^i + 1, \ldots, 2^{i+1} - 1\right\}$$

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so that

$$\frac{1}{b_k^r} \leq \frac{1}{2^{ir}}, \qquad k \in A_i.$$

(do not forget $b_k = k$.)

Since

$$A_i = \{k_{i-1} + 1, \ldots, k_i\},$$

we have

$$\mathsf{E}\left(\max_{k \in A_i} \left|\frac{S_k}{b_k}\right|^r\right)$$

Since

$$A_i = \{k_{i-1} + 1, \ldots, k_i\},$$

we have

$$\mathsf{E}\left(\max_{k \in A_i} \left|\frac{S_k}{b_k}\right|^r\right)$$
$$\leq \frac{1}{2^{ir}} \mathsf{E}\left(\max_{k \in A_i} |S_k|^r\right)$$

SINCE

$$A_i = \{k_{i-1} + 1, \ldots, k_i\},$$

we have

$$\begin{aligned} \mathsf{E}\left(\max_{k \in A_i} \left|\frac{S_k}{b_k}\right|^r\right) &\le \frac{1}{2^{ir}}\,\mathsf{E}\left(\max_{k \in A_i} |S_k|^r\right) \\ &\le \frac{1}{2^{ir}}\,\mathsf{E}\left(\max_{k \le k_i} |S_k|^r\right). \end{aligned}$$

Now

$$\mathsf{E}\left(\max_{k\le k_i}|S_k|^r\right)$$

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$$\mathsf{E}\left(\max_{k\le k_i}|S_k|^r\right)$$

$$\le \sum_{k=1}^{k_i}\lambda_k$$

$$= \sum_{j=0}^{i}\delta_j$$

where $\delta_0 = 0$,

$$\delta_j = \sum_{k=k_{j-1}+1}^{k_j}\lambda_k, \qquad j > 1,$$

or

$$\delta_j = \sum_{k\in A_i}\lambda_k, \qquad j > 1,$$

Therefore

$$\sum_{i=0}^{m-1} \mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right)$$

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$$\sum_{i=0}^{m-1} \mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right)$$

$$\leq \sum_{i=0}^{m-1} \frac{1}{2^{ir}}\,\mathsf{E}\left(\max_{k\le k_i}|S_k|^r\right)$$

Therefore

$$\sum_{i=0}^{m-1} \mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right)$$

$$\leq \sum_{i=0}^{m-1} \frac{1}{2^{ir}} \mathsf{E}\left(\max_{k\leq k_i}|S_k|^r\right)$$

$$\leq \sum_{i=0}^{m-1} \frac{1}{2^{ir}} \sum_{j=0}^{i} \delta_j$$

Therefore

$$\sum_{i=0}^{m-1} \mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right)$$

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$$\le \sum_{i=0}^{m-1} \frac{1}{2^{ir}} \sum_{j=0}^{i} \delta_j$$

$$\le \sum_{j=0}^{m-1} \delta_j \sum_{i=j}^{m-1} \frac{1}{2^{ir}}$$

Therefore

$$
\begin{aligned}
\sum_{i=0}^{m-1} \mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right)
&\le \sum_{i=0}^{m-1} \frac{1}{2^{ir}}\, \mathsf{E}\left(\max_{k\le k_i}|S_k|^r\right)\\
&\le \sum_{i=0}^{m-1} \frac{1}{2^{ir}} \sum_{j=0}^{i} \delta_j\\
&\le \sum_{j=0}^{m-1} \delta_j \sum_{i=j}^{m-1} \frac{1}{2^{ir}}\\
&\le \sum_{j=0}^{m-1} \frac{\delta_j}{2^{jr}} \sum_{i=0}^{\infty} \frac{1}{2^{ir}}.
\end{aligned}
$$

WHENCE

$$\mathsf{E}\left(\max_{k\leq n}\left|\frac{S_k}{b_k}\right|^r\right)$$

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WHENCE

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|^r\right) \le \sum_{i=0}^{m-1}\mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right) \le \text{const}\sum_{j=0}^{m-1}\frac{\delta_j}{2^{jr}}$$

WHENCE

$$
\begin{aligned}
\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|^r\right)
&\le \sum_{i=0}^{m-1}\mathsf{E}\left(\max_{k\in A_i}\left|\frac{S_k}{b_k}\right|^r\right)\\
&\le \text{const}\sum_{j=0}^{m-1}\frac{\delta_j}{2^{jr}}\\
&= \text{const}\sum_{j=0}^{m-1}\frac{1}{2^{jr}}\sum_{k\in A_j}\lambda_k
\end{aligned}
$$

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hence

$$\frac{1}{2^{jr}} \le \frac{2^r}{b_k^r}, \qquad k \in A_j.$$

Finally

$$\mathsf{E}\left(\max_{k\leq n}\left|\frac{S_k}{b_k}\right|^r\right)$$

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$$\le 2^r\,\text{const}\sum_{j=0}^{m-1}\sum_{k\in A_j}\frac{\lambda_k}{b_k^r}$$

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$$\le 2^r\,\text{const}\sum_{j=0}^{m-1}\sum_{k\in A_j}\frac{\lambda_k}{b_k^r}$$

$$= 2^r\,\text{const}\sum_{k\in A_0\cup A_1\cup\cdots\cup A_{m-1}}\frac{\lambda_k}{b_k^r}$$

Finally

$$
\begin{aligned}
\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|^r\right)
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&\le 2^r\,\text{const}\sum_{j=0}^{m-1}\sum_{k\in A_j}\frac{\lambda_k}{b_k^r}\\
&= 2^r\,\text{const}\sum_{k\in A_0\cup A_1\cup\cdots\cup A_{m-1}}\frac{\lambda_k}{b_k^r}\\
&= 2^r\,\text{const}\sum_{k=1}^{k_{m-1}}\frac{\lambda_k}{b_k^r}
\end{aligned}
$$

where

$$\text{const} = \sum_{i=0}^{\infty} \frac{1}{2^{ir}} = \frac{2^r}{2^r - 1}.$$

where

$$\text{const} = \sum_{i=0}^{\infty} \frac{1}{2^{ir}} = \frac{2^r}{2^r - 1}.$$

Thus

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_k}{b_k}\right|^r\right) \le \frac{2^{2r}}{2^r - 1}\sum_{k=1}^{n}\frac{\lambda_k}{b_k^r}.$$

□

Moments of the supremum

Corollary. *Assume that* $r > 0$ *and*

$$\mathsf{E}\,|S_n|^r < \infty \quad \textit{for all } n \geq 1.$$

Let

$$\mathsf{E}\left(\max_{k \leq n} |S_k|\right)^r \leq \sum_{k=1}^{n} \lambda_k$$

for all $n \geq 1$ *and some nonnegative numbers* $\{\lambda_k\}$.

If $\{b_n\}$ *is nondecreasing, then*

$$\mathsf{E}\left(\sup_{k \geq 1} \left|\frac{S_k}{b_k}\right|\right)^r \leq 4 \sum_{k=1}^{\infty} \frac{\lambda_k}{b_k^r}.$$

Fazekas, Klesov (1999).

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Fazekas, Klesov (1999).

Kolmogorov's type SLLN

Fazekas, Klesov (1999). *If a sequence* $\{b_n\}$ *is nondecreasing and* *unbounded* *and*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{b_k^r} < \infty,$$

KOLMOGOROV'S TYPE SLLN

Fazekas, Klesov (1999). *If a sequence* $\{b_n\}$ *is nondecreasing and unbounded and*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{b_k^r} < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad a.s.$$

Kolmogorov (1930). *Assume that the increments*

$$X_n = S_n - S_{n-1}$$

are independent and $\mathsf{E}\, X_n = 0$.

If a sequence $\{b_n\}$ *is nondecreasing and unbounded and*

$$\sum_{n=1}^{\infty} \frac{\mathsf{E}\, X_n^2}{b_n^2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad a.s.$$

Proof

By the corollary above

$$\mathsf{E}\left(\sup_{k\geq 1}\left|\frac{S_k}{b_k}\right|\right)^r < \infty,$$

whence

$$\sup_{k\geq 1}\left|\frac{S_k}{b_k}\right| < \infty \qquad \text{a.s.}$$

A TRICK

Let $\{\beta_k\}$ be a sequence such that

(1) $\beta_k \le \beta_{k+1}$;

(2) $\beta_k \to \infty$ as $k \to \infty$;

(3) $\beta_k = o(b_k)$;

(4) $$\sum_{k=1}^{\infty} \frac{\lambda_k}{\beta_k^r} < \infty.$$

As above

$$\sup_{k\geq 1}\left|\frac{S_k}{\beta_k}\right| < \infty \qquad \text{a.s.}$$

$$\sup_{k\geq 1}\left|\frac{S_k}{\beta_k}\right| < \infty \qquad \text{a.s.}$$

Therefore

$$\limsup_{k\to\infty}\left|\frac{S_k}{b_k}\right|$$

As above

$$\sup_{k \geq 1} \left| \frac{S_k}{\beta_k} \right| < \infty \qquad \text{a.s.}$$

Therefore

$$\limsup_{k \to \infty} \left| \frac{S_k}{b_k} \right| = \limsup_{k \to \infty} \left| \frac{S_k}{\beta_k} \right| \cdot \frac{\beta_k}{b_k}$$

As above

$$\sup_{k\geq 1}\left|\frac{S_k}{\beta_k}\right| < \infty \qquad \text{a.s.}$$

Therefore

$$\limsup_{k\to\infty}\left|\frac{S_k}{b_k}\right| = \underbrace{\limsup_{k\to\infty}\left|\frac{S_k}{\beta_k}\right|}_{\text{bounded}} \cdot \underbrace{\frac{\beta_k}{b_k}}_{\text{goes to } 0}$$

As above

$$\sup_{k\geq 1}\left|\frac{S_k}{\beta_k}\right| < \infty \qquad \text{a.s.}$$

Therefore

$$\limsup_{k\to\infty}\left|\frac{S_k}{b_k}\right| = \underbrace{\limsup_{k\to\infty}\left|\frac{S_k}{\beta_k}\right|}_{\text{bounded}} \cdot \underbrace{\frac{\beta_k}{b_k}}_{\text{goes to 0}} = 0$$

almost surely.

$\square$

5.
ρ-MIXING

Kolmogorov–Rozanov coefficient

Let $\{X_n\}$ be a sequence of random variables. Let $\rho(n)$ be its Kolmogorov–Rozanov mixing coefficient, that is,

$$\rho(n) = \sup_{\substack{X \in L_2(\mathscr{F}_1^k), \\ Y \in L_2(\mathscr{F}_{k+n}^\infty), \\ k \geq 1}} \left| \frac{\operatorname{cov}[X, Y]}{\sqrt{\operatorname{var} X \cdot \operatorname{var} Y}} \right|.$$

Shao (1995) SLLN

Theorem. *Assume that*

(1) X_n *are identically distributed*;

(2) $\mathsf{E}\, X_n = 0$;

(3) $1 \le r < 2$;

(4) $\mathsf{E}\, |X_n|^r < \infty$;

(5) $\sum \rho(2^i) < \infty$.

Then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/r}} = 0 \qquad a.s.$$

Shao (1995) inequality

(a particular case)

Let the second moments exist and (perhaps) the distributions are different.

There are constants $c_1 > 0$ and $c_2 > 0$ such that

$$\mathsf{E}\left(\max_{k\le n} S_k^2\right) \le T_n^{(1)} \cdot T_n^{(2)},$$

where

$$T_n^{(1)} = c_1 e^{c_2 \varphi(n)},$$

$$\varphi(t) \stackrel{\text{def}}{=} \sum_{i\le[\log_2 t]} \rho(2^i),$$

$$T_n^{(2)} = \max_{k\le n} \mathsf{E}\, X_k^2.$$

WHAT HAPPENS IF $\sum \rho(2^i) = \infty$?

If

$$\mathsf{E}\,|X_n|^r e^{c_2 \varphi(|X_n|^r)} < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/r}} = 0 \qquad \text{a.s.}$$

Recall

$$\varphi(t) \stackrel{\text{def}}{=} \sum_{i \le [\log_2 t]} \rho(2^i).$$

Then

$$\varphi(t) \asymp \log\log t.$$

If

$$\mathsf{E}\,|X_n|^r \left[\,\log\left(1+|X_n|\right)\,\right]^{rc_2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/r}} = 0 \qquad \text{a.s.}$$

6.
PELIGRAD–ROSENTAL INEQUALITY AND SLLN

Let $r \geq 2$. Assume that

$$\rho(N) \leq \rho_0$$

for some $N \geq 1$ and ρ_0.

Then there is a constant D such that

$$\mathsf{E}\left(\max_{k \leq n} |S_k|^r\right) \leq D\left[\sum_{k=1}^{n} \mathsf{E}\,|X_k|^r + \left(\sum_{k=1}^{n} \mathsf{E}\,X_k^2\right)^{r/2}\right].$$

Case of $r = 4$

Put $\sigma_k^2 \stackrel{\text{def}}{=} \mathsf{E}\, X_k^2$. Then

$$\mathsf{E}\left(\max_{k \le n} S_k^4\right) \le D \left[\sum_{k=1}^{n} \mathsf{E}\, X_k^4 + \left(\sum_{k=1}^{n} \sigma_k^2\right)^2\right]$$

Case of $r = 4$

Put $\sigma_k^2 \stackrel{\text{def}}{=} \mathsf{E}\, X_k^2$. Then

$$\mathsf{E}\left(\max_{k \le n} S_k^4\right) \le \underbrace{D\left[\sum_{k=1}^{n} \mathsf{E}\, X_k^4 + \left(\sum_{k=1}^{n} \sigma_k^2\right)^2\right]}_{\sum_{k=1}^{n} \lambda_k}$$

We proceed with $D = 1$

$$\lambda_k = \mathsf{E}\, X_k^4 + \left(\sum_{i=1}^{k} \sigma_i^2 \right)^2 - \left(\sum_{i=1}^{k-1} \sigma_i^2 \right)^2$$

Then

$$\lambda_k = \mathsf{E}\, X_k^4 + \sigma_k^2 \left(\sum_{i=1}^{k} \sigma_i^2 + \sum_{i=1}^{k-1} \sigma_i^2 \right)$$

Then

$$
\begin{aligned}
\lambda_k &= \mathsf{E}\, X_k^4 + \sigma_k^2 \left(\sum_{i=1}^{k} \sigma_i^2 + \sum_{i=1}^{k-1} \sigma_i^2 \right) \\
&= \mathsf{E}\, X_k^4 + \left(\sigma_k^2\right)^2 + 2\sigma_k^2 \sum_{i=1}^{k-1} \sigma_i^2
\end{aligned}
$$

Then

$$
\begin{aligned}
\lambda_k &= \mathsf{E}\, X_k^4 + \sigma_k^2 \left(\sum_{i=1}^{k} \sigma_i^2 + \sum_{i=1}^{k-1} \sigma_i^2 \right) \\
&= \mathsf{E}\, X_k^4 + \left(\sigma_k^2\right)^2 + 2\sigma_k^2 \sum_{i=1}^{k-1} \sigma_i^2 \\
&\le 2\, \mathsf{E}\, X_k^4 + 2\sigma_k^2 \sum_{i=1}^{k-1} \sigma_i^2 .
\end{aligned}
$$

Theorem. *Let a sequence $\{b_n\}$ be nondecreasing and unbounded. If*

$$\sum_{n=1}^{\infty} \frac{\mathsf{E}\, X_n^4}{b_n^4} < \infty,$$

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{b_n^2} \cdot \frac{1}{b_n^2} \sum_{k=1}^{n-1} \sigma_k^2 < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad a.s.$$

Independent X_n

- Teicher (1968)
  (many extra assumptions), $b_n = n$
- Egorov (1970)
- $\dots\ \dots\ \dots\ \dots$

INDEPENDENT X_n

- Teicher (1968)
  (many extra assumptions), $b_n = n$
- Egorov (1970)
- ... ...

EXAMPLE

Let X_n be a $(0, \sigma_n^2)$ Gaussian random variable,

$$\sigma_n^2 \asymp \frac{n}{\ln n}.$$

Also let

$$b_n = n.$$

Then

$$\sum \frac{\sigma_n^2}{b_n^2} \asymp \sum \frac{n}{n^2 \ln n} = \infty.$$

THEN

$$\sum \frac{\sigma_n^2}{b_n^2} \asymp \sum \frac{n}{n^2 \ln n} = \infty.$$

On the other hand

$$\sum_{k=1}^{n-1} \sigma_k^2 \asymp \frac{n^2}{\ln n},$$

$$\sum_{n=2}^{\infty} \frac{\sigma_n^2}{b_n^4} \sum_{k=1}^{n-1} \sigma_k^2 \asymp \sum \frac{n}{n^4 \ln n} \cdot \frac{n^2}{\ln n}$$

$$= \sum \frac{n}{(\ln n)^2} < \infty.$$

Moreover,

$$\mathsf{E}\, X_n^4 = 3\sigma_n^2 \asymp \frac{n}{\ln n},$$

whence

$$\sum \frac{\mathsf{E}\, X_n^4}{b_n^4} \asymp \frac{n}{n^4 \ln n} < \infty.$$

7.
PELIGRAD (1999)

STOPPED SUMS

Peligrad (1999) inequality

Let $\{X_n\}$ be a sequence of random variables with the mixing coefficient $\rho(n)$.

Put

$$\psi(t) \stackrel{\text{def}}{=} \sum_{i\le[\log_2 t]} \rho\left(\left[2^{i/3}\right]\right).$$

Theorem. *Assume that $\{X_n\}$ are centered.*

Let τ be a stopping time.

Then there is an absolute constant $K > 0$ such that

$$\mathsf{E}\, S_\tau^2 \mathbb{I}_{\{\tau\le n\}} \le K e^{8\psi(n)} \sum_{k=1}^{n} \mathsf{E}\, X_k^2.$$

Since the random variables $\xi_k \stackrel{\text{def}}{=} |S_\tau| \mathbb{1}_{\{\tau \le k\}}$ are nondecreasing in k,

$$\mathsf{E}\left(\max_{k \le n} |S_\tau| \mathbb{1}_{\{\tau \le k\}}\right)^2 \le K e^{8\psi(n)} \sum_{k=1}^{n} \mathsf{E}\, X_k^2.$$

Hájek–Rényi inequality

ψ is bounded

If ψ is a bounded function, then

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_\tau \mathbb{1}_{\{\tau\le k\}}}{b_k}\right|\right)^2 \le C\sum_{k=1}^{n}\frac{\mathsf{E}\,X_k^2}{b_k^2}$$

where

$$C = 4K\sup_{n\ge 1} e^{8\psi(n)}.$$

ψ is bounded

Conjecture. *Let*

$$B_n = \sum_{k=1}^{n} \mathsf{E}\, X_k^2.$$

Petrov bounds for stopped sums

ψ is bounded

Conjecture. *Let*

$$B_n = \sum_{k=1}^{n} \mathsf{E}\, X_k^2.$$

Assume that $B_n \to \infty$ *as* $n \to \infty$*.*

ψ is bounded

Conjecture. *Let*

$$B_n = \sum_{k=1}^{n} \mathsf{E}\, X_k^2.$$

Assume that $B_n \to \infty$ as $n \to \infty$.

Then, for several sequences of stoping times $\{\tau_k\}$,

$$\lim_{n\to\infty} \frac{S_{\tau_k}\, \mathbb{I}_{\{\tau_k \le k\}}}{\sqrt{B_k \big(\ln B_k\big)^{1+\varepsilon}}} = 0 \qquad a.\ s.$$

whatever $\varepsilon > 0$.

Proof

Let

$$b_k \overset{\text{def}}{=} \sqrt{B_k \left(\ln B_k \right)^{1+\varepsilon}}.$$

Proof

Let

$$b_k \stackrel{\text{def}}{=} \sqrt{B_k \left(\ln B_k \right)^{1+\varepsilon}}.$$

Then

$$\sum_{k=1}^{\infty} \frac{\mathsf{E}\, X_k^2}{b_k^2} = \sum_{k=1}^{\infty} \frac{\mathsf{E}\, X_k^2}{B_k \left(\ln B_k \right)^{1+\varepsilon}} < \infty,$$

Let

$$b_k \stackrel{\text{def}}{=} \sqrt{B_k \big(\ln B_k\big)^{1+\varepsilon}}.$$

Then

$$\sum_{k=1}^{\infty} \frac{\mathsf{E}\, X_k^2}{b_k^2} = \sum_{k=1}^{\infty} \frac{\mathsf{E}\, X_k^2}{B_k \big(\ln B_k\big)^{1+\varepsilon}} < \infty,$$

since

$$\frac{\mathsf{E}\, X_k^2}{B_k \big(\log B_k\big)^{1+\varepsilon}} \leq \int_{B_{k-1}}^{B_k} \frac{dx}{x\,(\ln x)^{1+\varepsilon}}.$$

Hájek–Rényi inequality

ψ is unbounded

If ψ is a bounded function, then

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_\tau \mathbb{1}_{\{\tau\le k\}}}{b_k}\right|\right)^2 \le 4K\sum_{k=1}^{n}\frac{\lambda_k}{b_k^2}$$

where

$$\lambda_k = e^{8\psi(k)}\sum_{i=1}^{k}\mathsf{E}\,X_i^2 - e^{8\psi(k-1)}\sum_{i=1}^{k-1}\mathsf{E}\,X_i^2$$

ψ is unbounded

If ψ is a bounded function, then

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_\tau \mathbb{1}_{\{\tau\le k\}}}{b_k}\right|\right)^2 \le 4K\sum_{k=1}^{n}\frac{\lambda_k}{b_k^2}$$

where

$$\begin{aligned}\lambda_k &= e^{8\psi(k)}\sum_{i=1}^{k}\mathsf{E}\,X_i^2 - e^{8\psi(k-1)}\sum_{i=1}^{k-1}\mathsf{E}\,X_i^2\\ &= g(k)\sum_{i=1}^{k}\mathsf{E}\,X_i^2 - g(k-1)\sum_{i=1}^{k-1}\mathsf{E}\,X_i^2\end{aligned}$$

ψ is unbounded

If ψ is a bounded function, then

$$\mathsf{E}\left(\max_{k\le n}\left|\frac{S_\tau \mathbb{1}_{\{\tau\le k\}}}{b_k}\right|\right)^2 \le 4K\sum_{k=1}^{n}\frac{\lambda_k}{b_k^2}$$

where

$$\begin{aligned}\lambda_k &= e^{8\psi(k)}\sum_{i=1}^{k}\mathsf{E}\,X_i^2 - e^{8\psi(k-1)}\sum_{i=1}^{k-1}\mathsf{E}\,X_i^2\\ &= g(k)\sum_{i=1}^{k}\mathsf{E}\,X_i^2 - g(k-1)\sum_{i=1}^{k-1}\mathsf{E}\,X_i^2\\ &\qquad \pm g(k)\sum_{i=1}^{k-1}\mathsf{E}\,X_i^2.\end{aligned}$$

Finally

$$\lambda_k = g(k)\,\mathsf{E}\,X_k^2 + \big(g(k) - g(k-1)\big) \sum_{i=1}^{k-1} \mathsf{E}\,X_i^2 .$$

SLLN for stopped sums

ψ is unbounded

If

$$\sum_{k=1}^{\infty} g(k) \frac{\mathsf{E}\, X_k^2}{b_k^2} < \infty,$$

$$\sum_{k=2}^{\infty} \frac{g(k) - g(k-1)}{b_k^2} \sum_{i=1}^{k-1} \mathsf{E}\, X_i^2 < \infty,$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad \text{a.s.}$$

ψ is unbounded

$$\lim_{n\to\infty} \frac{S_{\tau_k}\,\mathbb{I}_{\{\tau_k\le k\}}}{\sqrt{e^{8\psi(k)}B_k\big(\ln B_k k\big)^{1+\varepsilon}}} = 0 \qquad \text{a. s.}$$

whatever $\varepsilon > 0$.

8.
TAILS

Let a sequence $\{S_n\}$ be convergent almost surely,

$$\lim_{n\to\infty} S_n \stackrel{\text{a.s.}}{=} S.$$

Then

$$\lim_{n\to\infty} (S_n - S) \stackrel{\text{a.s.}}{=} 0.$$

Let a sequence $\{S_n\}$ be convergent almost surely,

$$\lim_{n\to\infty} S_n \overset{\text{a.s.}}{=} S.$$

Then

$$\lim_{n\to\infty} (S_n - S) \overset{\text{a.s.}}{=} 0.$$

Let $\{b_n\}$ be a nondecreasing and unbounded sequence of real numbers.

Let a sequence $\{S_n\}$ be convergent almost surely,

$$\lim_{n\to\infty} S_n \overset{\text{a.s.}}{=} S.$$

Then

$$\lim_{n\to\infty} (S_n - S) \overset{\text{a.s.}}{=} 0.$$

Let $\{b_n\}$ be a nondecreasing and unbounded sequence of real numbers.

If

$$\lim_{n\to\infty} b_n(S_n - S) \overset{\text{a.s.}}{=} 0,$$

then S_n converges to S faster than b_n approaches ∞.

Using the increments,

$$X_n = S_n - S_{n-1},$$

we write

$$S = \sum_{k=1}^{\infty} X_k,$$

$$S - S_n = \sum_{k=n+1}^{\infty} X_k.$$

WE WANT TO STUDY

$$\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} X_k \overset{\text{a.s.}}{=} 0,$$

i.e. the *rate of convergence* of the series

$$\sum X_k.$$

Chow, Teicher (1973).

- weighted i.i.d.

Barbour (1974).

- CLT

Budianu (1981).

- Kolmogorov LIL

Rosalsky (1983), Klesov (1983).

- SLLN

Notation

$$\zeta_n \stackrel{\text{def}}{=} \sum_{k=n}^{\infty} X_k.$$

Main assumption

Let $r > 0$.
There are nonnegative $\{\lambda_k\}$ such that

$$\mathsf{E}\left(\sup_{k\geq n} |\zeta_n|^r\right) \leq \sum_{k=n}^{\infty} \lambda_k.$$

for all $n \geq 1$.

Hájek–Rényi inequality

Theorem. *If a sequence* $\{b_n\}$ *is nondecreasing and* $b_k \geq 0$, *then*

$$\mathsf{E}\left(\sup_{k\geq n} |b_k \zeta_n|^r\right) \leq \sum_{k=n}^{\infty} b_k^r \lambda_k.$$

KOLMOGOROV SLLN FOR TAILS

Theorem. *Let a sequence* $\{b_n\}$ *be nondecreasing and* <u>*unbounded*</u>*.*

If

$$\sum_{k=1}^{\infty} b_k^r \lambda_k < \infty,$$

then

$$\lim_{n\to\infty} b_n \zeta_n \stackrel{\text{a.s.}}{=} 0.$$

Corollary. *Let $\{X_n\}$ be pairwise orthogonal and zero mean.*

Let a sequence $\{b_n\}$ be nondecreasing and unbounded.

If

$$\sum_{k=1}^{\infty} b_k^2 \big(\log(2k)\big)^2 \,\mathsf{E}\, X_k^2 < \infty,$$

then

$$\lim_{n\to\infty} b_n \zeta_n \overset{\text{a.s.}}{=} 0.$$

Series of stationary terms

Let $\{\xi_k\}$ be a *wide sense stationary* sequence,

$$\mathsf{E}\,\xi_k = 0,$$
$$\mathsf{E}\,\xi_k\xi_{k+n} \quad \text{does not depend on } k,$$
$$R_1(n) \stackrel{\text{def}}{=} \mathsf{E}\,\xi_0\xi_n.$$

GAPOSHKIN (1976)

studied the almost sure convergence of

$$\sum_{k=1}^{\infty} a_k \xi_k .$$

Gaposhkin (1976)

Set

$$R_2(n) \stackrel{\text{def}}{=} \mathsf{E}\left(\frac{S_n}{n}\right)^2$$
$$= \frac{1}{n^2}\left[nR_1(n) + 2\sum_{k=1}^{n-1}(n-k)R_1(k)\right].$$

If

$$\sum_{k=1}^{\infty} a_k^2 k R_2(k)\,(\log k)^2 < \infty,$$

then

$$\sum_{k=1}^{\infty} a_k \xi_k \quad \textit{converges a.s.}$$

Theorem. *Let $a_n b_n$ regularly varying. If*

$$\sum_{k=1}^{\infty} b_k^2 a_k^2 k R_2(k) (\log k)^2 < \infty,$$

then

$$b_n \sum_{k=n}^{\infty} a_k \xi_k \to 0 \qquad a.s.$$

9.
FUNCTION SERIES

MIKOSCH (1988)

Let w be a Wiener process,

$$w(t) = \sum_{k=1}^{\infty} \xi_k S_k(t),$$
$$S_k(t) = \int_0^t \varphi_k(s)\,ds.$$

How quickly the sums

$$\sum_{k=1}^{n} \xi_k S_k(t)$$

converge to w?

$$b_n \sum_{k=n} \xi_k S_k(t) \to 0 \qquad \text{a.s.}$$

in the uniform norm.

Kotelnikov–Shannon series

Let ξ be a stationary process. Assume it is mean square continuous.

Kotelnikov–Shannon series

Let ξ be a stationary process. Assume it is mean square continuous. If there is $\ell > 0$ such that $(-\ell, \ell)$ is the support of its spectral function, then

$$\xi(t) = \sum_{k=-\infty}^{\infty} \xi(\pi k/\ell) \frac{\sin(\ell t - \pi k)}{\ell t - \pi k}$$

in L_2 for all $t \in \mathbf{R}$. We agree that

$$\frac{\sin(0)}{0} = 1.$$

Kotelnikov–Shannon series

Let ξ be a stationary process. Assume it is mean square continuous. If there is $\ell > 0$ such that $(-\ell, \ell)$ is the support of its spectral function, then

$$\xi(t) = \sum_{k=-\infty}^{\infty} \xi(\pi k/\ell) \frac{\sin(\ell t - \pi k)}{\ell t - \pi k}$$

in L_2 for all $t \in \mathbf{R}$. We agree that

$$\frac{\sin(0)}{0} = 1.$$

Remark. If ξ is Gaussian, the convergence is almost sure.

Combining with the above Gaposhkin results, we find the rate of convergence of the Kotelnikov–Shannon series.

10.
GAUSSIAN CASE

Buldygin (≈1970)

Let $\{S_n\}$ be a sequence of standard Gaussian random variables,

$$\mathsf{E}\, S_n = 0,$$
$$\mathsf{E}\, S_n^2 = 1.$$

Theorem.

$$\frac{S_n}{b_n} \to 0 \qquad a.s.$$

if and only if

$$\frac{S_{n_k}}{b_{n_k}} \to 0 \qquad a.s.$$

for subsequences $\{n_k\}$ belonging to a certain finite class.

11.
BLOCKS

- **Bernstein (≈1930)**;
  dependent increments
- **Prohorov (1950)**;
  independent increments,
  $b_n = n$;
- **Loéve (1959)**;
  independent increments,

$$b_{n_{k+1}} \leq c b_{n_k};$$

- **Tomkins (≈1980)**;
- **Petrov, Martikainen (1980)**;
  independent increments,
  if and only if statement.

12.
RANDOM FIELDS

1. Prinsgeim convergence.

2. Another convergence.

3. Normalizing sequences

$$b_{n+1} - b_n \geq 0.$$

4. Normalizing sequences

$$b_{n+1} \geq b_n.$$

5. Blocks.