Estimating the Degree of Activity of jumps in High Frequency Financial Data

joint with Yacine Aït-Sahalia

Aim and setting

- An underlying process $X = (X_t)_{t \geq 0}$,
- observed at *equally spaced* discrete times :

$$X_0,\ X_{\Delta_n},\ \cdots\ , X_{i\Delta_n},\ \cdots$$

- on a *fixed* time interval $[0, T]$

Assuming X has jumps on $[0, T]$, determine the "degree of activity" of the jumps, when the time lag Δ_n goes to 0:

  - finitely many ?
  - or, if infinitely many, how "infinite" is it ?

Measuring the degree of activity - 1

If X is a Lévy process with Lévy measure F and "tail" $\overline{F}(x) = F(\{y : |y| > x\})$:

- $\overline{F}(x) =$ mean number of jumps $\Delta X_t = X_t - X_{t-}$ with size $|\Delta X_t| > x$ over $[0, 1]$

- For all $t > 0$ the equivalence holds:

$$\sum_{s \leq t} |\Delta X_s|^r < \infty \text{ a.s.} \quad \Leftrightarrow \quad \int (|x[^r \wedge 1) F(dx) < \infty$$

The set I of all r as such has the form

$$I = (\beta, \infty), \quad \text{or} \quad I = [\beta, \infty)$$

for some $\beta \in [0, 2]$, and $2 \in I$.

β is the Blumenthal-Getoor index, introduced in 1961.

β is a sensible measure of the jump activity, since

$$\lim_{x \to 0} \; x^{\beta+\varepsilon} \; \overline{F}(x) = 0, \qquad \limsup_{x \to 0} \; x^{\beta-\varepsilon} \; \overline{F}(x) = \infty$$

When X is stable, the BG index equals the stable index, and $\beta \notin I$

Measuring the degree of activity - 2

X is an Itô semimartingale, that is

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \text{JUMPS}$$

$$\text{JUMPS} = \int_0^t \int_{\{|x|\leq 1\}} x(\mu-\nu)(ds,dx) + \int_0^t \int_{\{|x|>1\}} x\mu(ds,dx)$$

Here μ is the jump measure of X, and its predictable compensator ν can be factorized as

$$\nu(\omega; dt, dx) = dt \; F_t(\omega, dx).$$

Assumptions:

• on b and σ: they are locally bounded (random or not, dependent on X or not).
• on F_t: see later.

Instantaneous index:

$$I_t^i = \{r \geq 0 : \int (|x|^r \wedge 1) F_t(dx) < \infty\}, \qquad \beta_t^i = \inf(I_t^i)$$

Index over $[0, t]$:

$$I_t \ = \ \{r \geq 0 : \int_0^t ds \int (|x|^r \wedge 1) F_t(dx) < \infty\}, \qquad \beta_t = \inf(I_t)$$

Warning: Those are random.

An unrealistic situation: the path of X_t is fully observed on $[0, T]$

Then:

- $\sigma(\omega)_t$ is known for $t \in [0, T]$
- $I(\omega)_T$ is also known, because (outside a null set) $r \in I_T$ iff $\sum_{s \leq T} |\Delta X_s|^r < \infty$.

Apart from I_T, the measures F_t are "essentially" unknown, even under the (strong) additional assumption that $F_t(\omega, dx) = F(dx)$ is non-random, independent of time.

This motivates the estimation of β_T, which is about the most we can infer, concerning the measures F_t, even in this unrealistic situation.

The main challenge

Consider the special case $X = \sigma W + Y$, where Y is a β-stable process, so $\beta_t(\omega) = \beta$.

Any increment $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ satisfies

$$\Delta_i^n X = \sigma \Delta_n^{1/2} W_1 + \Delta_n^{1/\beta} Y_1$$

(equality in law). Then:

- Recalling $\beta < 2$ and $\Delta_n \to 0$, with a large probability $\Delta_i^n X$ is close to $\sigma \Delta_n^{1/2} W_1$ in law. those increments give essentially no information on Y, and are of "order of magnitude" $\Delta_n^{1/2}$

- However if Y has a "big" jump at time S, the corresponding increment is close to ΔY_S.

Hence, one has to throw away all "small" increments. However, β is related to the behavior of F near 0, hence to the "very small" jumps of Y.

In practice one uses only increments bigger than a cutoff level

$$\alpha\Delta_n^{\varpi} \qquad \text{for some} \quad \varpi \in (0, 1/2).$$

Asymptotically:

• those increments are big because, since $\Delta_n^{1/2} << \Delta_n^{\varpi}$, the main contribution is due to Y.

• those increments mostly contain a single "big" jump, of size of order at least Δ_n^{ϖ}.

• we still get some information on small jumps, because $\Delta_n^{\varpi} \to 0$.

The same heuristics works for Itô semimartingales. This leads to consider, for fixed $\varpi \in (0, 1/2)$ and $\alpha > 0$, the functionals

$$U(\varpi, \alpha, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} 1_{\{|\Delta_i^n X| > \alpha \Delta_n^{\varpi}\}}.$$

which simply counts the number of increments whose magnitude is greater than $\alpha \Delta_n^{\varpi}$.

This way, we are retaining only those increments of X that are predominantly made of contributions due to a single jump

The key property

Again $X = \sigma W + Y$ with Y β-stable for a while. Essentially, $U(\varpi, \alpha, \Delta_n)_t$ is the same as, or close to, the number $V(\varpi, \alpha, \Delta_n)_t$ of jumps of Y which are bigger than $\alpha\Delta_n^{\varpi}$, in the interval $[0, t]$.

- $V(\varpi, \alpha, \Delta_n)_t$ is a Poisson random variable with parameter

$$Ct/\alpha^{\beta}\ \Delta_n^{\beta\varpi}.$$

($C =$ suitable constant). Hence

- $\Delta_n^{\beta\varpi} V(\varpi, \alpha, \Delta_n)_t \ \rightarrow\ C/\alpha^{\beta}$ (in probability),

- $\frac{1}{\Delta_n^{\beta\varpi/2}}\Big(\Delta_n^{\beta\varpi} V(\varpi, \alpha, \Delta_n)_t - C/\alpha^{\beta}\Big) \rightarrow \mathcal{N}(0, C/\alpha^{\beta})$ (in law).

These properties carry over to $U(\varpi, \alpha, \Delta_n)_t$ in the case above, and also to more general semimartingales, subject to the following assumption, where $0 \leq \beta' < \beta < 2$ are non-random:

We have for all (ω, t):

$$F_t = F'_t + F''_t + F'''_t,$$

where

- F'_t is locally of the $\beta-$stable form

$$F'_t(dx) \;=\; \frac{1}{|x|^{1+\beta}} \left(a_t^{(+)} 1_{\{0<x\leq z_t^{(+)}\}} + a_t^{(-)} 1_{\{-z_t^{(-)}\leq x<0\}} \right) \; dx,$$

  for some predictable non-negative processes $a_t^{(+)}, a_t^{(-)}, z_t^{(+)}$ and $z_t^{(-)}$.

- F''_t has a density, and has a Blumenthal-Getoor index $\leq \beta/2$ and $F''_t(\mathbb{R}) = 0$ if $F'_t(\mathbb{R}) = 0$.

- F'''_t is singular and has a Blumenthal-Getoor index $\leq \beta'$.

- Plus some (weak) technical conditions on $a_t^{(+)}, a_t^{(-)}, z_t^{(+)}$ and $z_t^{(-)}$.

For example, any process of the following form satisfies the assumption

$$dX_t = b_t dt + \sigma_t dW_t + \delta_{t-} dY_t + \delta'_{t-} dY'_t$$

where:

- δ and δ' are càdlàg adapted processes

- Y is $\beta-$stable

- Y' is any Lévy process with Blumenthal-Getoor index less that $\beta/2$.

THEOREM Under the previous assumptions, and if

$$A_t \;=\; = \frac{1}{\beta}\int_0^t \left(a_s^{(+)} + a_s^{(-)}\right) ds,$$

- $\Delta_n^{\beta\varpi} U(\varpi, \alpha, \Delta_n)_t \;\to\; A_t/\alpha^\beta$ in probability,

- $\frac{1}{\Delta_n^{\beta\varpi/2}}\Big(\Delta_n^{\beta\varpi} U(\varpi, \alpha, \Delta_n)_t - A_t/\alpha^\beta\Big)$ converges stably in law to a variable which, conditionally on the process X is centered Gaussian with variance A_t/α^β.

We also have the joint convergence (stably in law) for two or more values of α and/or ϖ.

The estimators

We pick $\varpi \in (0, 1/2)$ and $0 < \alpha < \alpha'$, and define

$$\widehat{\beta}_n(\varpi, \alpha, \alpha') = \frac{\log(U(\varpi, \alpha, \Delta_n)_T / U(\varpi, \alpha', \Delta_n)_T)}{\log(\alpha'/\alpha)},$$

By the first part of the theorem, we have consistency:

$$\widehat{\beta}_n(\varpi, \alpha, \alpha') \xrightarrow{\mathbb{P}} \beta,$$

in restriction to the set where $A_T > 0$.

Another family of consistent estimator is

$$\widehat{\beta}'_n(\varpi, \alpha) = \frac{\log(U(\varpi, \alpha, \Delta_n)_T / U(\varpi, \alpha, 2\Delta_n)_T)}{\varpi \log 2}.$$

A central limit theorem

The second part of the key theorem, yields

THEOREM As soon as $\varpi < \frac{1}{2+\beta} \bigwedge \frac{2}{5\beta}$, and in restriction to the set $\{AT > 0\}$,

1) the variables

$$\frac{1}{\Delta_n^{\varpi\beta/2}} \left(\widehat{\beta}_n(\varpi, \alpha, \alpha') - \beta\right)$$

converge stably in law to a variable which conditionally on the process X is centered Gaussian with variance $(\alpha'^{\beta} - \alpha^{\beta})/A_T(\log(\alpha'/\alpha))^2$,

2) the variables

$$\frac{\log(\alpha'/\alpha)}{\left(\frac{1}{U(\varpi,\alpha',\Delta_n)_t} - \frac{1}{U(\varpi,\alpha,\Delta_n)_t}\right)^{1/2}} \; \left(\widehat{\beta}_n(\varpi,\alpha,\alpha') - \beta\right)$$

converge stably in law to a standard normal variable independent of X.

(similar results hold for the other family of estimators).

- The qualifier “in restriction to the set $\{A_T > 0\}$” is essential in this statement.
  - On the (random) set $\{A_T > 0\}$, the jump activity index is β.
  - On the complement set $\{A_T = 0\}$, the number β is not the jump activity index for X on $[0, T]$. We do not know even the behavior of $\widehat{\beta}_n(\varpi, \alpha, \alpha')$ in probability, not to speak about a central limit theorem. However we suspect that any convergent subsequence as a limit strictly smaller than $\beta/2$.

- These results are model-free in a sense, because the drift and the volatility processes are totally unspecified; on the other hand the assumptions on the Lévy measures F_t are quite strong.

- When those assumptions fail, we do not know how to prove the results, even in the case where X is a Lévy process.

0.1. Simulation Results

- The data generating process is $dX_t/X_0 = \sigma_t dW_t + dY_t$

- Y is a pure jump process, $\beta-$stable or Compound Poisson ($\beta = 0$).

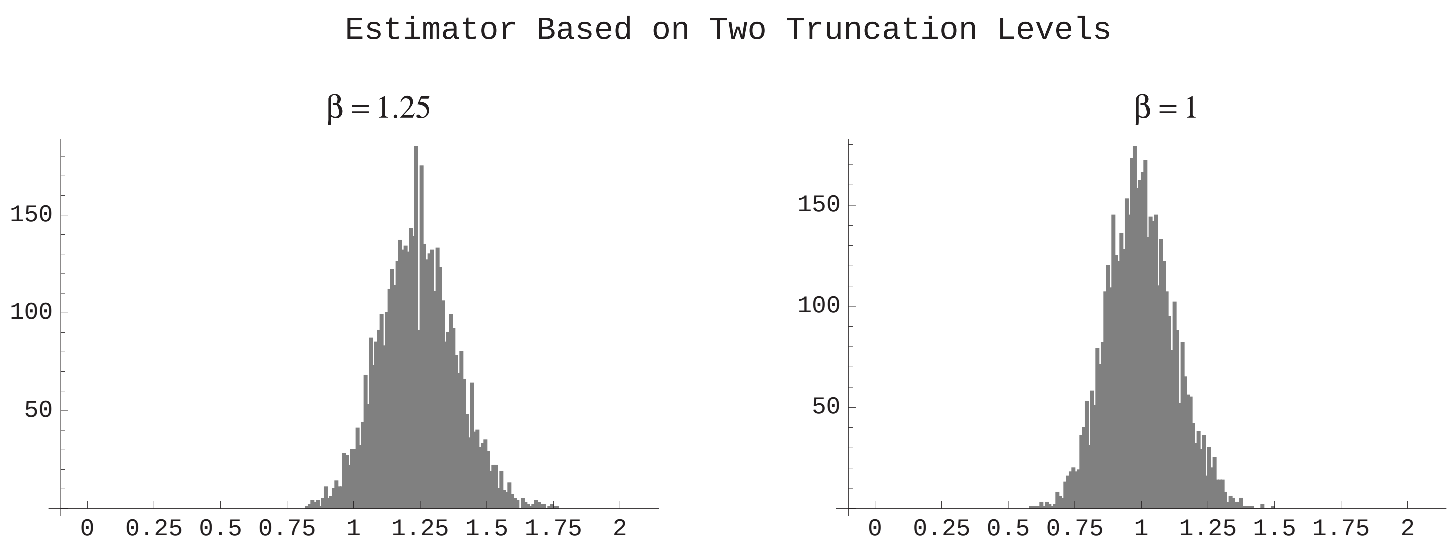
- Stochastic volatility $\sigma_t = v_t^{1/2}$

$$dv_t = \kappa(\eta - v_t)dt + \gamma v_t^{1/2} dB_t + dJ_t,$$

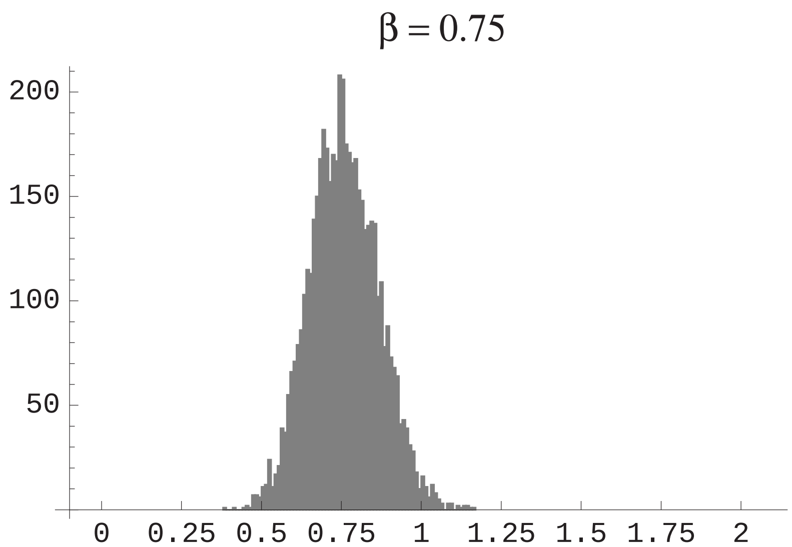
- Leverage effect: $E[dW_t dB_t] = \rho dt$, $\rho < 0$

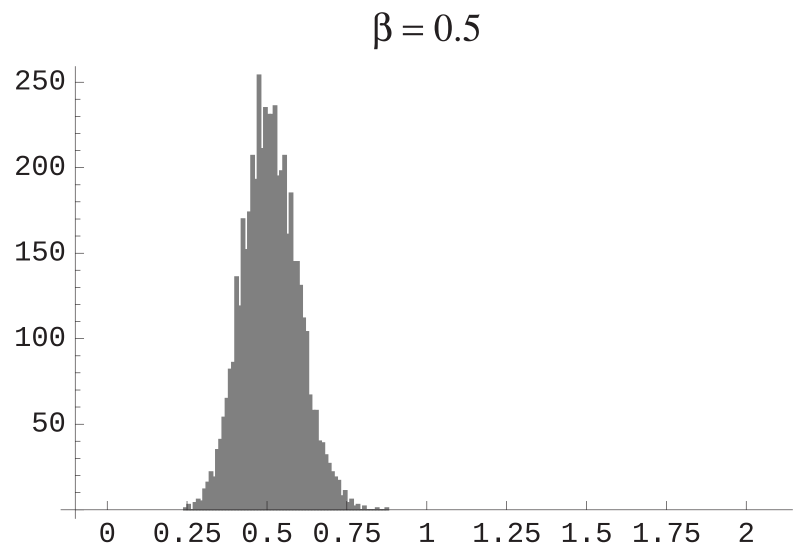
- With jumps in volatility: J is a compound Poisson process with uniform jumps.

Simulations: $\beta = 1.25$ and $\beta = 1$

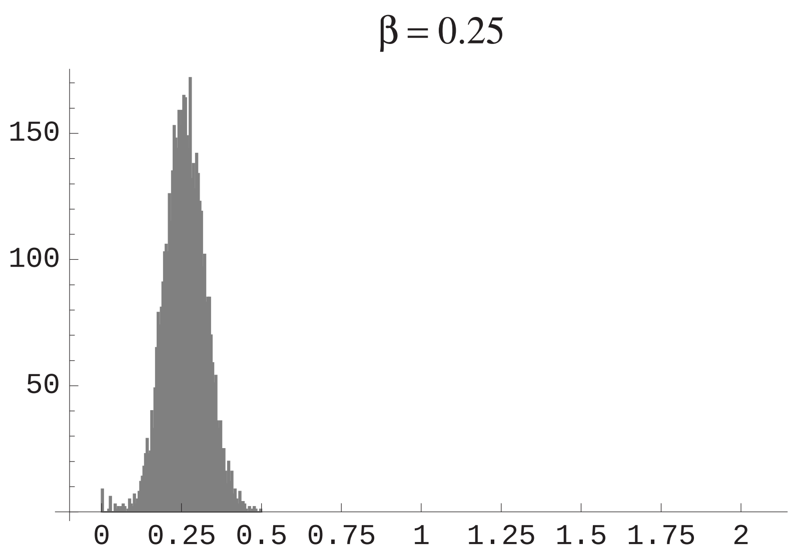


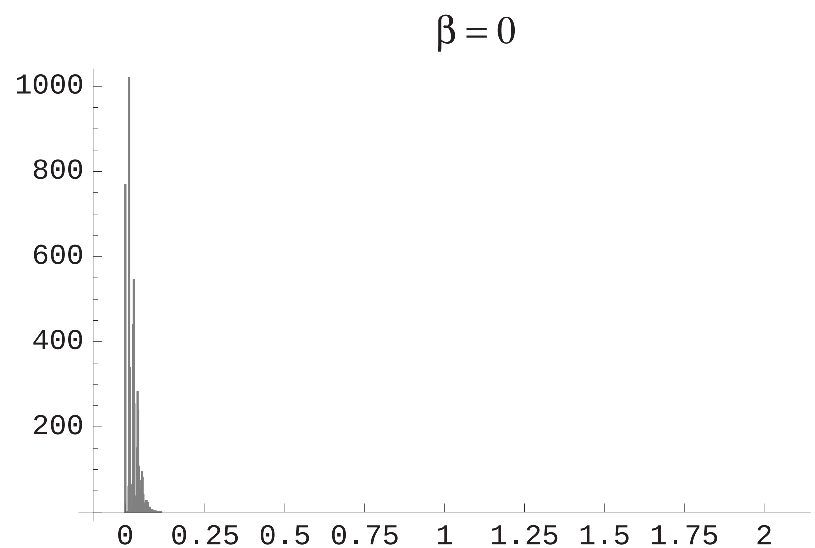
Simulations: $\beta = 0.75$ and $\beta = 0.5$





Simulations: $\beta = 0.25$ and $\beta = 0$

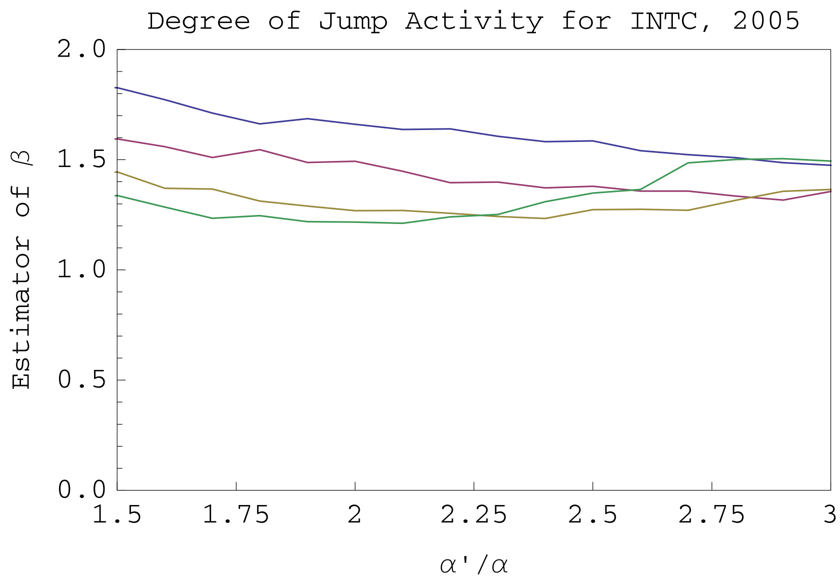


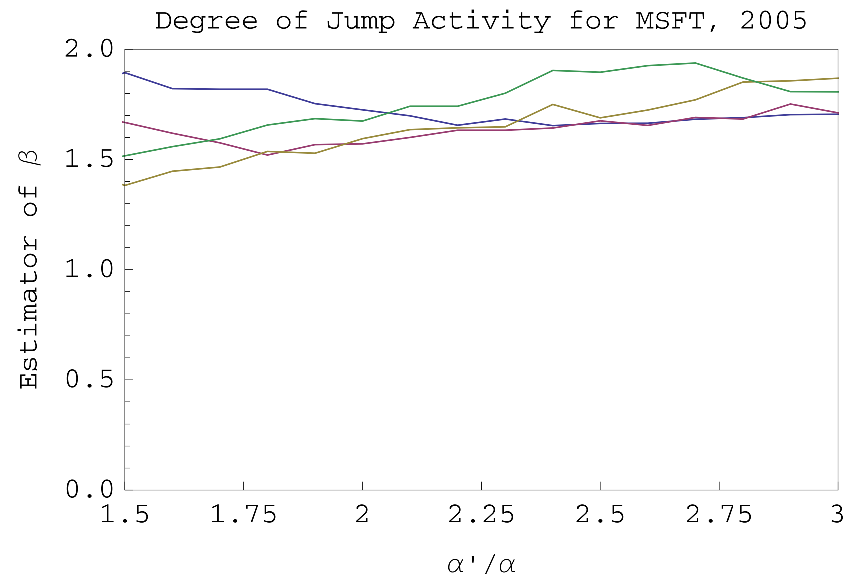


0.2. Empirical Results: Intel & Microsoft 2005

INTC									
Δ_n	2 sec			5 sec			15 sec		
α	4	5	6	4	5	6	4	5	6
Qtr 1	1.70	1.69	1.69	1.86	1.87	1.76	1.61	1.36	1.46
Qtr 2	1.06	1.06	1.05	1.23	1.13	1.09	1.09	1.13	1.14
Qtr 3	1.15	1.20	1.40	1.20	1.21	1.18	1.27	1.34	1.45
Qtr 4	1.32	1.51	1.59	1.54	1.35	1.42	1.77	1.72	1.42
All Year	1.30	1.35	1.40	1.44	1.36	1.32	1.40	1.36	1.32

	MSFT								
Δ_n	2 sec			5 sec			15 sec		
α	4	5	6	4	5	6	4	5	6
Qtr 1	1.72	1.92	1.94	1.74	1.86	1.86	1.75	1.89	2.00
Qtr 2	1.59	1.60	1.43	1.60	1.48	1.56	1.47	1.17	1.27
Qtr 3	1.50	1.60	1.63	1.52	1.54	1.63	1.66	1.81	1.97
Qtr 4	1.64	1.79	1.72	1.82	1.66	1.65	1.71	1.37	1.24
All Year	1.60	1.71	1.66	1.66	1.62	1.66	1.65	1.54	1.68


Degree of Jump Activity for INTC, 2005
Estimator of β
2.0
1.5
1.0
0.5
0.0
1.5
1.75
2
2.25
2.5
2.75
3
α'/α


Degree of Jump Activity for MSFT, 2005
Estimator of β
2.0
1.5
1.0
0.5
0.0
1.5
1.75
2
2.25
2.5
2.75
3
α'/α