Group Representations and High-Resolution Central Limit Theorems for Subordinated Spherical Random Fields

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CMB RADIATION

"THE GREATEST SCIENTIFIC DISCOVERY OF THE 20TH CENTURY, AND PERHAPS OF ALL TIMES"

(S. HAWKING, UNIVERSITY OF CAMBRIDGE)

COSMIC MICROWAVE BACKGROUND RADIATION

- **PW 1965 - NOBEL PRIZE 1978**
- **COBE 1993 - NOBEL PRIZE 2006**
- **WMAP EXPERIMENT: http://map.gsfc.nasa.gov/ (DATA RELEASES 2003 AND 2006)**
- **PLANCK EXPERIMENT: http://www.rssd.esa.int/Planck (LAUNCHED 2008)**
- **http://arxiv.org/astro-ph**
- **http://www.fisica.uniroma2.it/~cosmo/**

Isotropic Random Fields On A Sphere

$$T(\theta,\varphi)\ ,\ \ 0\leq\theta\leq\pi\quad 0\leq\varphi<2\pi$$

$$ET(\theta,\varphi)=0\ ,\ \ ET^2(\theta,\varphi)<\infty$$

T is **isotropic**, i.e.

$$T\left(x\right)\overset{\mathbf{LAW}}{=}T\left(gx\right),$$

for every **rotation** $g\in SO\left(3\right)$.

Spherical Harmonics

$$Y_{lm}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\varphi}\text{ , for }m\geq 0\text{ ,}$$

Associated Legendre Polynomials

$$P_{lm}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \text{ , } P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l,$$

$$m = 0, 1, 2, \ldots, l \text{ , } l = 1, 2, 3, \ldots \text{ .}$$

Spectral Representations

$$T(\theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \varphi),$$

Inversion Formula

$$a_{lm} = \int_{-\pi}^{\pi} \int_{0}^{\pi} T(\theta, \varphi) Y_{lm}^*(\theta, \varphi) \sin\theta d\theta d\varphi \text{ .}$$

Wigner's D Matrices (SO(3))

$$D^l_{m_1 m_2}(\alpha,\beta,\gamma) = \exp(-im_1\alpha) d^l_{m_1 m_2}(\beta) \exp(-im_2\gamma) \ ,$$

$$0 \le \alpha < 2\pi \ , \ 0 \le \beta \le \pi \ , \ 0 \le \gamma < 2\pi$$

$$D^l_{0 m_1}(\alpha,\beta,\gamma) = \sqrt{\frac{2l+1}{4\pi}} Y_{lm}(\beta,\alpha)$$

Under Isotropy (BM06), BMV(07)

$$(a_{l.}) \stackrel{d}{=} D^l(\alpha,\beta,\gamma)(a_{l.}) \text{ for all } \alpha,\beta,\gamma \ .$$

Whence (Angular Power Spectrum)

$$E a_{lm} \overline{a_{l'm'}} = \delta_l^{l'} \delta_m^{m'} C_l \ , \text{ for all } m, l \ .$$

Subordination
Subordinated field

$$F\,[T]\,(x) = F\,(T\,(x))\,, \quad x \in \mathbb{S}^2,$$

F square integrable. To simplify, $\boxed{F = H_q}$, H_q the qth **Hermite polynomial**. (No real loss of generality, since every F is just a series of the H_q's).

For every $q \geq 2$,

$$H_q\,[T]\,(x) = T^{(q)}\,(x) = \sum_{l \geq 0} \sum_{m=-l}^{l} a_{l,m}\,(q) \times Y_{lm}\,(x) = \sum_{l \geq 0} T_l^{(q)}\,(x)\,,$$

where $\{a_{l,m}\,(q)\}$ complex-valued, **non-Gaussian** array.

Frequency components

For $l \geq 0$,

$$T_l^{(q)}(x) = \sum_{m=-l}^{l} a_{l,m}(q) \times Y_{lm}(x) = \sum_{m=-l}^{l} \left\{ \int_{\mathbb{S}^2} H_q[T](z) \overline{Y_{lm}(z)} dz \right\} Y_{lm}(x),$$

the lth **frequency component** of $H_q[T]$.

Angular resolution of $180^\circ / L_{\text{MAX}}$ can just observe the **truncated** harmonic developement of $T^{(q)}$, namely

$$\sum_{l=0}^{L_{\text{MAX}}} T_l^{(q)}(x).$$

The main problem

PROBLEM: *Find conditions on the power spectrum* $\{C_l : l \geq 0\}$ *of the underlying Gaussian field* T*, to have that the finite dimensional distributions of*

$$\hat{T}_l^{(q)}(x) := \frac{T_l^{(q)}(x)}{\mathbf{Var}\left[T_l^{(q)}(x)\right]^{1/2}} = \frac{\sum_{m=-l}^{l} a_{l,m}(q) Y_{lm}(x)}{\mathbf{Var}\left[T_l^{(q)}(x)\right]^{1/2}}$$

are *“close to Gaussian”**, as* $l \to +\infty$.

High-resolution (or **high-frequency**) **Central Limit Theorem.**

Remark #1

By isotropy, the covariance of the field

$$\hat{T}_l^{(q)}(x) = \frac{T_l^{(q)}(x)}{\mathbf{Var}\left[T_l^{(q)}(x)\right]^{1/2}}$$

is always $R_{q,l}(x,y) = P_l(< x, y >)$, where P_l is the lth Legendre polynomial, and $< x, y >$ is the angle between x and y.

In general, the quantity $P_l(< x, y >)$ **does not converge** as $l \to +\infty$, so we can only look for results about Gaussian approximations, **but not for proper functional Central Limit Theorems.**

Remark #2
For every l, the span of

$$\{Y_{lm} : m = -l, ..., l\}$$

is an **irreducible representation** of $SO(3)$. Indeed,

$$\{\text{span}\{Y_{lm} : m = -l, ..., l\} : l \geq 0\} \simeq \widehat{SO(3)}.$$

This explains the appearance of Clebsch-Gordan coefficients later in the talk.

Motivations from Cosmology (Cosmic Microwave Background radiation – CMB)

Asymptotic statistical procedures on the CMB at **higher and higher frequencies** (or resolutions). Two questions related to our analysis.

1. For which models it is meaningful to test for Gaussianity at high frequencies?
2. For which models it is meaningful to use a Gaussian likelihood at high frequencies?

Our tools

For every l, the frequency field

$$\hat{T}_l^{(q)}(x) = \mathbf{Var}\left[T_l^{(q)}(x)\right]^{-1/2} \sum_{m=-l}^{l} \left\{ \int_{\mathbb{S}^2} H_q[T](z)\, \overline{Y_{lm}(z)} dz \right\} Y_{lm}(x)$$

linear functional of Hermite type transformations of T.

Hence, $\hat{T}_l^{(q)}(x)$ same law as

$$I_q^{\mathbf{W}}\left(h_{(l,q,x)}\right), \quad x \in \mathbb{S}^2,$$

where $\mathbf{W}$ appropriate Gaussian measure on $(A, \mathcal{A})$, $h_{(l,q,x)}$ symmetric kernel on A^q, $I_q^{\mathbf{W}}$ qth **Wiener-Itô integral** w.r.t. $\mathbf{W}$.

Absence of covariance conditions - Peccati, 2007, Nualart and Peccati (2005).

Theorem *Let* $I_k(l) = (I_{d_1}(l), ..., I_{d_k}(l))$*,* $l \geq 1$*, be a sequence of vectors of multiple Wiener Itô integrals (*d_i *does not change with* l*) such that*

$$\mathbb{E}\left[I_{d_i}(l)^4\right] - 3\mathbb{E}\left[I_{d_i}(l)^2\right]^2 \longrightarrow 0, \quad l \to +\infty.$$

Then, for every compact $M \subset R^k$

$$\sup_{\lambda \in M} \left| \mathbb{E}\left[\exp\left(\mathrm{i}\left\langle \lambda, \mathbf{I}_k(l)\right\rangle\right)\right] - \mathbb{E}\left[\exp\left(\mathrm{i}\left\langle \lambda, \mathbf{N}_k(l)\right\rangle\right)\right]\right| \longrightarrow 0,$$

where, $\forall l$*,* $\mathbf{N}_k(l)$ *is a* k*-dimensional Gaussian vector with the same covariance structure as* $\mathbf{I}_k(l)$*.*

No assumptions on the covariance of $\mathbf{I}_k(l)$. If $\mathbf{I}_k(l)$ has bounded variances, the Prokhorov distance between $\mathbf{N}_k(l)$ and $\mathbf{I}_k(l)$ converges to zero.

$$\hat{T}_l^{(q)}(x) = \mathbf{Var}\left[T_l^{(q)}(x)\right]^{-1/2} \sum_{m=-l}^{l} \left\{ \int_{\mathbb{S}^2} H_q[T](z) \overline{Y_{lm}(z)} dz \right\} Y_{lm}(x)$$

is basically a multiple integral $I_q^{\mathbf{W}}\left(h_{(l,q,x)}\right)$
'Asymptotically close to Gaussianity' if

$$\mathbb{E}\left[\hat{T}_l^{(q)}(x)^4\right] - 3\mathbb{E}\left[\hat{T}_l^{(q)}(x)^2\right]^2 \to 0. \qquad (*)$$

Proof - **Diagram Formulae** - for higher order cumulants lead to **generalized Gaunt integrals**

$$\int_{\mathbb{S}^2} Y_{l_1 m_1}(z) \cdots Y_{l_q m_q}(z) \overline{Y_{lm}(z)} dz.$$

Computed by using **Clebsch-Gordan matrices**.

Clebsch-Gordan matrices

Recall $\mathbf{D}_l$ the $(2l+1) \times (2l+1)$ matrix associated with the representation of $SO(3)$

$$\text{span}\left\{Y_{lm} : m = -l, ..., l\right\}.$$

Unitary Clebsch-Gordan matrices $\mathbf{C}_{l_1 l_2}$ are defined via the relations

$$\mathbf{D}_{l_1} \otimes \mathbf{D}_{l_2} = \mathbf{C}_{l_1 l_2} \left[\bigoplus_{l=|l_1-l_2|}^{l_1+l_2} \mathbf{D}_l\right] \mathbf{C}^{*}_{l_1 l_2}.$$

Non-zero elements $\mathbf{C}_{l_1 l_2}$ (real-valued)

$$C_{l_1 m_1 l_2 m_2}^{lm} \quad \begin{aligned} l &= |l_1 - l_2|, ..., l_1 + l_2 \\ -l &\leq m \leq l \text{ ; } -l_1 \leq m_1 \leq l_1 \text{ ; } -l_2 \leq m_2 \leq l_2 \end{aligned}$$

They are **probability amplitudes** in quantum mechanics.

Gaunt integrals are **convolutions** of CG coefficients.

Specific Clebsch–Gordan Coefficients

Central role is played by

$$C_{l_1 0 l_2 0}^{l0}, \quad l_1, l_2, l \geq 0.$$

Since Clebsch-Gordan matrices are unitary, for fixed l_1, l_2

$$l \longmapsto \left(C_{l_1 0 l_2 0}^{l0} \right)^2$$

is a probability over $\mathbb{N} \simeq \widehat{SO(3)}$.

Conditions for Central limit results?

The circle (Marinucci and Peccati, SPA 2007)

Isotropic Gaussian field on the circle $S^1 = [0, 2\pi)$

$$V(\theta) = \sum_{k \in \mathbb{Z}} a_k \exp(\mathrm{i}k\theta), \quad \theta \in [0, 2\pi).$$

Write $C_k = \mathbb{E}|a_k|^2$. Suppose $\sum C_k = \mathbb{E}V(\theta)^2 = 1.$

For $q \geq 2$, consider the Hermite Fourier coefficients

$$a_k(q) = \int_0^{2\pi} H_q(V(\theta)) \exp(-\mathrm{i}k\theta) \frac{d\theta}{2\pi}, \quad k \in \mathbb{Z}.$$

Consider i.i.d. random variables X_i, $i \geq 1$, such that $X_0 = 0$ and $\mathbb{P}[X_1 = k] = C_k = \mathbb{E}|a_k|^2$, as well as the random walk

$$Z_m = \sum_{i=0,1,...,m} X_i, \quad m \geq 0.$$

PROPOSITION (Marinucci and Peccati, 2007). *For $q \geq 2$, as $k \to +\infty$,*

$$\frac{a_k(q)}{\mathbf{Var}(a_k(q))^{1/2}} \xrightarrow{\mathbf{LAW}} N_{\mathbb{C}} \quad \text{(complex Gaussian)}$$

if, and only if, for every $p = 1, ..., q-1$

$$\lim_{k\to\infty} \sup_{\lambda} \mathbb{P}[Z_p = \lambda \mid Z_q = k] = 0$$

Clebsch-Gordan walks

Call $\Gamma_l = (2l+1)\, C_l$, and suppose that $\mathbb{E}T(x)^2 = \sum (2l+1)\, C_l = \sum \Gamma_l = 1$. One can define a **random walk** $\{Z_l\}$ on $\mathbb{N}$ (but indeed on $\widehat{SO(3)}$) as follows.

$$\begin{aligned} \mathbb{P}[Z_1 = l] &= \Gamma_l \\ \mathbb{P}[Z_{m+1} = l \mid Z_m = \lambda] &= \sum_{l_0} \Gamma_{l_0} \left(C^{l0}_{l_0 0 \lambda 0} \right)^2 . \end{aligned}$$

This can be represented by coupling of angular momenta in a quantum mechanical system of increasing complexity.

(For other walks on $\widehat{SO(3)}$, see Roynette (1974) and Guivarc'h, Keane and Roynette (1975))

The case $q = 2$
Sufficient condition to have that

$$T_l^{(2)}(x) = \mathbf{Var}\left[T_l^{(2)}(x)\right]^{-1/2} \sum_{m=-l}^{l} \left\{ \int_{\mathbb{S}^2} H_2[T](z)\, \overline{Y_{lm}(z)} dz \right\} Y_{lm}(x)$$

is close to Gaussian for $l \to \infty$

$$\lim_{l \to \infty} \sup_{\lambda \geq 0} \mathbb{P}\left[Z_1 = \lambda \mid Z_2 = l\right] = 0.$$

The case $q = 3$
A sufficient condition to have that

$$T_l^{(3)}(x) = \mathbf{Var}\left[T_l^{(3)}(x)\right]^{-1/2} \sum_{m=-l}^{l} \left\{ \int_{\mathbb{S}^2} H_3[T](z) \overline{Y_{lm}(z)} dz \right\} Y_{lm}(x)$$

is close to Gaussian when $l \to +\infty$ is that

$$\lim_{l\to\infty} \sup_{\lambda \geq 0} \mathbb{P}[Z_1 = \lambda \mid Z_3 = l] = \lim_{l\to\infty} \sup_{\lambda \geq 0} \mathbb{P}[Z_2 = \lambda \mid Z_3 = l] = 0.$$

Similar (but still partial) results have been proved for general $q \geq 4$.

A duality on the torus and on the sphere

On tori and on the sphere, one observes the basic duality:

- If $C_l \sim l^{\alpha} \exp(-\beta l)$, then the Gaussian approximation takes place.
- If $C_l \sim l^{-\gamma}$, then the Gaussian approximation does not hold.

References

[1] D. Marinucci and G. Peccati (2007). High-frequency asymptotics for subordinated stationary fields on an Abelian compact group. To appear in: *Stochastic Processes and their Applications.*

[2] D. Marinucci and G. Peccati (2007). Group representations and high-frequency central limit theorems for spherical random fields. Arxiv Preprint 0706.2851.