Self-similar random fields and rescaled random balls models

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Outlines

1 Random balls models

1.1 Mathematical description

We start with a family of grains $X_j + B(0, R_j)$ in $\mathbb{R}^d$ where

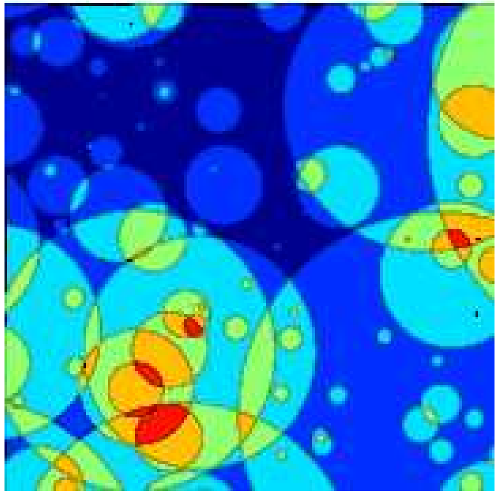
$$(X_j, R_j)_j \text{ is a Poisson point process in } \mathbb{R}^d \times \mathbb{R}^+,$$

with intensity measure $n(\mathrm{d}x, \mathrm{d}r) = \mathrm{d}x F(\mathrm{d}r)$ where F is a σ-finite non-negative measure on $\mathbb{R}^+$ such that

$$\int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty.$$

Let N be the associated Poisson random measure on $\mathbb{R}^d \times \mathbb{R}^+$.
For $A \in \mathcal{B}\left(\mathbb{R}^d \times \mathbb{R}^+\right)$ with $n(A) < \infty$

$$N(A) = \#\left\{j; (X_j, R_j) \in A\right\} \sim \mathcal{P}(n(A)).$$



Associated shot-noise random field

One can define the random field X on $\mathbb{R}^d$ by

$$X(t) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{B(x,r)}(t) N(dx, dr)$$
$$= \# \text{ grains containing } t \in \mathbb{R}^d.$$

Examples

$\mathbf{d} = 1 \longrightarrow X(t) =$ **numbers of connections to a server at time** t

$\mathbf{d} = 2 \longrightarrow X(t) =$ **discretized gray level** at point t in a picture

$\mathbf{d} = 3 \longrightarrow X(t) =$ **mass density** of a 3D granular media in t

Main properties

- X is stationary and isotropic
- $\mathbb{E}X(t) = \int_{\mathbb{R}^d \times \mathbb{R}+} \mathbf{1}_{B(x,r)}(t) \mathrm{d}x F(\mathrm{d}r) = c_d \int_{\mathbb{R}+} r^d F(\mathrm{d}r)$
- $\mathrm{Cov}(X(t), X(t')) = \int_{\mathbb{R}+} |B(t,r) \cap B(t',r)| \, F(\mathrm{d}r)$

1.2 Generalizations

➥ For V a Borel set of $\mathbb{R}^d$,

$$X(V) \quad = \quad \int_{\mathbb{R}^d \times \mathbb{R}+} |V \cap B(x,r)|\, N(dx, dr) = \text{ weighted coverage of } V.$$

➥ For $\mu \in \mathcal{M} = \{\text{signed measures } \mu \text{ on } \mathbb{R}^d; |\mu|(\mathbb{R}^d) < +\infty\}$,

$$\boxed{X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}+} \mu(B(x,r))\ N(\mathrm{d}x, \mathrm{d}r)}$$

- $\mathbb{E}\left(X(\mu)\right) = \int_{\mathbb{R}^d \times \mathbb{R}+} \mu(B(x,r))\mathrm{d}xF(\mathrm{d}r) = \mu\left(\mathbb{R}^d\right) c_d \int_{\mathbb{R}+} r^d F(\mathrm{d}r)$
- $\mathrm{Var}\left(X(\mu)\right) = \int_{\mathbb{R}^d \times \mathbb{R}+} \mu(B(x,r))^2 \mathrm{d}xF\mathrm{d}r \leq |\mu|\left(\mathbb{R}^d\right)^2 c_d \int_{\mathbb{R}+} r^d F(\mathrm{d}r)$

Prop: X is a continuous random linear functional on $\mathcal{M}$ ie

$$\boxed{X \ : \ \mathcal{M} \to L^2(\Omega) \text{ is continuous}}$$

Remark: $X(t) = X(\delta_t)$ for any $t \in \mathbb{R}^d$

2 Scaling behavior

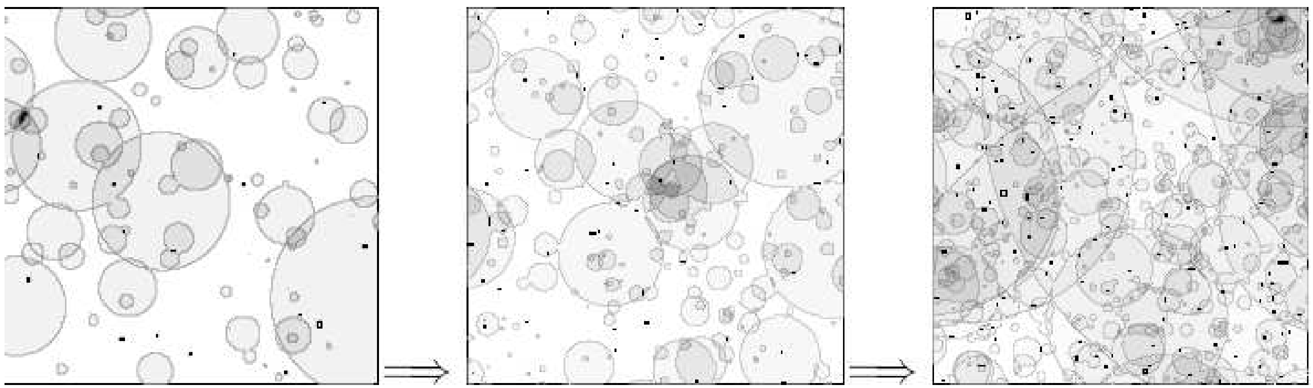
Let us multiply the radii by $\rho > 0$ and the intensity measure by $\lambda(\rho) > 0$:

$$n(\mathrm{d}x, \mathrm{d}r) = \mathrm{d}x F(\mathrm{d}r) \curvearrowright n_{\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r) = \lambda(\rho) \mathrm{d}x F_\rho(\mathrm{d}r)$$

$F_\rho(\mathrm{d}r)$ = image measure of $F(\mathrm{d}r)$ by the change of scale $r \mapsto \rho r$.

$$\boxed{X_\rho(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r)) \; N_{\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r)}$$

$$\begin{cases} \text{zoom-in:} & \rho \to +\infty \quad \text{(small grain assumption)} \\ \text{zoom-out:} & \rho \to 0 \quad \text{(large grain assumption)} \end{cases}$$



We are looking for a normalization term $n(\rho)$ s.t.

$$\frac{X_\rho(.) - \mathbb{E}(X_\rho(.))}{n(\rho)} \xrightarrow{fdd} W(.),$$

when $\rho \to 0$ (zoom-out) or $\rho \to +\infty$ (zoom-in). By linearity, it holds if

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right) \longrightarrow \mathbb{E}\left(\exp\left(iW(\mu)\right)\right),$$

for all μ in a convenient subspace of $\mathcal{M}$. Let us write

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right) = \exp\left(\int_{\mathbb{R}+} \lambda(\rho)\varphi_\rho(r)F_\rho(\mathrm{d}r)\right).$$

with

$$\varphi_\rho(r) = \int_{\mathbb{R}^d} \Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right)\mathrm{d}x \text{ and } \Psi(v) = e^{iv} - 1 - iv.$$

- Gaussian limit: $n(\rho) \to +\infty$, $\varphi_\rho(r) \sim -\frac{1}{2n(\rho)^2}\int_{\mathbb{R}^d} \mu(B(x,r))^2\mathrm{d}x$.
- Poisson limit: $n(\rho) \to n_0$, $\varphi_\rho(r) \sim \int_{\mathbb{R}^d} \Psi\left(\frac{\mu(B(x,r))}{n_0}\right)\mathrm{d}x$.

2.1 Power law assumptions

Intensity $\mathbf{A}(\beta)$: for $\beta \neq d$, assume $F(\mathrm{d}r) = f(r)\mathrm{d}r$ with

$$f(r) \sim C_\beta r^{-\beta-1}\ , \text{ as } \ r \longrightarrow 0^{d-\beta} = \begin{cases} r \longrightarrow +\infty, & \beta > d \text{ (zoom-out)} \\ r \longrightarrow 0, & \beta < d \text{ (zoom-in)} \end{cases} .$$

We consider F satisfying $\mathbf{A}(\beta)$ and $\int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty$.

Lemma: Under $\mathbf{A}(\beta)$ if g is a continuous function on $\mathbb{R}^+$ such that $|g(r)| \leq C \min(r^q, r^p)$, for some $0 < p < \beta < q$, then

$$\int_{\mathbb{R}^+} g(r) F_\rho(\mathrm{d}r) \underset{\rho \to 0^{\beta-d}}{\sim} C_\beta \rho^\beta \int_{\mathbb{R}^+} g(r) r^{-\beta-1} \mathrm{d}r.$$

Finite energy measures

$$\mathcal{I}_\alpha(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-z'|^{d-\alpha}\mu(\mathrm{d}z)\mu(\mathrm{d}z')$$

- For $\beta > d$, $\mathcal{M}_\beta = \{\mu \in \mathcal{M} \ ; \ \exists\alpha > \beta \ , \ \mathcal{I}_\alpha(|\mu|) < +\infty\}$
- For $\beta < d$, $\mathcal{M}_\beta = \left\{\mu \in \mathcal{M} \ ; \ \exists\alpha < \beta \ , \ \mathcal{I}_\alpha(|\mu|) < +\infty \text{ and } \int \mu = 0 \ \right\}$

Prop: for $\beta \in (d-1, 2d)$ with $\beta \neq d$, when $\mu \in \mathcal{M}_\beta$

$$\int_{\mathbb{R}^+\times\mathbb{R}^d} \mu(B(x,r))^2 r^{-\beta-1}\mathrm{d}r\mathrm{d}x = c_\beta \mathcal{I}_\beta(\mu) < +\infty.$$

Remark: -$c_\beta > 0$ for $d < \beta < 2d$ and $c_\beta < 0$ for $d-1 < \beta < d$;

-for any $t \in \mathbb{R}^d$, the Dirac mass $\delta_t \notin \mathcal{M}_\beta$;

-for any $t \in \mathbb{R}^d$, when $d-1 < \beta < d$, $\delta_t - \delta_0 \in \mathcal{M}_\beta$.

2.2 Gaussian limit

Theorem: Assume

- $d-1<\beta<2d$ such that $\beta \neq d$
- $\mathbf{A}(\beta)\ : F(\mathrm{d}r) \sim C_\beta r^{-\beta-1}\ \mathrm{d}r$ as $r \to 0^{d-\beta}$
- $\lambda(\rho)\rho^\beta \longrightarrow +\infty$ as $\rho \to 0^{\beta-d}$

Then for all $\mu \in \mathcal{M}_\beta$, as $\rho \to 0^{\beta-d} = \begin{cases} \rho \to 0, & \beta > d \text{ (zoom-out)} \\ \rho \to +\infty & \beta < d \text{ (zoom-in)} \end{cases}$

$$\frac{X_\rho(\mu)-\mathbb{E}\big(X_\rho(\mu)\big)}{\sqrt{\lambda(\rho)\rho^\beta}} \xrightarrow{fdd} W_\beta(\mu)$$

The limit field W_β is the centered Gaussian continuous random linear functional on $\mathcal{M}_\beta$ with

$$\mathrm{Cov}\left(W_\beta(\mu), W_\beta(\nu)\right) = c_\beta \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-z'|^{d-\beta}\mu(\mathrm{d}z)\nu(\mathrm{d}z')$$

Properties of W_β

Notations: For $\mu \in \mathcal{M}$,

- for $a > 0$: $\mu_a(A) := \mu(a^{-1}A)$ for any Borel set A;
- for $s \in \mathbb{R}^d$: $\tau_s\mu(A) := \mu(A - s)$ for any Borel set A.

Definition: A random field X, defined on $\mathcal{S} \subset \mathcal{M}$ is said to be

- H self-similar, for $H \in \mathbb{R}$, if

$$\forall \mu \in \mathcal{S}, \forall a > 0\ ,\ \mu_a \in \mathcal{S} \ \text{ and } \ X(\mu_a) \overset{fdd}{=} a^H X(\mu)\,.$$

- translation invariant, if

$$\forall \mu \in \mathcal{S}, \forall s \in \mathbb{R}^d\ ,\ \tau_s\mu \in \mathcal{S} \ \text{ and } \ X(\tau_s\mu) \overset{fdd}{=} X(\mu)\,.$$

Prop: Let $\beta \in (d-1, 2d) \smallsetminus \{d\}$. The field W_β defined on $\mathcal{M}_\beta$ is

- self-similar with index $H = \frac{d-\beta}{2} \in (-d/2, 1/2) \smallsetminus \{0\}$
- translation invariant.

2.3 Poisson limit

Theorem: Assume

- $d-1<\beta<2d$ such that $\beta\neq d$
- $\mathbf{A}(\beta)\ :\ F(\mathrm{d}r)\sim C_\beta r^{-\beta-1}\ \mathrm{d}r$ as $r\to 0^{d-\beta}$
- $\lambda(\rho)\rho^\beta \longrightarrow \sigma_0^{d-\beta}$ as $\rho\to 0^{\beta-d}$

Then for all $\mu\in\mathcal{M}_\beta$, as $\rho\to 0^{\beta-d}=\begin{cases}\rho\to 0, & \beta>d \text{ (zoom-out)}\\ \rho\to+\infty & \beta<d \text{ (zoom-in)}\end{cases}$

$$\boxed{X_\rho(\mu)-\mathbb{E}\left(X_\rho(\mu)\right)\xrightarrow{fdd} J_\beta(\mu_{\sigma_0})}$$

The limit field J_β is the centered continuous random linear functional on $\mathcal{M}_\beta$ defined as

$$J_\beta(\mu)=\int_{\mathbb{R}^d}\int_{\mathbb{R}+}\mu\left(B(x,r)\right)\widetilde{N_\beta}(\mathrm{d}x,\mathrm{d}r),$$

where $\widetilde{N_\beta}$ is a compensated Poisson random measure with intensity $C_\beta \mathrm{d}x\ r^{-\beta-1}\mathrm{d}r$, and μ_{σ_0} is defined by $\mu_{\sigma_0}(A)=\mu\left(\sigma_0^{-1}A\right)$.

Properties of J_β

Let $\beta \in (d-1, 2d) \setminus \{d\}$. Then,

- J_β is not Gaussian, not self-similar.
- J_β is aggregate similar with index $H = \frac{d-\beta}{2} \in (-d/2, 1/2) \setminus \{0\}$:

$$\forall m \geq 1, \quad \sum_{i=1}^{m} J_\beta^i(\mu) \overset{fdd}{=} J_\beta(\mu_{a_m}) \quad \text{for} \quad a_m = m^{1/2H},$$

  where $\left(J_\beta^i\right)$ are iid copies of J_β.

- J_β is translation invariant.
- J_β has the same covariance as W_β (H second order self-similar).

Remark: W_β is also aggregate-similar with index H.

3 Self-similar random fields

3.1 Dobrushin's characterization

Schwartz classes: $\mathcal{S}(\mathbb{R}^d) = \{\varphi : \mathbb{R}^d \to \mathbb{R} \, , \, \mathcal{C}^\infty \, , \, \text{rapidely} \searrow\}$

$$\begin{aligned} \mathcal{S}_n(\mathbb{R}^d) &= \{\varphi \in \mathcal{S}(\mathbb{R}^d); \int_{\mathbb{R}^d} z^j \varphi(z) dz = 0 \, , \, j \in \mathbb{N}^d \, , \, |j| < n\} \\ &= \{\varphi \in \mathcal{S}(\mathbb{R}^d); D^j \widehat{\varphi}(0) = 0 \, , \, j \in \mathbb{N}^d \, , \, |j| < n\} \end{aligned}$$

where $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(z) \mathrm{d}z \in \mathcal{S}(\mathbb{R}^d)$. Since $\left(\mathcal{S}_n(\mathbb{R}^d)\right)_n \searrow$, set

$$\mathcal{S}_0(\mathbb{R}^d) := \mathcal{S}(\mathbb{R}^d) \text{ and } \mathcal{S}_\infty(\mathbb{R}^d) := \bigcap_n \mathcal{S}_n(\mathbb{R}^d).$$

Identifying $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\varphi(z)dz \in \mathcal{M}$, $\forall \beta \in (d-1, 2d) \setminus \{d\}$

$$\mathcal{S}_n(\mathbb{R}^d) \subset \mathcal{M}_\beta \text{ for all } n \geq 1, \text{ and } n \geq 0 \text{ if } \beta > d$$

Theorem [Dobrushin (79)]. Let $n \geq 0$ and X be a centered continuous random linear functional on $\mathcal{S}_n(\mathbb{R}^d)$.

X is translation invariant and self-similar of order $H \in \mathbb{R}$

if and only if

$$\begin{aligned} \mathrm{Cov}\,(X(\varphi), X(\psi)) &= \int_{S^{d-1}} \int_0^{+\infty} \widehat{\varphi}(r\theta)\overline{\widehat{\psi}}(r\theta) r^{-2H-1} \mathrm{d}r \mathrm{d}\sigma(\theta) \\ &+ \sum_{|j|=|k|=n} A_{j,k} \alpha_j(\varphi)\overline{\alpha_k(\psi)}, \forall \varphi, \psi \in \mathcal{S}_n(\mathbb{R}^d) \end{aligned}$$

where

- $\sigma = 0$ if $H \geq n$;
- $(A_{j,k})_{|j|=|k|=n} = 0$ if $H \neq n$
- $\alpha_j(\varphi) = \int_{\mathbb{R}^d} \varphi(x) x^j \mathrm{d}x = i^{|j|} D^j \widehat{\varphi}(0)$.

Corollary: Let $\beta \in (d-1, 2d) \setminus \{d\}$ and $H = \frac{d-\beta}{2} \in (-d/2, 1/2) \setminus \{0\}$. $\exists k_\beta > 0$ s.t.,

$$\begin{aligned} \mathrm{Cov}\,(W_\beta(\varphi), W_\beta(\psi)) &= \mathrm{Cov}\,(J_\beta(\varphi), J_\beta(\psi)) \\ &= c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{2H} \varphi(z)\psi(z')\, \mathrm{d}z\mathrm{d}z' \\ &= k_\beta \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\overline{\widehat{\psi}}(\xi)\, |\xi|^{-2H-d}\, \mathrm{d}\xi, \end{aligned}$$

$\forall \varphi, \psi \in \mathcal{S}_1(\mathbb{R}^d)$ if $0 < H < 1/2$, $\mathcal{S}(\mathbb{R}^d)$ if $-d/2 < H < 0$.

Notation: Let $H \in \mathbb{R} \setminus \mathbb{Z}$ and consider (cf Dobrushin) B_H Gaussian s.t.

$$\mathrm{Cov}\,(B_H(\varphi), B_H(\psi)) = k_H \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\overline{\widehat{\psi}}(\xi)\, |\xi|^{-2H-d}\, \mathrm{d}\xi, \quad \forall \varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^d).$$

Remark: B_H is the only Gaussian generalized random field isotropic, translation invariant and self-similar of order H. Moreover

$$\boxed{B_H = W_{d-2H} \text{ for all } H \in (-d/2, 1/2) \setminus \{0\}}$$

Let $H \in \mathbb{R} \setminus \mathbb{Z}$ and $m = \left[H + \frac{1}{2}\right] \in \mathbb{Z}$ s.t. $H \in [m - 1/2, m + 1/2)$.

Define the operator $(-\Delta)^{-\frac{m}{2}} : \mathcal{S}_\infty(\mathbb{R}^d) \longrightarrow \mathcal{S}_\infty(\mathbb{R}^d)$ by

$$\widehat{(-\Delta)^{-\frac{m}{2}}\varphi}(\xi) = |\xi|^{-m}\widehat{\varphi}(\xi).$$

Theorem: For $\beta_H = d - 2(H - m)$ assume

- $\mathbf{A}(\beta_H)\ : F(\mathrm{d}r) \underset{r \to 0^{H-m}}{\sim} C_\beta r^{-\beta_H - 1}\ \mathrm{d}r;$
- $\lambda(\rho)\rho^{\beta_H} \underset{\rho \to 0^{m-H}}{\longrightarrow} +\infty.$

Then for all $\varphi \in \mathcal{S}_\infty$, as $\rho \to 0^{m-H} = \begin{cases} \rho \to 0, & H < m \text{ (zoom-out)} \\ \rho \to +\infty & H > m \text{ (zoom-in)} \end{cases}$

$$\frac{X_\rho((-\Delta)^{-m/2}\varphi) - \mathbb{E}\big(X_\rho((-\Delta)^{-m/2}\varphi)\big)}{\sqrt{\lambda(\rho)\rho^{\beta_H}}} \xrightarrow{fdd} B_H(\varphi).$$

Let $H \in \mathbb{R} \setminus \mathbb{Z}$ and $m = \left[H + \frac{1}{2}\right] \in \mathbb{Z}$ s.t. $H \in [m - 1/2, m + 1/2)$.

Theorem: For $\beta_H = d - 2(H - m)$ assume

- $\mathbf{A}(\beta_H) \ : F(\mathrm{d}r) \underset{r \to 0^{H-m}}{\sim} C_\beta r^{-\beta_H - 1} \, \mathrm{d}r$;
- $\lambda(\rho)\rho^{\beta_H} \underset{\rho \to 0^{m-H}}{\longrightarrow} \sigma_0^{2(H-m)}$.

Then for all $\varphi \in \mathcal{S}_\infty$, as $\rho \to 0^{m-H} = \begin{cases} \rho \to 0, & H < m \text{ (zoom-out)} \\ \rho \to +\infty & H > m \text{ (zoom-in)} \end{cases}$

$$\boxed{X_\rho((-\Delta)^{-m/2}\varphi) - \mathbb{E}\left(X_\rho((-\Delta)^{-m/2}\varphi)\right) \xrightarrow{fdd} P_H(\sigma_0^m \varphi_{\sigma_0}),}$$

where

$$P_H(\varphi) = J_{\beta_H}((-\Delta)^{-m/2}\varphi), \quad \varphi \in \mathcal{S}_\infty(\mathbb{R}^d).$$

3.2 Representation

Let $H \in \mathbb{R}^+ \setminus \mathbb{N}$.

There exists $\widetilde{B_H} : \mathbb{R}^d \to L^2(\Omega)$ a representation of B_H on $\mathcal{S}_\infty(\mathbb{R}^d)$ ie

$$\forall\, \varphi \in \mathcal{S}_\infty(\mathbb{R}^d), \quad B_H(\varphi) \overset{L^2(\Omega)}{=} \int_{\mathbb{R}^d} \widetilde{B_H}(t)\varphi(t)\mathrm{d}t$$

Moreover, for any $t, s \in \mathbb{R}^d$,

$$\mathrm{Cov}\left(\widetilde{B_H}(t), \widetilde{B_H}(s)\right) =$$

$$k_H \int_{\mathbb{R}^d} \left(e^{-it\cdot\xi} - \sum_{k=0}^{\lceil H\rceil -1} \frac{(it\cdot\xi)^k}{k!}\right) \overline{\left(e^{-is\cdot\xi} - \sum_{k=0}^{\lceil H\rceil -1} \frac{(is\cdot\xi)^k}{k!}\right)} |\xi|^{-d-2H} \mathrm{d}\xi$$

$$= c_H \left(|t-s|^{2H} - \sum_{|l|=0}^{\lceil H\rceil -1} \frac{(-1)^{|l|}}{l!} \left(s^l D^l |t|^{2H} + t^l D^l |s|^{2H} \right) \right)$$

Rk: there also exists a representation $\widetilde{P_H} : \mathbb{R}^d \to L^2(\Omega)$ of P_H on $\mathcal{S}_\infty(\mathbb{R}^d)$.

Characterization of $\widetilde{B_H}$ for $H > 0$

$\widetilde{B_H}$ is the only Gaussian random field that satisfies

- stationary n^{th} increments for $n = \lceil H \rceil$
- self-similar of order H
- isotropic
- $D^j \widetilde{B_H}(0) = 0$ a.s for all $|j| < \lceil H \rceil$

➥ For $H \in (0, 1)$,

$\widetilde{B_H}$ =Fractional Brownian field of Hurst parameter H.

➥ For $H \in (0, 1/2)$ and $\beta_H = d - 2H \in (d-1, d)$,

$$\left\{ \widetilde{B_H}(x); x \in \mathbb{R}^d \right\} \overset{fdd}{=} \{ W_{\beta_H}(\delta_x - \delta_0); x \in \mathbb{R}^d \}.$$

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