Infinite variance stable limits for dependent sequences [1]

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Stable limits for an iid sequence

- For an iid real-valued sequence (X_t) consider the partial sums $S_n = X_1 + \cdots + X_n$, $n \geq 1$.
- Using classical limit theory for sums of independent random variables, e.g. Gnedenko, Kolmogorov (1954), Feller (1971), Petrov (1975, 1996), one can show that there exist sequences $0 < a_n \to \infty$ and $b_n \in \mathbb{R}$ and a random variable Y with non-degenerate law H such that

$$a_n^{-1}(S_n - b_n) \xrightarrow{d} Y \sim H$$

if and only if either $f(x) = EX^2 I_{\{|X| \leq x\}}$, $x > 0$, is slowly varying or X is regularly varying with index $\alpha \in (0, 2)$, i.e., there exist $p, q \geq 0$ with $p + q = 1$ and a slowly varying function L such that

$$P(X > x) \sim p\,\frac{L(x)}{x^\alpha} \quad \text{and} \quad P(X \le -x) \sim q\,\frac{L(x)}{x^\alpha}\,,$$

- $H = H_\alpha$, $\alpha \in (0, 2]$, is α-stable in the convolution sense, i.e. for any $n \ge 2$ and an iid sequence (Y_t) with common distribution H, there exist $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$c_n^{-1}(Y_1 + \cdots + Y_n - d_n) \stackrel{d}{=} Y\,.$$

- Moreover, for $\alpha \in (0, 2)$, (a_n) can be chosen such that $P(|X| > a_n) \sim n^{-1}$ and $b_n = n\,EXI_{\{|X| \le a_n\}}$.
- Classical proofs are based on characteristic function arguments.
- An alternative way of proving this result goes back to LePage, Woodroofe, Zinn (1981), Resnick (1986); see also Resnick (2007).

- Since regular variation of X for any $\alpha > 0$ is equivalent to the weak convergence of the point processes

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t} \stackrel{d}{\rightarrow} N = \sum_{t=1}^{\infty} \varepsilon_{J_t} \sim \mathrm{PRM}(\mu)$$

for some Poisson random measure N with mean measure μ on $\overline{\mathbb{R}} \setminus \{0\}$ given by

$$\mu(dx) = [p\, x^{-\alpha} I_{\{x>0\}} + q\, |x|^{-\alpha} I_{\{x<0\}}]\, dx\,.$$

- The mapping $T_\epsilon : M_p \rightarrow \mathbb{R}$ given by

$$T_\epsilon(m) = T_\epsilon(\sum_t \varepsilon_{j_t}) = \sum_t j_t I_{\{|j_t|>\epsilon\}}$$

is a.s. continuous relative to the distribution of N for every $\epsilon > 0$.

- Hence

$$T_\epsilon(N_n) = \sum_{t=1}^{n} (a_n^{-1} X_t) I_{(\{|a_n^{-1} X_t| > \epsilon\}} \xrightarrow{d} T_\epsilon(N) = \sum_{t=1}^{\infty} J_t I_{\{|J_t| > \epsilon\}} .$$

- For $\alpha \in (0,2)$ the right-hand side has a limit as $\epsilon \downarrow 0$ (with additional centering for $\alpha \in [1,2)$): *series representation of an α-stable random variable.*

- Example. Assume $p = 1$ (X is totally skewed to the right) and $\alpha \in (0,1)$. Then $N = \sum_{t=1}^{\infty} \varepsilon_{\Gamma_i^{-1/\alpha}}$, where $0 < \Gamma_1 < \Gamma_2 < \cdots$ are the points of a homogeneous Poisson process. Hence

$$T_\epsilon(N) = \sum_{t=1}^{\infty} \Gamma_t^{-1/\alpha} I_{\{|\Gamma_t^{-1/\alpha}| > \epsilon\}} \xrightarrow{\text{a.s.}} \sum_{t=1}^{\infty} \Gamma_t^{-1/\alpha} \quad \text{as } \epsilon \downarrow 0 .$$

which represents an α-stable random variable.

- **It finally suffices to show that**

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} P(|a_n^{-1}S_n - T_\epsilon(N_n) - E(\cdot)| > \delta) = 0\,, \quad \delta > 0\,,$$

e.g. by showing that var$(a_n^{-1}S_n - T_\epsilon(N_n))$ can be made small.

Generalizations to dependent sequences

Linear processes.

- Recall the definition of a linear process

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{1}$$

for sequences of suitable constants ψ_j, $j \in \mathbb{Z}$, and an iid sequence (Z_t).

- If Z is regularly varying with index $\alpha > 0$, i.e.,

$$P(Z > x) \sim p \frac{L(x)}{x^\alpha} \quad \text{and} \quad P(Z \le -x) \sim q \frac{L(x)}{x^\alpha},$$

and the series (1) converges a.s. then X is regularly varying with index $\alpha > 0$.[2]

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[2] The converse is not true in general; see Jacobsen, Mikosch, Rosiński, Samorodnitsky (2009).

- In a series of papers, Davis and Resnick (1985, 1986) proved that the sequence of the partial sums $(a_n^{-1} S_n)$ has a stable limit for $\alpha \in (0,2)$. They also showed the joint convergence for

$$\sum_{t=1}^{n} \Big(a_n^{-1} X_t, a_n^{-1} X_t^2, \widetilde{a}_n^{-1} X_t X_{t+1}, \ldots, \widetilde{a}_n^{-1} X_t X_{t+h} \Big) - \mathrm{b}_n$$

towards a mixed stable distribution. This was achieved by using the weak convergence of the underlying point processes and a continuous mapping argument.

- Phillips and Solo (1992) used the structure of a linear process to show that, under general weak dependence conditions,

$$a_n^{-1} \left(\sum_{t=1}^{n} X_t - \sum_{j=0}^{n} \psi_j \sum_{t=1}^{n} Z_t \right) \xrightarrow{P} 0 \,,$$

thus the stable CLT for (X_t) follows from the one for (Z_t).

- **Kasahara, Maejima, Vervaat (1988) also considered stable FCLTs in the case of strong dependence.**

Mixing conditions.

- Let (X_t) be a strictly stationary sequence with partial sum process $S_n = X_1 + \cdots + X_n$, $n \geq 1$.
- Davis and Hsing (1995) proved stable limit theory by using the point process approach.
- Davis and Hsing (1995) require the mixing condition $\mathcal{A}(a_n)$ in terms of the point processes

$$N_{nm} = \sum_{t=1}^{m} \varepsilon_{a_n^{-1} X_t} \quad \text{and} \quad N_n = N_{nn} = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t} .$$

- They require closeness of the Laplace functionals

$$E \mathrm{e}^{-\int f dN_n} - \left(E \mathrm{e}^{-\int f dN_{nm}} \right)^{k_n} \to 0 ,$$

where $m = m_n \to \infty$, $k_n = [n/m] \to \infty$.

- Bartkiewicz et al. (2010) prove stable limit theory by using characteristic functions.
- Bartkiewicz et al. (2010) require a mixing condition in terms of the characteristic functions

$$\varphi_n(x) = E\mathrm{e}^{ixa_n^{-1}S_n} \quad \text{and} \quad \varphi_{nm}(x) = E\mathrm{e}^{ixa_n^{-1}S_m}.$$

- They require closeness of the characteristic functions

$$\varphi_n(x) - \Big(\varphi_{nm}(x)\Big)^{k_n} \to 0\,,$$

where $m = m_n \to \infty$, $k_n = [n/m] \to \infty$.

- Conditions of this type as well as $\mathcal{A}(a_n)$ follow from strong mixing with suitable rates.

- **These conditions imply that the corresponding limits, if they exist, are *infinitely divisible.***

Conditions on the tails.

- To ensure convergence to an infinite variance stable limit, we require regular variation of the finite-dimensional distributions of (X_t) as in Davis and Hsing (1995) and Bartkiewicz et al. (2010):[3] There exist $\alpha \geq 0$ and, for every $h \geq 1$, a non-constant vector Θ_h on the unit sphere of $\mathbb{R}^h$ such that for $\mathbf{Y}_h = (X_1, \ldots, X_h)$, as $x \to \infty$,

$$\frac{P(|\mathbf{Y}_h| > x\,c)}{P(|\mathbf{Y}_h| > x)} \to c^{-\alpha}, \quad c > 0\,,$$

and

$$P(\mathbf{Y}_h/|\mathbf{Y}_h| \in \cdot \mid |\mathbf{Y}_h| > x) \xrightarrow{w} P(\Theta_h \in \cdot)\,.$$

[3] Regular variation is not necessary for partial sum convergence of a strictly stationary sequence; Surgailis (2004), Gouëzel (2004)

- **We say that (X_t) is regularly varying with index $\alpha > 0$.**
- **An *equivalent definition* is the following: for every $h \geq 1$, there exists a non-null Radon measure μ_h on $\overline{\mathbb{R}}^h \backslash \{0\}$ such that**

$$n\, P(a_n^{-1} \mathbf{Y}_h \in \cdot) \xrightarrow{v} \mu_h(\cdot)\,,$$

where (a_n) satisfies $P(|X| > a_n) \sim n^{-1}$.

- **The measure μ_h satisfies $\mu_h(t\, A) = t^{-\alpha} \mu_h(A)$, $t > 0$, for some $\alpha \geq 0$.**

- **Examples. Infinite variance stable stationary processes.**

  ARMA/linear processes with iid regularly varying noise.

  Stochastic recurrence equations $X_t = A_t X_{t-1} + B_t$ **with iid non-negative** $((A_t, B_t))$ Kesten (1973), Goldie (1991).

  GARCH processes $X_t = \sigma_t Z_t$ **with iid noise** (Z_t) **with infinite support.**

  Stochastic volatility processes with regularly varying noise (Z_t).

  Transformed Gaussian stationary sequence such that the one-dimensional marginals are regularly varying.

- If (X_t) is regularly varying with index $\alpha > 0$ so are the linear combinations of any finite segment of this sequence: for A bounded away from zero with a smooth boundary,

$$n\,P(a_n^{-1} S_d \in A) \to \mu_d(\{\mathrm{x} \in \mathbb{R}^d : x_1 + \cdots + x_d \in A\})\,.$$

- In particular, for $d \geq 1$,

$$P(S_d > x) \sim p(d)\,P(|X| > x) \quad \text{and} \quad P(S_d \leq -x) \sim q(d)\,P(|X| > x)\,.$$

- $(p(d))_{d\geq 1}$ and $(q(d))_{d\geq 1}$ measure the strength of dependence in (X_t) with respect to the tails of partial sums.
- Example. For (X_t) iid and $X > 0$, $P(S_d > x) \sim d\,P(X > x)$.

  For $X_t = X > 0$, $P(S_d > x) = P(d\,X > x) \sim d^{\alpha}\,P(X > x)$.

Main result

Assumptions.

- **The strictly stationary sequence (X_t) is mixing in the sense**

$$\varphi_n(x) - \Big(\varphi_{nm}(x)\Big)^{k_n} \to 0\,, \quad x \in \mathbb{R}\,,$$

where $m = m_n \to \infty$, $k_n = [n/m] \to \infty$.

- **(X_t) is regularly varying with index $\alpha \in (0,2)$**
- **An anti-clustering and a centering condition hold.**
- **The following limits exist**[4]

(Jak) $\quad p = \lim_{d\to\infty} [p(d) - p(d-1)] \quad$ **and** $\quad q = \lim_{d\to\infty} [q(d) - q(d-1)]\,.$

[4] This condition was introduced in Jakubowski (1993,1997).

- **Then $p, q \geq 0$ and for (a_n) with $P(|X| > a_n) \sim n^{-1}$, $a_n^{-1} S_n \xrightarrow{d} Y_\alpha$, where Y_α is α-stable with characteristic function ψ_α[5] given by**

$$
\begin{aligned}
& -\log \psi_\alpha(x) \\
&= |x|^\alpha \frac{\Gamma(2-\alpha)}{1-\alpha} \left((p+q)\cos(\pi\alpha/2) - i\,\mathrm{sign}(x)\,(p-q)\sin(\pi\alpha/2)\right) \\
&= \chi_\alpha(x, p, q)\,, \quad x \in \mathbb{R}\,.
\end{aligned}
$$

[5]Shown for $\alpha \neq 1$ only.

- The condition (Jak) implies that $p = \lim_{d\to\infty} d^{-1}p(d)$ and $q = \lim_{d\to\infty} d^{-1}q(d)$ exist.
- Examples. m_0-dependence: $p = p(m_0+1) - p(m_0)$, $q = q(m_0+1) - q(m_0)$.

  Stochastic volatility model: $X_t = \sigma_t Z_t$ with stationary Gaussian $\log \sigma_t$ and iid regularly varying (Z_t): $p = dp - (d-1)p$ and $q = dq - (d-1)q$.

  Stochastic recurrence equations: $X_t = A_t X_{t-1} + B_t$ with iid non-negative $((A_t, B_t))$. Let $E[A^\kappa] = 1$ have the (unique) solution $\alpha > 0$. Then (X_t) is regularly varying with index α

and

$$P(X > x) \sim c_0\, x^{-\alpha}\,, \quad x \to \infty\,.$$

- With $\Pi_t = A_1 \cdots A_t$, $t \geq 1$,

$$(X_1, \ldots, X_d) = X_0\,(\Pi_1, \ldots, \Pi_d) + R_d$$

where X_0 is independent of $R_d, \Pi_1, \ldots, \Pi_d$.

- Hence, with $T_d = \sum_{i=1}^{d} \Pi_i$, by a result of Breiman (1965)

$$P(S_d > x) \sim P(X_0\, T_d > x) \sim P(X_0 > x) E[T_d^\alpha]$$

and $p(d) = E[T_d^\alpha]$. Since $E[A^\alpha] = 1$,

$$\begin{aligned} p(d+1) - p(d) &= E[T_{d+1}^\alpha] - E[T_d^\alpha] = E[A_{d+1}^\alpha (1 + T_d)^\alpha] - E[T_d^\alpha] \\ &= E[(1 + T_d)^\alpha - T_d^\alpha] \to E[(1 + T_\infty)^\alpha - T_\infty^\alpha]\,. \end{aligned}$$

- Although $E[T_\infty^\alpha] = \infty$,

$$d^{-1} E[T_d^\alpha] = E[d^{-1/\alpha} T_d]^\alpha \rightarrow E[(1+T_\infty)^\alpha - T_\infty^\alpha] < \infty\,.$$

- Squared GARCH processes can be embedded in stochastic recurrence equations. Similar results hold for (X_t^2) and (σ_t^2) and also for (X_t).

Main idea of proof

- In view of the mixing condition it follows that $(a_n^{-1} S_n)$ has the same limit as $(a_n^{-1} \sum_{i=1}^m S_{mi})$, where S_{mi}, $i = 1, \dots, k_n$, are iid copies of S_m.
- For this triangular array, it suffices to show that

$$k_n \, (\varphi_{nm}(x) - 1) = k_n \, \log \varphi_{nm}(x) + o(1) \to \log \psi_\alpha(x) = -\chi_\alpha(x, p, q) \,.$$

- **Key lemma.** Under regular variation of (X_t) and with the anti-clustering condition,

$$\lim_{d\to\infty} \limsup_{n\to\infty} \left| k_n \, (\varphi_{nm}(x) - 1) - n \left(\varphi_{nd}(x) - \varphi_{n,d-1}(x) \right) \right| = 0 \,, \quad x \in \mathbb{R} \,.$$

- Under regular variation of S_d,

$$n \left(\varphi_{nd}(x) - 1 \right) \to -\chi_\alpha(x, p(d), q(d)), \quad x \in \mathbb{R} \,,$$

and

$$\chi_\alpha(x, p(d), q(d)) - \chi_\alpha(x, p(d-1), q(d-1))$$

$$= \chi_\alpha(x, p(d) - p(d-1), q(d) - q(d-1))$$

$$\longrightarrow \chi_\alpha(x, p, q)$$

Related work

- Balan and Louhichi (2009) use the point process process approach for partial sums of triangular arrays of dependent random variables to show convergence towards infinitely divisible laws.
- Buraczewski, Damek, Guivarc'h (2009,2010) prove limit theory for multivariate stochastic recurrence equations $X_t = A_t X_{t-1} + B_t$ without extra mixing conditions.
- Tyran-Kamińska (2010) proves a FCLT with stable Lévy motion under the condition

$$P(|X_j| > x \mid |X_0| > x) \to 0\,, \quad j \geq 1\,,$$

which is necessary under the J_1-topology.

- Basrak, Krizmanić and Segers (2010) **prove a FCLT with stable Lévy motion in the M_1-topology under $\mathcal{A}(a_n)$ and using the point process approach.**

FIGURE 1. Til lykke.