On the existence of some ARCH(∞) processes.

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I. ARCH(∞) processes.

II. IARCH(∞) processes.

III. The FIGARCH($0, d, 0$) process.

IV. Open questions.

I. ARCH(∞) processes

An ARCH(∞) process is the solution of a stochastic recurrence equation

$$\begin{aligned} X_n &= \sigma_n \epsilon_n \,, \\ \sigma_n^2 &= a_0 + \textstyle\sum_{k=1}^{\infty} a_k X_{n-k}^2 \,, \end{aligned} \tag{1}$$

where $a_0 > 0$, $a_j \geq 0$ and $\{\epsilon_n\}$ is an i.i.d. sequence with zero expectation and unit variance and $\sigma_n \geq 0$.

Engle (1982), Robinson (1991).

The solution is said causal if σ_n^2 is measurable with respect to the sigma-field $\mathcal{F}_{n-1}^{\epsilon}$ generated by ϵ_k, $k < n$.

We are interested in finding causal and stationary and non trivial solutions to Equation (1).

- **The weakly stationary case**

A weakly stationary solution exists if and only if $\sum_{j=1}^{\infty} a_j < 1$. It is then given by the Volterra series expansion

$$\sigma_n^2 = a_0 \sum_{j=1}^{\infty} \sum_{k_1,\ldots,k_j=1}^{\infty} a_{k_1} \ldots a_{k_j} \epsilon^2{}_{n-k_1} \ldots \epsilon^2{}_{n-k_1-\cdots-k_j} \,, \qquad (2)$$

which is almost surely convergent and

$$E[\sigma_n^2] = a_0 \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_k \right)^j = a_0 \, \frac{A_1}{1 - A_1} \,,$$

with $A_1 = \sum_{j=1}^{\infty} a_j$.

Exemple: GARCH(p,q) processes.

GARCH(p,q) are a parametric subfamily of ARCH(∞) processes. They are solutions of a stochastic recurrence equation

$$\begin{aligned} X_n &= \sigma_n \epsilon_n \,, \\ \sigma_n^2 &= a_0 + \textstyle\sum_{k=1}^{q} a_k X_{n-k}^2 + \sum_{j=1}^{p} b_j \sigma_{n-j}^2 \,, \end{aligned} \tag{3}$$

A weakly stationary solution exists if and only if

$$\sum_{k=1}^{q} a_k + \sum_{j=1}^{p} b_j < 1 \,.$$

Bollerslev (1986).

• Fourth order property and weak convergence

Giraitis and Surgailis (2002) give a necessary and sufficient condition for finite fourth moment. This condition implies that

$$\sum_{k=1}^{\infty} |\mathrm{cov}(X_0^2, X_n^2)| < \infty \, .$$

Since an ARCH(∞) is *associated*, this implies that the partial sum process

$$n^{-1/2} \sum_{k=1}^{[nt]} \{X_k^2 - E[X_0^2]\}$$

converges in the skorohod space $\mathcal{D}$ to the Brownian motion.

The squares of a fourth order stationary ARCH process cannot long exhibit memory.

II. Integrated ARCH processes

An ARCH process is said to be integrated (IARCH) if it satisfies Equation (1) with

$$\sum_{j=1}^{\infty} a_j = 1 \ .$$

Consequence: If a stationary solution exists, then

$$E[X_n^2] = \infty \ .$$

Conditions for existence of causal and stationary solutions depend on

- the sequence $\{a_j\}$;

- the distribution of the noise $\{\epsilon_n\}$.

Exemple: IGARCH(p, q) process.

An IGARCH(p, q) process is a process that satisfies (3) with

$$\sum_{k=1}^{p} a_k + \sum_{j=1}^{q} b_j = 1 \; . \tag{4}$$

Bougerol and Picard (1992) gave a necessary and sufficient condition in terms of the top Lyapunov exponent of some random matrices.

This necessary and sufficient implies some necessary and some sufficient conditions.

- $\sum_{j=1}^{p} b_j < 1$ is a necessary condition.

- If the distribution of ϵ_0 has unbounded support, if it has no mass at 0 and if all the coeficients a_j and b_j are positive, then Equation (3) has a unique strictly stationary solution in the integrated case (4).

A general result for IARCH(∞).
Kazakevičius and Leipus (2003).

- If $a_j \leq C r^j$ for some $r \in (0, 1)$;
- If $E[\log_+(\epsilon_0^2)] < \infty$;

Then there exists a causal stationary solution.

Open questions
- What if the coefficients a_j decay slowly?
- Tails and memory properties?

III. The FIGARCH(0,d,0) process.

In order to model long memory in volatility, Baillie et al. (1996) consider Equation (1) with coefficients

$$a_j(d) = (-1)^{j-1}\binom{d}{j} = \frac{d(1-d)\cdots(j-1-d)}{j!}\,. \qquad (5)$$

Equivalently

$$\sigma_n^2 = a_0 + (I - (I-L)^d)X_n^2$$

Does there exist a stationary solution?

- The econometric literature takes it for granted.

- Kazakevičius and Leipus (2003) doubted it because the top Lyaponov exponent of such a process would be zero, which rules out existence of a stationary solution in the IGARCH case.

- **A simple approach**

Denote $\mu_p = E[\epsilon_0^{2p}]$ and $A_p = \sum_{j=1}^{\infty} a_j^p$.

The Volterra series expansion (2) is always well-defined since every terms in it are positive. Define

$$S_n = a_0 \sum_{j=1}^{\infty} \sum_{k_1,\ldots,k_j=1}^{\infty} a_{k_1} \ldots a_{k_j} \epsilon_{n-k_1}^2 \ldots \epsilon_{n-k_1-\cdots-k_j}^2 .$$

This series is either infinite or almost surely convergent.

In both cases, for any $p \in (0,1)$, we can apply the inequality

$$(a+b)^p \leq a^p + b^p \ ,$$

which yields

$$S_n^p \leq a_0 \sum_{j=1}^{\infty} \sum_{k_1,\ldots,k_j=1}^{\infty} a_{k_1}^p \ldots a_{k_j}^p \epsilon_{n-k_1}^{2p} \ldots \epsilon_{n-k_1-\cdots-k_j}^{2p} \ .$$

Hence

$$E[S_n^p] \leq a_0^p \sum_{j=1}^{\infty} \sum_{k_1,\ldots,k_j=1}^{\infty} a_{k_1}^p \ldots a_{k_j}^p \ \mu_{2p}^j = a_0^p \sum_{j=1}^{\infty} (A_p \ \mu_{2p})^j \ .$$

Thus a sufficient condition for the convergence of the series is

$$A_p \, \mu_{2p} < 1 \,. \tag{6}$$

This is possible since $A_1 = \mu_1 = 1$ imply $A_p > 1$ but $\mu_{2p} < 1$.

- Robinson and Zaffaroni (2006).

If this holds, define $\sigma_n = \sqrt{S_n}$ and $X_n = \sigma_n \epsilon_n$.

Then X_n is a causal and stationary solution to Equation (1) such that

$$E[|X_n|^q] < \infty$$

for all $q < 2$ and $E[X_n^2] = \infty$.

- A necessary and sufficient condition for (6)

Assume that A_{p^*} for some $p^* < 1$.

Condition (6) holds for some $p \in (0,1)$ if and only if

$$\sum_{j=1}^{\infty} a_i \log(a_i) + E[\epsilon_0^2 \log(\epsilon_0^2)] \in (0,\infty] \,. \qquad (7)$$

- This holds for any coefficients a_j in the following cases:

  - $E[\epsilon_0^2 \log(\epsilon_0^2)] = +\infty$ (but recall that $E[\epsilon_0^2] = 1$);

  - if $P(\epsilon_0 = 0) > 0$.

- Proof: The function

$$q \mapsto \log \sum_{j=1}^{\infty} a_j^q + \log E[\epsilon_0^{2q}]$$

is convex and the quantity in (7) is its left derivative at 1.

- **Application to the FIGARCH process**

Assume that $P(\epsilon_0^2 = 1) < 1$.

Then there exists $d^* \in [0,1)$ such that, for all $d \in (d^*, 1)$,

$$\sum_{j=1}^{\infty} (-1)^{j+1} \binom{d}{j} \log \left((-1)^{j+1} \binom{d}{j} \right) + E[\epsilon_0^2 \log(\epsilon_0^2)] > 0.$$

hence the FIGARCH equation has a unique causal stationary solution that satisfies $E[|X_n|^{2p}] < \infty$ for all $p < 1$.

- Proof: the first term in the left hand side tends to zero as d tends to 1.

IV. Open questions.

• A weaker condition than (7) is needed. Preferably, a condition which does not tie the coefficients a_j and the distribution of ϵ_0.

• Exact tail behaviour: We know that $E[|X_n|^q] < \infty$ for all $q < 2$. But we do not know the exact tail behaviour of X_n^2. Is it regularly varying with index -1?

• Short or long Memory?

- For $q \leq 2$, does the partial sum process of X_n^q converge to a process with *independent* or *dependent* increments?
- For $q < 1$, is the series $\sum_{n=1}^{\infty} |\mathrm{cov}(|X_0|^q, |X_n|^q)|$ summable or not summmable?

- **Statistical inference**

Robinson and Zaffaroni (2006) studied parametric estimation for ARCH(∞) processes that include the FIGARCH$(0, d, 0)$ process under Assumption (6) and for $d > 1/2$.

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