Multifractal random walks as fractional Wiener integrals

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MOTIVATIONS

Scale invariance property : Find a stochastic process $(X_t)_{t\in[0,1]}$ such that the moments of the fluctuations at different scales behave as

$$\mathbb{E}\left(|X(t+\tau)-X(\tau)|^q\right)\sim C_q(\tau)\tau^{\xi(q)}.$$

Definition : If $\xi(q)=qH$, X is a monofractal process.
Exemple : Fractional Brownian motion.

Definition : If the scaling exponent $q\mapsto\xi(q)$ is non linear, X is a multi fractal process.

Related to the multifractal formalism through the Legendre transformation.
Applications :

- stock prices in finance (Lee at al 06)
- teletraffic in the Internet (Barral at al 05)
- ranging from turbulence in hydrodynamics (Mandelbrot 74)

Multifractal processes

Random non negative multifractal measures (MRM)

- multiplicative binomial cascades of Mandelbrot (74)
- compound Poisson cascades of Barral at al (02)
- log-infinitely divisble multifractal measures (Bacry at al 02)

Multifractal random walk (MRW) :

Two approaches

- by subordination $X(t) = B(M([0,t])$ where M is a multifractal measure, B self similar process with stationnary increments, independent of M
- by integration $X(t) = \int_0^t Q(u)dB(u)$, N stationary nonnegative multifractal noise (Barral at al (02))

If B is a Brownian motion the two approaches give the same process (Bacry Muzy) (03)
Muzy at al (02), Ludena (06) introduce the case B fractional Brownian motion

Infinitely divisible cascading noise (IDC)

Definition An IDC noise is a family $((Q_r(t),\ t \in \mathbb{R})_r)$ of processes

$$Q_r(t) = \frac{e^{M(C_r(t))}}{\mathbb{E}(e^{M(C_r(t))})},$$

where $C_r(t) = \{(t', r')\ : r \leq r' \leq 1,\ t - \frac{r'}{2} \leq t' < t + \frac{r'}{2}\}$, M is an infinitely divisible, independentely scattered random measure,

$\mathbb{E}(e^{qM(A)}) = e^{-\rho(q)m(A)}$

$$dm(t,r) = \begin{cases} dt\left(c\frac{dr}{r^2} + c^*\delta_1\right) & \text{if } 0 < r \leq 1, \\ 0 & \text{if } r > 1. \end{cases}$$

Lemma : Moments
$\mathbb{E}(Q_r(0)^q) = e^{-\varphi(q)m(C_r(0))}$,
$\mathbb{E}(Q_r(t)Q_r(s))e^{-\varphi(2)m(C_r(t)\cap C_r(s))}$,
$\varphi(q) = \rho(q) - q\rho(1)$ is concave, $\varphi(0) = \varphi(1) = 0$ and $\varphi(2) < 0$.

Proposition : Exact scale invariance Bacry Muzy (03)
$(c^* = c)$,
$\{Q_{rt}(tu)\}_{u\in[0,1]} =^d e^{\Omega_t}\{Q_r(u)\}_{u\in[0,1]}$ where Ω is independent of M and $\mathbb{E}(e^{q\Omega_t}) = t^{c\varphi(q)}$.

Fractional Wiener integral

Let B^k be a fractional Brownian motion with Hurst parameter $\kappa + 1/2 \in (0,1)$,

Definition : The Wiener fractional integral $(\mathcal{I}^\kappa(f),\ f \in \mathcal{L}^\kappa)$ is the Gaussian process with covariance function $\mathbb{E}(\mathcal{I}^\kappa(f)\mathcal{I}^\kappa(g)) =< f, g >_\kappa$, where

- $\kappa > 0$ (Pipiras Taqqu (03)) not complete inner product space
  $\mathcal{L}^\kappa = \{f\ : \int\int_{[0,t]^2} |f(u)|f(v)||u-v|^{2\kappa-1}dudv < \infty\}$
  with inner product
  $< f, g >_\kappa = \int\int_{[0,t]^2} f(u)g(v)|u-v|^{2\kappa-1}dudv$,
- $\kappa < 0$
  $\mathcal{L}^\kappa = \{g \in L^2[0,t],\ g^\kappa_{||} \in L^2|0,T]\}$
  $g^\kappa_{||}(u) = \frac{|g(u)|(t-u)^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t |g(s)-g(u)|(u-s)_+^{\kappa-1}ds$
  with inner product $< f, g >_\kappa =< f^\kappa, g^\kappa >_{L^2[0,t]}$ and f^κ defined as $f^\kappa_{||}$ without absolute values.

Lemma : For $s < s'$, $\mathcal{I}^\kappa(\mathbf{1}_{[s,s']}) = B^\kappa(s) - B^\kappa(s')$.

MRW

Lemma : (Scale invariant case $c^* = 0$)
For $r > 0$ $Q_r \in \mathcal{L}^\kappa$ and $\mathbb{E}(< Q_r, Q_r >_\kappa) \sim C(t)n(r)^2$ where

$$n(r)^2 = \begin{cases} 1 & \text{if } c\varphi(2) + 2\kappa > 0, \\ -\ln r & \text{if } c\varphi(2) + 2\kappa = 0, \\ r^{c\varphi(2)+2\kappa} & \text{if } c\varphi(2) + 2\kappa < 0. \end{cases}$$

Notation : $Z_r^\kappa(t) = \int_0^t Q_r(u) dB^\kappa(u)$.
Theorem : Existence results
For fixed $t \in \mathbb{R}$; $(Z_r^\kappa(t))_r$ converges in $L^2(\Omega)$ if and only if $c\varphi(2) + 2\kappa > 0$.

Definition : The limit process, $(Z_t^\kappa)_{t>0}$ is called Multifractal random walk as fractional integral.

Question :
Convergence of $(n(r)^{-1} Z_r^\kappa(t))_r$ when $c\varphi(2) + 2\kappa > 0$?
Not in $L^2(\Omega)$.

MRW The Brownian cases

Proposition

1. Let B be a Brownian motion,

   $(r^{-\frac{c\varphi(2)}{2}} \int_0^t Q_r(u) dB(u),\ t \in [0,1])_r$ converge in law the sens of finite dimensional marginal in law.

2. Conditionnaly to $\sigma(M(A),\ A$ Borelian), for fixed t, $(\int_0^t Q_r(u) dB_u)_r$ converges weakly to 0.

Lemma : The following processes are non negative martingales

- for fixed u, $(Q_r(u))_r$,
- $(r^{-c\frac{\varphi(2)}{2}} \int_0^1 Q_r(u)^2 du)_r$,
- for all nonnegative $f \in L^2[0,1]$, $(\int_0^1 f(u) Q_r(u) du)_r$

and converge almost surely.

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proof :

1. Conditionnaly to $\sigma(M(A), A$ Borelian),
   $(r^{-c\varphi(2)}\int_0^{t_i} Q_r(u)dB(u))_r)_{i=1}^n$ is a family of Gaussian random vectors with covariance matrix
   $(r^{-c\varphi(2)}\int_0^{t_i\wedge t_j} Q_r(u)^2du)_{i,j}$ which converge when r goes to 0, when it converges in law.
2. $(r^{-c\frac{\varphi(2)}{2}}Q_r)_r$ converges weakly to 0 in $L^2|0,1]$ since
   - the family of its norm $(r^{-c\varphi(2)}\int_0^1 Q_r(u)^2du)$ converges
   - and if for $j = 0, ..2^k$, $k \in \mathbb{N}$ and $\xi_{j,k}$ is the Haar function
     $$\xi_{j,k} = 2^{j/2}\left[\mathbf{1}_{[\frac{2k-2}{2^{j+1}},\frac{2k-1}{2^{j+1}}[} - \mathbf{1}_{[\frac{2k-1}{2^{j+1}},\frac{2k}{2^{j+1}}[}\right]]$$
     then $(\int_0^1 \xi_{j,k}(u)Q_r(u)du)_r$ converges.
   - Since $\varphi(2) < 0$, $(r^{-c\varphi(2)}\int_0^1 \xi_{j,k}(u)Q_r(u)du)_r$ converges to 0.

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proof :

Let $F \in L^2(\Omega, \sigma(B))$, having a stochastic gradient
$\mathbb{E}_M(F \int_0^1 Q_r(u) dB_u) = \mathbb{E}_M(< DF, Q_r)_{L^2(du)}$
converge to 0.
$\mathbb{E}_M([\int_0^1 Q_r(u) dB_u]^2) = \int_0^1 Q_r(u)^2 du$ converges,
conclude to weak convergence using density of
$\{F, \in L^2(\Omega, \sigma(B))$, having a stochastic gradient $\}$ in $L^2(\Omega, \sigma(B))$.

MRW when $c\varphi(2) + 2\kappa < 0$

Proposition :
For fixed t,
for all $F \in L^\infty(\Omega, \mathbb{P})$ $(\mathbb{E}(Fn(r)^{-1} \int_0^t Q_r(u) dB_u^\kappa)_r)$ converges to 0.

Proof : Let $F = F_Q \times F_{B^\kappa}$ such that

- F_Q bounded $\sigma(M(A)$, A Borelian) mesurable,
- F_{B^κ} bounded $\sigma(B^\kappa)$ mesurable, having a stochastic gradient with respect to B^κ, $DF_{B^\kappa} \in L^2(du)$.

Integration by part yields
$\mathbb{E}(Fn(r)^{-1} \int_0^t Q_r(u) dB_u^\kappa) = \mathbb{E}(F_Q < DF_{B^\kappa}, Q_r >_{L^2(du)})$.
and conclude with a density argument.
Remark : $\kappa \neq 0$, not weak convergence conditionnaly to $\sigma(M(A)$, A Borelian) since no almost sure convergence result for $(< Q_r, Q_r >_{\mathcal{L}^\kappa})_r$.

Numerical synthesis

Numerical simulation of $(Z_r^\kappa(n/N))_{n=1}^N$ using Riemann sums :

- $(B^\kappa(k/K)_{k=0}^{K-1})$ using circulant embedding method (Bardet
- at al) 03),
- $(Q_r(k/K)_{k=0}^{K-1})$ using Chainais at al (05)
- $N < K = RN$ and $r >> 1/K$.

$Z_r^\kappa(n/N) \sim \sum_{k=1}^{K-1} Q_r(k/K) \left(B^\kappa(k/K) - B^\kappa(k-1/K)\right),$

Properties of MRW

Proposition Exact scale invariance case
For $c\varphi(2) + 2\kappa > 0$, t $\in [0, 1]$,
E(—Z$_\kappa(t)|^q) = C_q t^{(\kappa+\frac{1}{2})q+c\varphi(q)}$.

Proposition : Scale invariance case
Under suitable conditions, $c\varphi(2) + 2\kappa > 0$, t $\in [0, 1]$, for $q > 0$ there exists $\overline{C_q}$, C_q,

$$C_q t^{c\varphi(q)+(\kappa+1/2)q} \leq \mathbb{E}(|Z_\kappa(t)|^q) \leq \overline{C_q} t^{c\varphi(q)+(\kappa+1/2)q}.$$