Asymptotic results for empirical measures of weighted sums of independent random variables

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Outline

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Circulant random matrices

Let (X_n) be a sequence of random variables and consider the **symmetric circulant random** matrix

$$A_n = \frac{1}{\sqrt{n}} \begin{pmatrix} X_1 & X_2 & X_3 & \cdots & X_{n-1} & X_n \\ X_2 & X_3 & X_4 & \cdots & X_n & X_1 \\ X_3 & X_4 & X_5 & \cdots & X_1 & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_n & X_1 & X_2 & \cdots & X_{n-2} & X_{n-1} \end{pmatrix}.$$

Goal. Asymptotic behavior of the empirical spectral distribution

$$F_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{\{\lambda_k \leqslant x\}}.$$

Trigonometric weighted sums

We shall make use of

$$r_n = \left[\frac{n-1}{2} \right]$$

and of the trigonometric weighted sums

$$S_{n,k} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t \cos\left(\frac{2\pi kt}{n}\right),$$
$$T_{n,k} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t \sin\left(\frac{2\pi kt}{n}\right).$$

Eigenvalues

Lemma (Bose-Mitra)

The eigenvalues of A_n are given by

$$\lambda_0 = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t,$$

if n is even

$$\lambda_{n/2} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (-1)^{t-1} X_t,$$

and for all $1 \leqslant k \leqslant r_n$

$$\lambda_k = -\lambda_{n-k} = \sqrt{S_{n,k}^2 + T_{n,k}^2}.$$

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Theorem (Bose-Mitra, 2002)

Assume that (X_n) *is a sequence of* **iid** *random variables such that* $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_1^2] = 1$, $\mathbb{E}[|X_1|^3] < \infty$. *Then, for each* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}[(F_n(x) - F(x))^2] = 0$$

where F is given by

$$F(x) = \frac{1}{2}\begin{cases} \exp(-x^2) & \text{if} \quad x \leqslant 0, \\ 2 - \exp(-x^2) & \text{if} \quad x \geqslant 0. \end{cases}$$

Remark. F is the symmetric Rayleigh CDF.

Empirical periodogram

Let (X_n) be a sequence of random variables and consider the **empirical periodogram** defined, for all $\lambda \in [-\pi, \pi[$, by

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} e^{-it\lambda} X_t \right|^2 .$$

At the **Fourier frequencies** $\lambda_k = \dfrac{2\pi k}{n}$, we clearly have

$$\boldsymbol{I_n(\lambda_k) = S_{n,k}^2 + T_{n,k}^2.}$$

Goal. Asymptotic behavior of the empirical distribution

$$F_n(x) = \frac{1}{r_n} \sum_{k=1}^{r_n} \mathbf{1}_{\{I_n(\lambda_k) \leqslant x\}}.$$

Theorem (Kokoszka-Mikosch, 2000)

Assume that (X_n) is a sequence of **iid** *random variables such that $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$. Then, for each $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{E}[(F_n(x) - F(x))^2] = 0$$

where F is the exponential CDF.

Remark. (F_n) also converges uniformly in probability to F.

Almost sure central limit theorem

Let (X_n) be a sequence of **iid** random variables such that $\mathbb{E}[X_n] = 0$ and $\mathbb{E}(X_n^2) = 1$ and denote

$$S_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t.$$

Theorem (Lacey-Phillip, 1990)

The sequence (X_n) satisfies an **ASCLT** *which means that for any $x \in \mathbb{R}$*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} \mathbf{1}_{\{S_t \leqslant x\}} = \Phi(x) \quad a.s.$$

where Φ stands for the standard normal CDF.

Outline

Assumptions

Let $(\mathbf{U}^{(n)})$ be a family of real rectangular $r_n \times n$ matrices with $1 \leqslant r_n \leqslant n$, satisfying for some constants $C, \delta > 0$

(A_1)
$$\max_{1 \leqslant k \leqslant r_n,\, 1 \leqslant t \leqslant n} |u_{k,t}^{(n)}| \leqslant \frac{C}{(\log(1+r_n))^{1+\delta}},$$

(A_2)
$$\max_{1 \leqslant k, l \leqslant r_n} \left| \sum_{t=1}^{n} u_{k,t}^{(n)} u_{l,t}^{(n)} - \delta_{k,l} \right| \leqslant \frac{C}{(\log(1+r_n))^{1+\delta}}.$$

Let $(\mathbf{V}^{(n)})$ be such a family and assume that $(\mathbf{U}^{(n)}, \mathbf{V}^{(n)})$ satisfies

(A_3)
$$\max_{1 \leqslant k, l \leqslant r_n} \left| \sum_{t=1}^{n} u_{k,t}^{(n)} v_{l,t}^{(n)} \right| \leqslant \frac{C}{(\log(1+r_n))^{1+\delta}}.$$

Trigonometric weights

(A_1) to (A_3) are fulfilled in many situations. For example, if

$$r_n \leqslant \left[\frac{n-1}{2}\right]$$

and for the trigonometric weights

$$\boldsymbol{u_{k,t}^{(n)} = \sqrt{\frac{2}{n}} \cos\left(\frac{2\pi kt}{n}\right)},$$

$$\boldsymbol{v_{k,t}^{(n)} = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi kt}{n}\right)}.$$

We shall make use of the sequences of weighted sums

$$S_{n,k} = \sum_{t=1}^{n} u_{k,t}^{(n)} X_t \qquad \text{and} \qquad T_{n,k} = \sum_{t=1}^{n} v_{k,t}^{(n)} X_t.$$

Uniform strong law

Theorem (Bercu-Bryc, 2007)

Assume that (X_n) *is a sequence of* **independent** *random variables such that* $\mathbb{E}[X_n] = 0$, $\mathbb{E}[X_n^2] = 1$, $\sup \mathbb{E}[|X_n|^3] < \infty$. *If* (A_1) *and* (A_2) *hold, we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{r_n} \sum_{k=1}^{r_n} 1_{\{S_{n,k}\leqslant x\}} - \Phi(x) \right| = 0 \quad a.s.$$

In addition, under (A_3), *we also have for all* $(x, y) \in \mathbb{R}^2$,

$$\lim_{n\to\infty} \frac{1}{r_n} \sum_{k=1}^{r_n} 1_{\{S_{n,k}\leqslant x, T_{n,k}\leqslant y\}} = \Phi(x)\Phi(y) \quad a.s.$$

Sketch of proof

For all $s, t \in \mathbb{R}$, let

$$\Phi_n(s,t) = \frac{1}{r_n} \sum_{k=1}^{r_n} \exp(isS_{n,k} + itT_{n,k}).$$

Lemma (Bercu-Bryc, 2007)

Under (A_1) to (A_3), one can find some constant $C(s,t) > 0$ such that for n large enough

$$\mathbb{E}\left[\left|\Phi_n(s,t) - \Phi(s,t)\right|^2\right] \leqslant \frac{C(s,t)}{(\log(1 + r_n))^{1+\delta}}$$

where $\Phi(s,t) = \exp(-(s^2 + t^2)/2)$.

Lemma (Lyons, 1988)

Let $(Y_{n,k})$ be a sequence of **uniformly bounded** *$\mathbb{C}$-valued random variables and denote*

$$Z_n = \frac{1}{r_n} \sum_{k=1}^{r_n} Y_{n,k}.$$

Assume that for some constant $C > 0$,

$$\mathbb{E}[|Z_n|^2] \leqslant \frac{C}{(\log(1 + r_n))^{1+\delta}}.$$

Then, (Z_n) converges to zero a.s.

By the two lemmas, we have for all $s, t \in \mathbb{R}$,

$$\lim_{n \to \infty} \Phi_n(s, t) = \Phi(s, t) \quad \text{a.s.}$$

Corollary

The result of Bose and Mitra on empirical spectral distributions of **symmetric circulant random matrices** *holds with probability one*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0 \quad a.s.$$

Corollary

The result of Kokoszka and Mikosch on empirical distributions of **periodograms at Fourier frequencies** *holds with probability one if* $\sup \mathbb{E}[|X_n|^3] < \infty$

$$\lim_{n\to\infty} \sup_{x\geqslant 0} |F_n(x) - (1 - \exp(-x))| = 0 \quad a.s.$$

Sketch of proof

The empirical measure

$$\nu_n = \frac{1}{r_n} \sum_{k=1}^{r_n} \delta_{(S_{n,k}, T_{n,k})} \Rightarrow \nu \quad \text{a.s.}$$

where ν stands for the product of two independent normal probability measure. Let h be the continuous mapping

$$h(x, y) = \frac{1}{2}(x^2 + y^2).$$

F_n is the CDF of $\mu_n = \nu_n(h)$. Consequently, as $\mu_n \Rightarrow \mu$ a.s. where $\mu = \nu(h)$ is the exponential probability measure

$$\lim_{n \to \infty} F_n(x) = 1 - \exp(-x) \quad \text{a.s.}$$

In all the sequel, we only deal with **trigonometric weights**. We shall make use of Sakhanenko's strong approximation lemma.

Definition. A sequence (X_n) of independent random variables satisfies Sakhanenko's condition if $\mathbb{E}[X_n] = 0$, $\mathbb{E}[X_n^2] = 1$ and for some constant $a > 0$,

$$(S) \qquad \sup_{n\geqslant 1} a\,\mathbb{E}[|X_n|^3 \exp(a|X_n|)] \leqslant 1.$$

Remark. Sakhanenko's condition is stronger than Cramér's condition as it implies for all $|t| \leqslant a/3$

$$\sup_{n\geqslant 1} \mathbb{E}[\exp(tX_n)] \leqslant \exp(t^2).$$

A keystone lemma

Lemma (Sakhanenko, 1984)

Assume that (X_n) is a sequence of **independent** *random variables satisfying (S). Then, one can construct a sequence (Y_n) of* **iid** $\mathcal{N}(0,1)$ *random variables such that, if*

$$S_n = \sum_{t=1}^{n} X_t \qquad \text{and} \qquad T_n = \sum_{t=1}^{n} Y_t$$

then, for some constant $c > 0$,

$$\mathbb{E}\left[\exp\left(ac \max_{1 \leqslant t \leqslant n} |S_t - T_t|\right)\right] \leqslant 1 + na.$$

Theorem (Bercu-Bryc, 2007)

Assume that (X_n) *is a sequence of* **independent** *random variables satisfying (S). Suppose that* $(\log n)^2 r_n^3 = o(n)$. *Then, for all* $x \in \mathbb{R}$,

$$\frac{1}{\sqrt{r_n}} \sum_{k=1}^{r_n} (1_{\{S_{n,k} \leqslant x\}} - \Phi(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Phi(x)(1 - \Phi(x)))\,.$$

In addition, we also have

$$\sqrt{r_n} \sup_{x \in \mathbb{R}} \left| \frac{1}{r_n} \sum_{k=1}^{r_n} 1_{\{S_{n,k} \leqslant x\}} - \Phi(x) \right| \underset{n \to +\infty}{\xrightarrow{\mathcal{L}}} \mathcal{K}$$

where $\mathcal{K}$ *stands for the Kolmogorov-Smirnov distribution.*

Remark. $\mathcal{K}$ is the distribution of the supremum of the absolute value of the Brownian bridge. For all $x > 0$,

$$\mathbb{P}(\mathcal{K} \leqslant x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k \exp(-2k^2x^2).$$

Remark. One can observe that the CLT also holds if (S) is replaced by the assumption that for some $p > 0$

$$\sup_{n\geqslant 1} \mathbb{E}[|X_n|^{2+p}] < \infty,$$

as soon as

$$r_n^3 = o(n^{p/(2+p)}).$$

Relative entropy

We are interested in the **large deviation principle** for the random empirical measure

$$\mu_n = \frac{1}{r_n} \sum_{k=1}^{r_n} \delta_{S_{n,k}}.$$

The **relative entropy** with respect to the standard normal law with density ϕ is given, for all $\nu \in \mathcal{M}_1(\mathbb{R})$, by

$$I(\nu) = \int_{\mathbb{R}} \log \frac{f(x)}{\phi(x)} f(x) dx$$

if ν is absolutely continuous with density f and the integral exists and $I(\nu) = +\infty$ otherwise.

Theorem (Bercu-Bryc, 2007)

Assume that (X_n) is a sequence of **independent** *random variables satisfying (S). Suppose that $\log n = o(r_n)$, $r_n^4 = o(n)$. Then, (μ_n) satisfies an* **LDP** *with speed (r_n) and good rate function I,*

- **Upper bound:** *for any closed set $F \subset \mathcal{M}_1(\mathbb{R})$,*

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mathbb{P}(\mu_n \in F) \leqslant - \inf_{\nu \in F} I(\nu).$$

- **Lower bound:** *for any open set $G \subset \mathcal{M}_1(\mathbb{R})$,*

$$\liminf_{n\to\infty} \frac{1}{r_n} \log \mathbb{P}(\mu_n \in G) \geqslant - \inf_{\nu \in G} I(\nu).$$

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