Conference in honour of Magda Peligrad

Finkelstein's theorem for intermittent maps

Jérôme Dedecker, Université Paris 6, LSTA

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Some measures of dependence

$(X_i)_{i\in\mathbb{Z}}$: stationary sequence of real valued random variables.
Let $\mathcal{M}_0 = \sigma(X_i, i \le 0)$. Let F be the d. f. of X_i and $F_{X_k|\mathcal{M}_0}$ be the conditional d. f. of X_k given $\mathcal{M}_0$. Let $G_k = F_{X_k|\mathcal{M}_0} - F$. Define

$$
\begin{aligned}
\phi_{1,X}(k) &= \sup_{t\in\mathbb{R}} \|F_{X_k|\mathcal{M}_0}(t) - F(t)\|_\infty \,, \\
\phi_{2,X}(k) &= \phi_{1,X}(k) \vee \sup_{i>j\ge k} \sup_{t,s\in\mathbb{R}} \Big\| G_i(t)G_j(s) - \mathbb{E}(G_i(t)G_j(s)) \Big\|_\infty \,. \\
\alpha_{1,X}(k) &= \sup_{t\in\mathbb{R}} \|F_{X_k|\mathcal{M}_0}(t) - F(t)\|_1 \,, \\
\alpha_{2,X}(k) &= \alpha_{1,X}(k) \vee \sup_{i>j\ge k} \sup_{t,s\in\mathbb{R}} \Big\| G_i(t)G_j(s) - \mathbb{E}(G_i(t)G_j(s)) \Big\|_1 \,. \\
\beta_{1,X}(k) &= \Big\| \sup_{t\in\mathbb{R}} |F_{X_k|\mathcal{M}_0}(t) - F(t)| \Big\|_1 \,, \\
\beta_{2,X}(k) &= \beta_{1,X}(k) \vee \sup_{i>j\ge k} \Big\| \sup_{t,s\in\mathbb{R}} |G_i(t)G_j(s) - \mathbb{E}(G_i(t)G_j(s))| \Big\|_1 \,.
\end{aligned}
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Iterated transformations and Markov chains

- Let θ be a map from $[0,1]$ to itself, preserving a probability ν on $[0,1]$. The sequence $(\theta^i)_{i\geq 0}$ of random variables from $([0,1],\nu)$ to $[0,1]$ is strictly stationary.

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- Let K be the Perron-Frobenius operator of θ: for any functions h, f in $\mathbb{L}^2([0,1],\nu)$,

$$\nu(K(h)\cdot f) = \nu(h \cdot f \circ \theta)\,.$$

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- It is easy to see that $(\theta^0, \theta^1, \ldots, \theta^n)$ is distributed as $(X_n, X_{n-1}, \ldots, X_0)$, where $(X_i)_{i\geq 0}$ is a stationary Markov chain with invariant measure ν and transition kernel K.

Properties of the associated chains: Example 1

- Assume that θ is uniformly expanding with an unique a. c. invariant probability measure whose density h is such that

$$\frac{1}{h}\mathbf{1}_{h>0} \quad \text{is a BV function.}$$

Then, using the contraction properties of K in the space of BV functions (see for instance Broise (1996)), we have proved with C. Prieur (2005) that

$$\phi_{2,X}(n) \le C\rho^n, \quad \text{with } \rho < 1\,.$$

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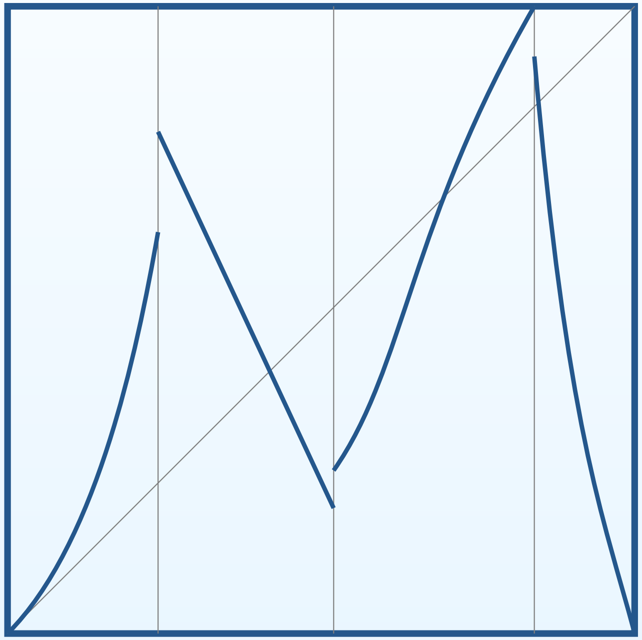
- Standard examples of uniformly expanding maps are
  - the β-transformations: $\theta(x) = \beta x - [\beta x]$, pour $\beta > 1$.
  - the Gauss map: $\theta(x) = x^{-1} - [x^{-1}]$.

Properties of the associated chains: Example 2

The graph of an intermittent map is as follows:



Behavior around zero: $\theta'(0) = 1$ and $\theta''(x) \sim cx^{\gamma-1}$ when $x \to 0$, for some $c > 0$ and $0 < \gamma < 1$.

Properties of the associated chains: Example 2

- An example of intermittent map is the LSV map:

$$\text{for } 0 < \gamma < 1, \quad \theta(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$

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- Starting from the properties of K in the space of Hölder functions (cf. Maume-Deschamps (2001)), we have proved with C. Prieur (2009) that, for any $\varepsilon > 0$,

$$\frac{A}{n^{\frac{1-\gamma}{\gamma}}} \leq \beta_{2,X}(n) \leq \frac{B(\varepsilon)}{n^{\frac{1-\gamma}{\gamma}-\varepsilon}} .$$

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- With S. Gouëzel and F. Merlevède (2008), we have proved that

$$\frac{A}{n^{\frac{1-\gamma}{\gamma}}} \leq \alpha_{2,X}(n) \leq \frac{B}{n^{\frac{1-\gamma}{\gamma}}} .$$

CLT for uniformly expanding maps

- Let $\mathcal{C}(M, p, \nu)$ be the closure of the convex hull of the set of functions f which are monotonic on some open interval of $]0, 1[$ and 0 elsewhere, and such that $\nu(|f|^p) \leq M$.

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- Let $\mathcal{C}(M,p,\nu)$ be the closure of the convex hull of the set of functions f which are monotonic on some open interval of $]0,1[$ and 0 elsewhere, and such that $\nu(|f|^p) \leq M$.

- Let $S_n(f) = \sum_{i=1}^n (f \circ \theta^i - \nu(f))$. If θ is uniformly expanding, and if $f \in \mathcal{C}(M,2,\nu)$, then, on $([0,1],\nu)$,

$$\frac{1}{\sqrt{n}} S_n(f) \quad \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,\sigma^2(f)),$$

where

$$\sigma^2(f) = \text{Var}_\nu(f) + 2\sum_{k>0} \text{Cov}_\nu(f, f\circ\theta^k)\,.$$

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- In particular, the CLT holds if f is monotonic on $]0,1[$, and $\int f^2(t)dt < \infty$.

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- Let H be a tail function. Let $\mathcal{F}(H, \nu)$ be the closure of the convex hull of the set of functions f which are monotonic on some open interval of $]0, 1[$ and 0 elsewhere, and such that $\nu(|f| > t) \leq H(t)$.

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- Let θ be intermittent, with $\gamma<1/2$. With S. Gouëzel and F. Merlevède (2008), we proved the CLT for $f\in\mathcal{F}(H,\nu)$, as soon as

$$\int_0^\infty x(H(x))^{\frac{1-2\gamma}{1-\gamma}}\,dx<\infty\,.$$

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- \- if $f\downarrow$ on $]0,1]$ and $f(x)\le Cx^{-a}$, the CLT holds if $a<\dfrac{1}{2}-\gamma$. The cut is optimal: see Gouëzel (2004).

  \- If $f\uparrow$ on $[0,1[$ and $f(x)\le C(1-x)^{-a}$, the CLT holds if

$$a<\frac{1}{2}-\frac{\gamma}{2(1-\gamma)}\,.$$

Proof of the CLT

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- If $\alpha_{1,X}(n) = O(n^{(\gamma-1)/\gamma})$, then

$$\sum_{k>0} \int_0^{\alpha_{1,X}(k)} Q^2(u)du < \infty \quad \text{as soon as} \quad \int_0^{\infty} x(H(x))^{\frac{1-2\gamma}{1-\gamma}}dx < \infty\,.$$

Empirical CLT for uniformly expanding maps.

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- If $\theta(x) = 2x - [2x]$ or if θ is the Gauss map, the ECLT follows from a general result for functions of ϕ-mixing sequences given in Billingsley (1968): $\sqrt{n}(F_{n,\theta} - F)$ converges in distribution to a Gaussian process G with covariance:

$$\text{Cov}(G(s), G(t)) = \sum_{k \ge 1} \text{Cov}_\nu(\mathbf{1}_{\theta \le t}, \mathbf{1}_{\theta^k \le s}) + \sum_{k > 1} \text{Cov}_\nu(\mathbf{1}_{\theta \le s}, \mathbf{1}_{\theta^k \le t}) \,.$$

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- The ECLT follows from a general result for functions of β-mixing sequences given in Borovkova, Burton and Dehling (2001).

An ECLT for β-dependent sequences.

- Let $(X_i)_{i\in\mathbb{Z}}$ be stationary, and $F_n(t) = n^{-1}\sum_{k=1}^{n} \mathbf{1}_{X_k\le t}$.

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- This result applies to uniformly expanding maps without the assumption of finite partition.

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- If θ is an intermittent map with $\gamma < 1/2$, then the ECLT holds.
- This result is no longer valid if $\gamma = 1/2$. For instance, if θ is the map

$$\theta(x) = \begin{cases} x(1+\sqrt{2x}) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$

one can prove that the finite dimensional marginals of the process $(n/\ln(n))^{1/2}(F_{n,\theta} - F)$ converges in distribution to those of the degenerated Gaussian process G defined by:

$$\text{for any } t \in [0,1], \quad G(t) = \sqrt{h(1/2)}(1 - F(t))\mathbf{1}_{t \neq 0} Z \,,$$

where Z is a standard normal and h is the density of ν.

Sketch of proof of the empirical CLT

- For the fidi convergence, let $t_1 < t_2 < \cdots < t_k$ and $(a_1, \ldots, a_k)$ in $\mathbb{R}^k$. Let $Y_i = a_1 \mathbf{1}_{X_i \leq t_1} + \cdots + a_k \mathbf{1}_{X_i \leq t_k}$ and $S_n(Y) = Y_1 + \cdots + Y_n$.

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- According to Gordin CLT (1973) for stationary ergodic sequences of bounded r.v.'s, $n^{-1/2}(S_n(Y) - \mathbb{E}(S_n(Y)))$ converges to a normal law provided that

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- Clearly $\|\mathbb{E}(Y_i|\mathcal{M}_0) - \mathbb{E}(Y_i)\|_1 \le (|a_1| + \cdots + |a_k|)\alpha_{1,X}(i)$, so that the fidi convergence holds as soon as $\sum_{i>0} \alpha_{1,X}(i) < \infty$.

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- For the tightness, one needs a new Rosenthal inequality for random variables with moments of order p, for p in $[2, 3]$.

Finkelstein's theorem for β-dependent sequences

Joint work with F. Merlevède (in progress).

- Let $(X_i)_{i\in\mathbb{Z}}$ be stationary, and $F_n(t) = n^{-1}\sum_{k=1}^{n}\mathbf{1}_{X_k\le t}$. Let $v_n = n^{1/2}(2\ln\ln(n))^{-1/2}$.

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- K is a compact set in $C(\mathbb{R}, \|\cdot\|_\infty)$ which can be described as follows. Let G be the Gaussian process with covariance Γ, then

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- Previous result by Philipp (1977), for functions of α-mixing sequences and Lacunary sequences.

Sketch of proof of Finkestein's theorem

- For the fidi conv., we prove that $v_n(F_n(t_1) - F_X(t_1), \ldots, F_n(t_d) - F_X(t_d))^t$ is a.s. relatively compact in $\mathbb{R}^d$ with set of limit points K^T, the restriction of K to $T = (t_1, \ldots, t_d)$. This follows from Theorem 1 of a joint paper with F. Merlevède (2009) as soon as $\sum_{k>0} \beta_2(k) < \infty$.

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- For $K \in \mathbb{N}$ let $\Pi_K(x) = 2^{-K}[2^K x]$. Let $\mu_k(t) = k(F_k(t) - F_Y(t))$. For the tightness, we prove that there exist positive numbers $(A_K)_{K\geq 1}$ tending to zero as K tends to infinity, such that

$$\sum_{n\geq 3} \frac{1}{n} \mathbb{P}\Big(\sup_{1\leq k\leq n} \sup_{t\in[0,1]} |\mu_k(t) - \mu_k(\Pi_K(t))| > A_K \sqrt{n \ln(\ln(n))} \Big) < \infty \,.$$

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- For the tightness, we go back to $[0,1]$ by setting $Y_i = F_b(X_i)$, where F_b is an appropriate d. f. related to the dependence structure of $(X_i)_{i\in\mathbb{Z}}$.
- For $K \in \mathbb{N}$ let $\Pi_K(x) = 2^{-K}[2^K x]$. Let $\mu_k(t) = k(F_k(t) - F_Y(t))$. For the tightness, we prove that there exist positive numbers $(A_K)_{K\geq 1}$ tending to zero as K tends to infinity, such that

$$\sum_{n\geq 3} \frac{1}{n} \mathbb{P}\Big(\sup_{1\leq k\leq n} \sup_{t\in[0,1]} |\mu_k(t) - \mu_k(\Pi_K(t))| > A_K \sqrt{n \ln(\ln(n))}\Big) < \infty\,.$$

- To prove this inequality, we combine the Rosenthal inequality mentioned above and a maximal inequality for α dependent sequences given in the paper with F. Merlevède (2009).

The case of intermittent maps

- Let θ be the LSV map, with $\gamma < 1/2$. The preceding result can be directly applied to the Markov chain associated to θ, but not to θ itself, because the identity between the distribution of $(\theta^0, \theta^1, \ldots, \theta^n)$ and that of $(X_n, X_{n-1}, \ldots, X_0)$ is not enough to prove almost sure results.

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- In fact, the tightness is not a problem, because of the inequality

$$\nu_\gamma\Big(\sup_{1\le k\le n}\sup_{t\in[0,1]}|\mu_{k,\theta}(t)-\mu_{k,\theta}(\Pi_K(t))|>\lambda\Big)$$
$$\le \mathbb{P}\Big(2\sup_{1\le k\le n}\sup_{t\in[0,1]}|\mu_{k,X}(t)-\mu_{k,X}(\Pi_K(t))|>\lambda\Big),$$

where $\mu_{k,\theta}(t) = k(F_{k,\theta}(t) - F(t))$.

Finite dimensional convergence

- Again, we want to control the almost sure behavior of $v_n(F_{n,\theta}(t_1) - F(t_1), \ldots, F_{n,\theta}(t_d) - F(t_d))^t$. We approximate the indicators $f_i = \mathbf{1}_{[0,t_i]}$ by Lipschitz functions $f_{i,\varepsilon}$ such that $\nu_\gamma(|f_i - f_{i,\varepsilon}|^2)$ tends to zero as ε tends to zero.

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- Melbourne and Nicol (2009): $v_n(n^{-1}S_n(f_{1,\varepsilon}), \ldots, n^{-1}S_n(f_{d,\varepsilon}))^t$ is a.s. relatively compact, and the limit set is the unit ball in the RKHS generated by the limiting covariance Γ_ε.

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- We first prove that Γ_ε converges to $(\Gamma(t_i, t_j))_{1 \le i,j \le d}$. Next, by the maximal inequality in the paper with F. Merlevède (2009),

$$\limsup_{n\to\infty} v_n \sum_{i=1}^{d} |F_{n,\theta}(t_i) - F(t_i) - n^{-1}S_n(f_i)| \le C(\varepsilon) \text{ a.s.}$$

with $C(\varepsilon) \to 0$. The result follows.

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