Weak Dependence, Models and Some Applications

A book on the subject coauthored with Dedecker, Lang, León, Louhichi and Prieur appeared as the LNS 190 (Springer) in july 2007

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LIMIT THEOREMS AND APPLICATIONS

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Independence of $A \in \sigma(P)$ and $B \in \sigma(F)$ is also written as:

$$\mathrm{Cov}(f(P), g(F)) = 0, \qquad \forall f, g, \quad \|f\|_\infty, \|g\|_\infty \le 1$$

(The variables P and F stand for Past and Future)

Mixing

$$\alpha(\sigma(P),\sigma(F)) = \sup_{\|f\|_\infty,\|g\|_\infty \le 1} |\mathrm{Cov}(f(P),g(F))|$$

$$X=(X_t)_{t\in\mathbb{Z}}: \qquad P=(X_{s_1},\ldots,X_{s_u}), \qquad F=(X_{t_1},\ldots,X_{t_v}),$$

$$s_1\le\cdots\le s_u,\ t_1\le\cdots\le t_u, \qquad r=t_1-s_u \text{ is large}$$

$$\alpha(r)=\sup_{u,v}\max_{\substack{s_1\le\cdots\le s_u\\ t_1\le\cdots\le t_u\\ r=t_1-s_u}}\alpha(\sigma(P),\sigma(F)) \to_{r\to\infty} 0 \quad \text{(Rosenblatt)}$$

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Examples of non-mixing models

Andrews-Rosenblatt (1984)'s AR-model, $X_t=\frac{1}{2}\left(X_{t-1}+\xi_t\right), \quad \xi_t\sim b\left(\frac{1}{2}\right)$ iid,

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Galton-Watson model with immigration, $X_t=\sum_{i=1}^{X_{t-1}} Y_{i,t}+\xi_t$, iid integer valued, $\mathbb{E}Y_{i,t}<1$.

General formulation [Doukhan & Louhichi, 1999]

$(X_t)_{t\in T}$ $(\in E)$, $f : E^u \to \mathbb{R}$ in $\mathcal{F}$ and $g : E^v \to \mathbb{R}$ in $\mathcal{G}$:

$$\begin{aligned} |\mathrm{Cov}\left(f(X_{i_1},\ldots,X_{i_u}), g(X_{j_1},\ldots,X_{j_v})\right)| &\le \Psi(f,g)\epsilon(r), \\ d_T\left(\{i_1,\ldots,i_u\},\{j_1,\ldots,j_v\}\right) \ge r, &\quad \epsilon(r) \downarrow 0 \end{aligned}$$

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Choices of $\mathcal{F},\mathcal{G}$, Ψ, $\epsilon(r)$ yield different weak dependence conditions. Now $T=\mathbb{Z}$,
$i_1 \le \cdots \le i_u < i_u + r \le j_1 \le \cdots \le j_v$, $\mathrm{Lip}\, f = \sup_{(y_1,\ldots,y_u)\neq(x_1,\ldots,x_u)} \dfrac{|f(y_1,\ldots,y_u)-f(x_1,\ldots,x_u)|}{\|y_1-x_1\|+\cdots+\|y_u-x_u\|}$.

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$$\begin{aligned} \Psi(f,g) &= v\mathrm{Lip}\, g, & \epsilon(r) &= \theta(r), & \mathcal{F} &= \{\|f\|_\infty \leq 1\}, \mathcal{G} = \{\mathrm{Lip}\, g < \infty\}, \\ &= u\mathrm{Lip}\, f + v\mathrm{Lip}\, g, & \epsilon(r) &= \eta(r), & &\text{noncausal}, \mathcal{F} = \mathcal{G} = \{\mathrm{Lip}\, g < \infty\}, \\ &= uv\mathrm{Lip}\, f \cdot \mathrm{Lip}\, g, & \epsilon(r) &= \kappa(r), \\ &= u\mathrm{Lip}\, f + v\mathrm{Lip}\, g + uv\mathrm{Lip}\, f \cdot \mathrm{Lip}\, g, & \epsilon(r) &= \lambda(r). \end{aligned}$$

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Other causal cases including dynamical systems are considered with sharp limit theorems in Prieur's talk (see also Merlevède talk).

Causal coefficients ($\mathcal{F}$ is the set of bounded functions)

- θ coefficients: $\Psi(f,g) = v\|f\|_\infty \mathrm{Lip}\,(g)$,
  $\theta_p(\mathcal{M}, X) = \sup\{\|\mathbb{E}(g(X)|\mathcal{M}) - \mathbb{E}g(X)\|_p \,/ \mathrm{Lip}\, g \le 1\}$,
  $(X_i)_{i\in\mathbb{Z}}$ in $\mathbb{L}^p$, $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ σ-algebras $(\sigma(X_j, j \le k))$.

$$\theta_{p,v}(r) = \max_{s\le v} \frac{1}{s} \sup_{i+r\le j_1\le\cdots\le j_s} \theta_p\left(\mathcal{M}_i, (X_{j_1}, \ldots, X_{j_s})\right), \;\; \theta(r) = \theta_{1,\infty}(r)$$

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- τ coefficients:
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  $\theta_p(\mathcal{M},X) \le \tau_p(\mathcal{M},X)$,
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- γ coefficients (projective measure):
  $\gamma_p(\mathcal{M}, X) = \|\mathbb{E}(X|\mathcal{M}) - \mathbb{E}(X)\|_p \, (\le \theta_p(\mathcal{M}, X))$, $\gamma_p(r) = \sup_{i\in\mathbb{Z}} \gamma_p(\mathcal{M}_i, X_{i+r})$.

Hereditarity properties ($Y_n = h(X_n)$)

$(X_n)_{n\in\mathbb{Z}}$ η, κ or λ-weakly dependent$\Rightarrow$ $(Y_n)_{n\in\mathbb{Z}}$, for h Lipschitz.
$(X_n)_{n\in\mathbb{Z}}$, $\mathbb{R}^k$-valued, $p>1$, $\exists c, C>0, a\in[1,p]$: $\max_{1\le i\le k}\|X_i\|_p \le C$,
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- $(X_n)_{n\in\mathbb{Z}}$ η-weak dependent, then $(Y_n)_{n\in\mathbb{Z}}$ also, and

$$\eta_Y(n) = O\left(\eta(n)^{\frac{p-a}{p-1}}\right) \ ;$$

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This condition is satisfied by polynomials with degree a.

Examples

- Having in mind linear processes, we set:

$$
\begin{aligned}
X_t = H((\xi_{t-j})_{j\in\mathbb{Z}}), & \qquad H \in \mathbb{L}^1(\mathbb{R}^{\mathbb{Z}}, \mu) \quad \xi \sim \mu \\
\mathbb{E}\left|H\left(\xi_j, j \in \mathbb{Z}\right) - H\left(\xi_j \mathbb{1}_{|j|<r}, j \in \mathbb{Z}\right)\right| & \leq \delta_r \downarrow 0 \ (r \uparrow \infty) \\
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- If $H(x_j, j\in\mathbb{Z})$ does not depend on x_j, $j<0$, it is θ-dependent and $\theta(r)=\delta_r$. Wu makes use of those models, however the main problem is perhaps to find such functions H ?

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We now come to an unformal *Botanic of the models*

Vector valued LARCH(∞) models

$$X_t = \xi_t\Big(a + \sum_{j=1}^{\infty} a_j X_{t-j}\Big) \tag{1}$$

- Bilinear (Giraitis, Surgailis, 2003) $X_t = \zeta_t\Big(a + \sum_{j=1}^{\infty} a_j X_{t-j}\Big) + b + \sum_{j=1}^{\infty} b_j X_{t-j}$

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- GARCH(p, q) (Engle, Granger) $r_t = \sigma_t \epsilon_t, \ \sigma_t^2 = \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \gamma_0 + \sum_{j=1}^{q} \gamma_j r_{t-j}^2$

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If $\phi = \|\xi_0\|_m \sum_j \|a_j\| < 1$, $A(s) = \|\xi_0\|_m \sum_{j\geq s} \|a_j\|$ a solution of (1) in $\mathbb{L}^m$ is

$$X_t = \xi_t\Big(a + \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k \geqslant 1} a_{j_1}\xi_{t-j_1} \cdots a_{j_k}\xi_{t-j_1-\cdots-j_k} a\Big) \quad (2)$$

$$\begin{array}{lllllll} \theta(t) & \leq & C'/t^b, & \text{if} & A(s) & \leq & Cs^{-b}, \\ \theta(t) & \leq & C'(q \vee \phi)^{\sqrt{t}}, & \text{if} & A(s) & \leq & Cq^s. \end{array}$$

General nonMarkov nonlinear models

- *Processes with infinite memory:* $X_t = F(X_{t-1}, X_{t-2}, X_{t-3}, \ldots; \xi_t)$ $(\xi_t)_{t\in\mathbb{Z}}$ iid, $F : (\mathbb{R}^d)^{\mathbb{N}} \times \mathbb{R}^D \to \mathbb{R}^d$. If $A = \|F(0,0,0,\ldots;\xi_t)\|_m < \infty$, $m \geq 1$ and $\|F(x_1,x_2,x_3,\ldots;\xi_t) - F(y_1,y_2,y_3,\ldots;\xi_t)\|_m \leq \sum_{j=1}^\infty a_j\|x_j - y_j\|$ with $e^{-\alpha} = \sum_{j=1}^\infty a_j < 1$ then existence in $\mathbb{L}^m$, stationarity and weak dependence hold with: $\theta(r) \leq C \inf_{N>0}\left(\sum_{j\geq N} a_j + e^{-\frac{\alpha r}{N}}\right)$

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  hold with: $\theta(r) \leq C \inf_{N>0} \left(\sum_{j\geq N} a_j + e^{-\frac{\alpha r}{N}} \right)$
- *Random fields with infinite interactions:* $X_t = F\Big((X_{t-j})_{j\in\mathbb{Z}^d\setminus\{0\}}; \xi_t\Big)$
  Causal set $C \subset \mathbb{Z}^d$: $0 \notin \widetilde{C} = \left\{ \sum_{i=1}^{k} n_i j_i \Big/ j_\ell \in C, n_\ell > 0, 1 \leq \ell \leq k < \infty \right\}$
  Causal sets are Singletons, half lines or an open half space with half of its boundary in $\mathbb{Z}^2$ (improves on the usual quadrants condition!) .

General nonMarkov nonlinear models

- Regression models $X_t = f(X_{t-\ell_1}, \ldots, X_{t-\ell_k}) + \zeta_t g(X_{t-\ell_1}, \ldots, X_{t-\ell_k}) + \xi_t$

$$\left.\begin{array}{rcl} \|f(x_1,\ldots,x_k) - f(y_1,\ldots,y_k)\| & \leq & \sum_{i=1}^k b_i \|x_i - y_i\| \\ \|g(x_1,\ldots,x_l) - g(y_1,\ldots,y_l)\| & \leq & \sum_{j=1}^l c_j \|x_j - y_j\| \end{array}\right\} \Rightarrow \eta(r) \leq$$

$C\left(e^{-\frac{\alpha}{2k}}\right)^r$ if $e^{-\alpha} = \sum_{i=1}^k (b_i + \|\zeta_0\|_\infty c_i) < 1$ or

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- Linear fields $X_t = \sum_{j \in C} A_t^j X_{t-j} + \xi_t$, $\left(\left(A_t^j\right)_{j\in C}, \xi_t\right)_{t \in \mathbb{Z}^d}$ iid, $\|\xi_0\|_m < \infty$,
  $\sum_{j\in C} \|A_0^j\|_\infty < 1$, $\sum_{j\in C} \|A_0^j\|_m < 1$ is enough for C causal

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  $\|\xi_0\|_m \sum_{s\in C} a_s < 1$ also for C causal.
- Mean fields type model $X_t = f\big(\xi_t, \sum_{s\in C} a_s X_{t-s}\big)$

$$\begin{array}{rclcl} \sup_u \|f(u,x) - f(u,y)\| & \leq & b\|x-y\|, & b\sum_{i\neq 0}\|a_i\| & < & 1, \\ \|f(\xi_0,x) - f(\xi_0,y)\|_m & \leq & b\|x-y\|, & b\sum_{i\neq 0}\|a_i\| & < & 1, \quad \text{for } C \text{ causal.} \end{array}$$

Models with dependent innovations

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- If $b_s \leq Cs^{-b}$,

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This applies *e.g.* to linear processes, H symmetric polynomial, non causal LARCH(∞), etc.

Integer valued models

Set for $a > 0$ and $X \in \mathbb{Z}$

$$a \circ X = sign(X) \sum_{i=1}^{|X|} Y_i$$

with Y_i independent of the context and iid with mean a.

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- Such models also provide a wide class of nonmixing processes.

Some Applications

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- censored data spectral analysis (Bahamonde, D., Moulines)
- random fields (reliability of large multicomponent systems) (Lang, Ycart, Truquet, Bulinskii & Shashkin)

Limit theorems are fundamental to prove consistencies

- Donsker invariance principles D., Louhichi (1999) and D., Wintenberger (2007); for causal cases, sharp results D. & Dedecker (2003), Dedecker & Prieur (2005) for the coefficient $\widetilde{\alpha}$,

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- empirical CLT (D., Louhichi, Lang, Dedecker, Prieur) under causal or noncausal assumptions,
- for iid rvs, Bernstein inequality writes $\mathbb{P}(S_n \geq t\sqrt{n}) \leq C \exp\left\{-\frac{t^2}{2\sigma^2 + K\frac{t}{\sqrt{n}}}\right\}$

  $\star$ With Louhichi we use moment combinatorics to get $\leq Ce^{-c\sqrt{t}}$,

  $\star$ Cumulant techniques give $\leq C \exp\left\{-\dfrac{t^2}{2\sigma^2 + K(t/\sqrt{n})^\alpha}\right\}$ with Neumann,

  $\star$ Dedecker & Prieur use coupling arguments under causality.

Lindeberg Method (D.,Bardet, Lang & Ragache, 2007)

Apologizes are done to one organizer but I will talk about a joint work he does not like!

$X_i = (X_{i,1}, \ldots, X_{i,d})$ is 0-mean, $A_k = \sum_{i=1}^k \mathbb{E}\left(\|X_i\|^{2+\delta}\right) < \infty, \delta \le 1$
for $Y_i \sim \mathcal{N}(0, \text{Var } X_i)$ independent and $f \in \mathcal{C}_b^3$ and $k \in \mathbb{N}^*$:

$$\Delta_k = \left|\mathbb{E}\left(f(X_1 + \cdots + X_k) - f(Y_1 + \cdots + Y_k)\right)\right| \tag{3}$$

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Lemma 1 [standard Lindeberg Lemma under independence, 1922]

$$\Delta_k \le 3\,\|f^{(2)}\|_\infty^{1-\delta}\,\|f^{(3)}\|_\infty^{\delta} \cdot A_k.$$

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Lemma 2 [Lindeberg Lemma under dependence]

Set $f(x) = e^{i<t,x>}$ for $t \in \mathbb{R}^d$, $T(k) = \sum_{j=1}^k \left|\text{Cov}(e^{i<t,X_1+\cdots+X_{j-1}>}, e^{i<t,X_j>})\right|$ then

$$\Delta_k \leq T(k) + 3\|t\|^{2+\delta} A_k.$$

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$(X_i)_{i\in\mathbb{N}}$ stationary 0-mean, with $(2+\delta)$-order moments ($\delta > 0$). If:

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then $\displaystyle \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i \xrightarrow[k\to\infty]{\mathcal{D}} \sigma W_t$ for some $\sigma^2 \geq 0$.

Subsampling

Proposition 1

$(X_i)_{i\in\mathbb{Z}}$ 0-mean, $(2+\delta)$-order stationary, $\delta > 0$.
If $(m_n)_{n\in\mathbb{N}}$ is such that $m_n \underset{n\to\infty}{\longrightarrow} \infty$ and $k_n = [n/m_n] \underset{n\to\infty}{\longrightarrow} \infty$,

$$S_{k_n,n} = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} X_{im_n} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_d(0,\Sigma), \qquad \Sigma = \mathrm{Cov}(X_0)$$

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if moreover

- $(X_i)_{i\in\mathbb{Z}}$ is a $\theta-$weakly dependent sequence and $\theta(m_n)\sqrt{k_n} \underset{n\to\infty}{\longrightarrow} 0$.
- $(X_i)_{i\in\mathbb{Z}}$ is a $\lambda-$weakly dependent sequence and $\lambda(m_n)k_n^{\frac{3}{2}} \underset{n\to\infty}{\longrightarrow} 0$.

Kernel density estimation

$(X_i)_{i\in\mathbb{N}}$ stationary with marginal density f_X. $K:\mathbb{R}\to\mathbb{R}$ bounded Lipschitz, $\int_{-\infty}^{\infty} K(t)\,dt = 1$

$$\widehat{f}_X^{(n)}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n}K\left(\frac{x-X_i}{h_n}\right) \quad \text{for } x\in\mathbb{R},\ h_n \underset{n\to\infty}{\longrightarrow} 0,\, nh_n \underset{n\to\infty}{\longrightarrow} \infty$$

Kernel density estimation

Proposition 2

If $\|f_X\| < \infty$, $\sup_{i\neq j} \|f_{i,j}\|_\infty < \infty$ (joint marginal densities), then

$$\sqrt{nh_n}\left(\widehat{f}_X^{(n)}(x) - \mathbb{E}\widehat{f}_X^{(n)}(x)\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, f_X(x)\int_{\mathbb{R}} K^2(t)\,dt\right),$$

Dependence	Condition of dependence	Condition on h_n
κ	$\kappa(r) = O(r^{-\kappa})$ with $\kappa > 6$	$h_n = o\left(n^{-2/(\kappa-4)}\right)$
η	$\eta(r) = O(r^{-\eta})$ with $\eta > 5$	$h_n = o\left(n^{-5/(2\eta-5)}\right)$
λ	$\lambda(r) = O(r^{-\lambda})$ with $\lambda > 6$	$h_n = o\left(n^{-\frac{2}{\lambda-4}\vee\frac{5}{2\lambda-5}}\right)$
θ	$\theta(r) = O(r^{-\theta})$ with $\theta > 3$	$h_n = o(1)$

$f_X \in \mathcal{C}^p(\mathbb{R})$, $p \in \mathbb{N}^*$. If $h_n = C \cdot n^{-1/(2p+1)}$ and, $\lambda > 5p+5$, $\theta > 3$, $\kappa > 4p+6$, or $\eta > 5p+5$ then

$$\sqrt{nh_n}\left(\widehat{f}_X^{(n)}(x) - f_X(x)\right) \to \mathcal{N}\left(\frac{f_X^{(p)}(x)}{p!}\int_{\mathbb{R}} t^p K(t)\,dt\,,\ f_X(x)\int_{\mathbb{R}} K^2(t)\,dt\right)$$

Here $\int_{\mathbb{R}} K(t)t^q\,dt = 0$ for $0 < q < p$ and $\int_{\mathbb{R}} K(t)t^p\,dt \neq 0$.

Subsampled kernel density estimation

If $m_n \underset{n\to\infty}{\longrightarrow} \infty$ and $k_n = \left[n/m_n\right] \underset{n\to\infty}{\longrightarrow} \infty$, set:

$$\widehat{f}_X^{(n,m_n)}(x) = \frac{1}{k_n}\sum_{i=1}^{k_n}\frac{1}{h_n}K\left(\frac{x - X_{im_n}}{h_n}\right), \quad \text{for } x \in \mathbb{R}$$

Subsampled kernel density estimation

If $m_n \underset{n\to\infty}{\longrightarrow} \infty$ and $k_n = [n/m_n] \underset{n\to\infty}{\longrightarrow} \infty$, set:

$$\widehat{f}_X^{(n,m_n)}(x) = \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{1}{h_n} K\left(\frac{x - X_{im_n}}{h_n}\right), \quad \text{for } x \in \mathbb{R}$$

Proposition 3

$$\sqrt{k_n h_n}\left(\widehat{f}_X^{(n,m_n)}(x) - \mathbb{E}\widehat{f}_X^{(n,m_n)}(x)\right) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \mathcal{N}\left(0, f_X(x)\int_{\mathbb{R}} K^2(t)\,dt\right)$$

Arbitrary dependence rates are required and they are related to the sampling window m_n.

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