Invariance principle, multifractional Gaussian processes and long-range dependence

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Contents

1 Fractional Brownian motion

A. N. Kolmogorov (1940), B. Mandelbrot and J. Van Ness (1968),...

Definition 1 *Let $H \in (0,1)$. The fractional Brownian motion $W_H = \{W_H(t)\}_{t \in \mathbb{R}}$ (fBm) with Hurst index H is the real centered Gaussian process such that*

$$\mathbb{E}[W_H(t)W_H(s)] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big).$$

Remark: if $H = 1/2$ then $W_{1/2}$ is a classical Brownian motion.

Property 1 ***(Self-similarity)***
W_H is self-similar with index H: for every $\gamma > 0$

$$\{W_H(\gamma t)\}_{t \in \mathbb{R}} \stackrel{d}{=} \{\gamma^H W_H(t)\}_{t \in \mathbb{R}}$$

Property 2 ***(Regularity)***
For every $H' < H$, there exists a modification of W_H which is locally Hölder continuous with exponent H'. In fact, the pointwise Hölder exponent at a point t_0 is a.s. equal to H, i.e.

$$\alpha_{W_H}(t_0) := \sup \left\{ \alpha \,:\, \lim_{\varepsilon \to 0} \frac{W_H(t_0 + \varepsilon) - W_H(t_0)}{|\varepsilon|^\alpha} = 0 \right\} = H.$$

Definition 2 ***(Increments of W_H: fractional white noise)***
The sequence of the increments of W_H which are defined for every n by

$$\delta W_H(n) := W_H(n+1) - W_H(n)$$

is called the fractional white noise (with index H).

Property 3 ***(Stationarity of increments)***
δW_H is stationary: for every $m, n \in \mathbb{N}$

$$\mathbb{E}[\delta W_H(m)\delta W_H(n)] = \mathbb{E}[\delta W_H(0)\delta W_H(n-m)]$$

From now we assume that $H > 1/2$.

Property 4 ***(Long-range dependence of increments)***
δW_H satisfies the long-range dependence relation

$$\sum_{n=0}^{\infty} \mathbb{E}[\delta W_H(0)\delta W_H(n)] = \infty.$$

More precisely: as $n \to \infty$

$$\mathbb{E}[\delta W_H(0)\delta W_H(n)] \sim H(2H-1)n^{2H-2}.$$

Remark: This is in dramatic contrast with the case of classical Brownian motion $W_{1/2}$!

2 Invariance principle

Property 5 ***(For classical Brownian motion*** $W_{1/2}$***)***
Let $\{X_n\}_n$ *be a stationary sequence of* ***independent*** *Gaussian random variables, then*

$$\overset{d}{\lim_{N\to\infty}} \left\{ \frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor Nt \rfloor} X_n \right\}_{t\geq 0} = \{c_0 W_{1/2}(t)\}_{t\geq 0}$$

with $c_0^2 = \mathbb{E}[X_0^2]$.

Also known as functional central limit theorem, Donsker theorem...

Property 6 *(For fBm W_H with $H > 1/2$)*
Let $\{X_n\}_n$ be a sequence of stationary Gaussian variables such that when $n \to \infty$

$$\mathbb{E}[X_0 X_n] \sim c n^{2H-2} \text{ with } c > 0 \left(\text{in particular } \sum_{n=0}^{\infty} \mathbb{E}[X_0 X_n] = \infty \right).$$

Then

$$\overset{d}{\lim_{N\to\infty}} \left\{ \frac{1}{N^H} \sum_{n=1}^{\lfloor Nt \rfloor} X_n \right\}_{t\geq 0} = \{c_0 W_H(t)\}_{t\geq 0}$$

with $c_0^2 = H^{-1}(2H-1)^{-1}c$.

To summarize about fractional Brownian motion

Interests of fBm:

- Generalizes cBm,
- provides model for long-range dependence,
- satisfies an invariance principle.

Drawback of fBm:

- Homogeneity of its properties (its pointwise Hölder exponent is constant...).

Consequence:

- Introduction of “multi”fractional processes...

3 Multifractional Brownian motion

(Independently introduced by R.F. Peltier and J. Lévy Vehel (1996) and by A. Benassi, S. Jaffard and D. Roux (1997))
Consider the set of fractional Brownian motions $\{W_H\}_{H\in(0,1)}$ defined by

$$W_H(t) = \frac{1}{C(H)} \int_{-\infty}^{\infty} \frac{e^{-ixt}-1}{|x|^{H+1/2}} \widehat{B}(dx)$$

where $\widehat{B}(dx)$ is the Fourier transform of a real Brownian measure $B(d\xi)$. We make

$$H \to h(t)$$

to get:

Definition 3 ***(Multifractional Brownian motion)***
Let a function $h : \mathbb{R} \to (0,1)$. The multifractional Brownian motion W_h with Hurst function h is defined by

$$W_h(t) := W_{h(t)}(t) = \frac{1}{C(h(t))} \int_{-\infty}^{\infty} \frac{e^{-ixt}-1}{|x|^{h(t)+1/2}} \widehat{B}(dx).$$

Assumption (A): There exists $\beta > 0$ such that h is $\beta-$Hölder and

$$\sup h < \beta.$$

Property 7 ***(Local self-similarity)***
Under (A), W_h is locally self-similar with function h: for every $t \geq 0$

$$\lim_{\varepsilon \to 0}^{d} \left\{ \frac{W_h(t+\varepsilon u) - W_h(t)}{\varepsilon^{h(t)}} \right\}_{u \geq 0} = \{c(t) W_{h(t)}(u)\}_{u \geq 0},$$

where c is a function.

Property 8 ***(Regularity)***
Under (A), for every t_0 the Hölder pointwise exponent $\alpha_{W_h}(t_0)$ of W_h is almost surely equal to $h(t_0)$.

Therefore, multifractional Brownian motion W_h with Hurst function

$$h : \mathbb{R} \rightarrow (1/2, 1)$$

could be a relevant alternative to fractional processes with $H \in (1/2, 1)$ to provide models for long-range dependences.

But, can W_h (or an other multifractional Gaussian process) serve as an universal Gaussian models for long-range dependence ?

4 The main result

Let

- $X = \{X_n(H), H \in (1/2, 1), n \in \mathbb{N}\}$ be a centered Gaussian field
- and $h : \mathbb{R} \to [a, b] \subset (1/2, 1)$ be a continuous function.

We also let the assumptions

- **Assumption (i)** For every $M > 0$ the map

  $$(j, k, H_1, H_2) \longmapsto \mathbb{E}[X_j(H_1)X_k(H_2)]$$

  is bounded on $\{(j, k) \in \mathbb{N}^2, |j - k| \leq M\} \times [a, b]^2$.

- **Assumption (ii)** There exists a continuous function $R : [a, b]^2 \to (0, \infty)$ such that when $j - k \to \infty$

  $$\mathbb{E}[X_j(H_1)X_k(H_2)] \sim R(H_1, H_2)(j - k)^{H_1+H_2-2}$$

  uniformly for $(H_1, H_2) \in [a, b]^2$.

For every $n, N \in \mathbb{N}$ we define

$$h_n^N = h\Big(\frac{n}{N}\Big).$$

Theorem 1 *Under Assumptions (i) and (ii),*

$$\lim_{N\to\infty}^{d} \left\{ \sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h_n^N)}{N^{h_n^N}} \right\}_{t\geq 0} = \{S_h(t)\}_{t\geq 0}$$

where S_h is a centered Gaussian process such that

$$\mathbb{E}[S_h(t)S_h(s)] \quad = \quad \int_0^t d\theta \int_0^s d\sigma\, \mathcal{R}(\theta, \sigma; h(\theta), h(\sigma))|\theta - \sigma|^{h(\theta)+h(\sigma)-2}$$

where $\mathcal{R}$ is defined for every t, s, H_1, H_2 by

$$\mathcal{R}(t, s; H_1, H_2) = R(H_1, H_2)1_{t\geq s} + R(H_2, H_1)1_{t<s}.$$

Remark: If $h \equiv H \in (1/2, 1)$, then Theorem 1 is the classical invariance principle for fBm.

Assumption (A'): There exists $\beta > 0$ such that h is $\beta-$Hölder.

Property 9 ***(Local self-similarity)***

Under (A'), S_h is locally self-similar with function h: for every $t \geq 0$

$$\lim_{\varepsilon \to 0}^{d} \left\{ \frac{S_h(t+\varepsilon u) - S_h(t)}{\varepsilon^{h(t)}} \right\}_{u \geq 0} = \{c(t) W_{h(t)}(u)\}_{u \geq 0},$$

where

$$c(t)^2 = \frac{R(h(t), h(t))}{(2h(t)^2 - h(t))}$$

Property 10 ***(Regularity)***

Under (A'), for every t_0 the Hölder pointwise exponent $\alpha_{S_h}(t_0)$ of S_h is almost surely equal to $h(t_0)$.

Remark: (A') is weaker than (A)!

Main ideas of the proof of Theorem 1: It consists in studying the limit of

$$\mathbb{E}\left[\sum_{n=1}^{\lfloor Nt\rfloor}\frac{X_n(h_n^N)}{N^{h_n^N}}\sum_{m=1}^{\lfloor Ns\rfloor}\frac{X_m(h_m^N)}{N^{h_m^N}}\right]=\sum_{n=1}^{\lfloor Nt\rfloor}\sum_{m=1}^{\lfloor Ns\rfloor}\frac{1}{N^{h_n^N+h_m^N}}\mathbb{E}[X_n(h_n^N)X_m(h_m^N)].$$

- Assumption (ii) $\Longrightarrow$ if $|n-m|$ is large, then

$$\frac{1}{N^{h_n^N+h_m^N}}\mathbb{E}[X_n(h_n^N)X_m(h_m^N)]\simeq\frac{1}{N^2}\mathcal{R}\left(\frac{n}{N},\frac{m}{N};h_n^N,h_m^N\right)\left|\frac{n}{N}-\frac{m}{N}\right|^{h_n^N+h_m^N-2}$$

- Assumption (i) $\Longrightarrow$ if $|n-m|$ is small, then

$$\frac{1}{N^{h_n^N+h_m^N}}\mathbb{E}[X_n(h_n^N)X_m(h_m^N)]\simeq 0$$

Therefore

$$\sum_{n=1}^{\lfloor Nt \rfloor} \sum_{m=1}^{\lfloor Ns \rfloor} \frac{1}{N^{h_n^N + h_m^N}} \mathbb{E}[X_n(h_n^N) X_m(h_m^N)]$$

$$\simeq \frac{1}{N^2} \sum_{n=1}^{\lfloor Nt \rfloor} \sum_{m=1}^{\lfloor Ns \rfloor} \mathcal{R}\left(\frac{n}{N}, \frac{m}{N}; h_n^N, h_m^N\right) \left|\frac{n}{N} - \frac{m}{N}\right|^{h_n^N + h_m^N - 2}$$

$$\longrightarrow \int_0^t d\theta \int_0^s d\sigma \, \mathcal{R}(\theta, \sigma; h(\theta), h(\sigma)) |\theta - \sigma|^{h(\theta) + h(\sigma) - 2}$$

as $N \to \infty$ using a Riemann sum convergence type theorem. □

To summarize

Now: we have got a Gaussian process which is

- limit of an invariance principle (by definition !),
- multifractional (local self-similarity and regularity properties),
- suitable for modelling long-range dependence (because obtained for $h : \mathbb{R}^+ \to (1/2, 1)$).

But: how is this process related to multifractional Brownian motion ?

Representation of S_h

Let us consider the set of fractional Brownian motions $\{W_H\}_{H\in(1/2,1)}$ defined by

$$W_H(t) = \frac{1}{C(H)} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{|x|^{H+1/2}} \widehat{B}(dx)$$

from the real Brownian measure $B(d\xi)$.
We let for every $H \in (1/2, 1)$

$$X_n(H) = W_H(n+1) - W_H(n) = \delta W_H(n).$$

We can easily check that such a field satisfies the assumptions of Theorem 1, then, there exists S_h such that

$$\overset{d}{\lim_{N\to\infty}} \left\{ \sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h_n^N)}{N^{h_n^N}} \right\}_{t\geq 0} = \{S_h(t)\}_{t\geq 0}$$

But also, we can compute explicitely:

$$
\begin{aligned}
\sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h_n^N)}{N^{h_n^N}} &= \sum_{n=1}^{\lfloor Nt \rfloor} \frac{1}{N^{h_n^N}} \Big(W_{h_n^N}(n) - W_{h_n^N}(n-1) \Big) \\
&=^d \sum_{n=1}^{\lfloor Nt \rfloor} \left(W_{h_n^N}\left(\frac{n}{N}\right) - W_{h_n^N}\left(\frac{n-1}{N}\right) \right) \\
&= \sum_{n=1}^{\lfloor Nt \rfloor} \left(W_{h_n^N}\left(\frac{n}{N}\right) - W_{h_{n-1}^N}\left(\frac{n-1}{N}\right) \right) \\
&\quad - \sum_{n=1}^{\lfloor Nt \rfloor} \left(W_{h_n^N}\left(\frac{n-1}{N}\right) - W_{h_{n-1}^N}\left(\frac{n-1}{N}\right) \right).
\end{aligned}
$$

We have

$$\sum_{n=1}^{\lfloor Nt \rfloor} \left(W_{h_n^N}\left(\frac{n}{N}\right) - W_{h_{n-1}^N}\left(\frac{n-1}{N}\right) \right)$$

$$= W_{h_{\lfloor Nt \rfloor}^N}\left(\frac{\lfloor Nt \rfloor}{N}\right) \longrightarrow W_{h(t)}(t)$$

and

$$\sum_{n=1}^{\lfloor Nt \rfloor} \left(W_{h_n^N}\left(\frac{n-1}{N}\right) - W_{h_{n-1}^N}\left(\frac{n-1}{N}\right) \right)$$

$$\simeq \sum_{n=1}^{\lfloor Nt \rfloor} (h_n^N - h_{n-1}^N) \left. \frac{\partial W_H(\frac{n-1}{N})}{\partial H} \right|_{H=h_{n-1}^N}$$

$$\simeq \frac{1}{N} \sum_{n=1}^{\lfloor Nt \rfloor} h'\left(\frac{n-1}{N}\right) \left. \frac{\partial W_H(\frac{n-1}{N})}{\partial H} \right|_{H=h_{n-1}^N}$$

$$\longrightarrow \int_0^t h'(\theta) \left. \frac{\partial W_H(\theta)}{\partial H} \right|_{H=h(\theta)} d\theta$$

Hence, we have

$$\overset{d}{\lim_{N\to\infty}} \left\{ \sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h_n^N)}{N^{h_n^N}} \right\}_{t\geq 0} = \left\{ W_{h(t)}(t) - \int_0^t h'(\theta) \frac{\partial W_H(\theta)}{\partial H}\bigg|_{H=h(\theta)} d\theta \right\}_{t\geq 0}.$$

Combined with

$$\overset{d}{\lim_{N\to\infty}} \left\{ \sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h_n^N)}{N^{h_n^N}} \right\}_{t\geq 0} = \{S_h(t)\}_{t\geq 0}$$

this gives

$$\begin{aligned} \{S_h(t)\}_{t\geq 0} &=^d \left\{ W_h(t) - \int_0^t h'(\theta) \frac{\partial W_H(\theta)}{\partial H}\bigg|_{H=h(\theta)} d\theta \right\}_{t\geq 0} \\ &=^d \{W_h(t) - \Phi(t)\}_{t\geq 0} \end{aligned}$$

where Φ is a $\mathcal{C}^1$ process.

But we can generalize for an arbitrary field $\{X_n(H)\}_{n,H}$:

Theorem 2 *Let $\{X_n(H)\}_{n,H}$ satisfying (i), (ii) and assume that R and h are sufficiently regular We have*

$$\{S_h(t)\}_{t\geq 0} \quad =^d \quad \left\{ \widetilde{W}(t, h(t)) - \int_0^t h'(\theta) \left. \frac{\partial \widetilde{W}(\theta, H)}{\partial H} \right|_{H=h(\theta)} d\theta \right\}_{t\geq 0}.$$

where $\{\widetilde{W}(t, H)\}_{t,H}$, is obtained by

$$\overset{d}{\lim_{N\to\infty}} \left\{ \frac{1}{N^H} \sum_{n=1}^{\lfloor Nt \rfloor} X_n(H) \right\}_{t,H} = \{\widetilde{W}(t, H)\}_{t,H}.$$

To summarize

$$
\begin{array}{ccccc}
 & & W_H & & \\
 & \text{integral representation} \swarrow & & \searrow \text{invariance principle} & \\
 & W_h & \text{generalizations} & S_h & \\
 & \searrow & & \swarrow & \\
 & & W_h - \Phi = S_h & &
\end{array}
$$