A Quadratic ARCH(∞) model with long memory and Lévy stable behavior of squares

Donatas Surgailis

Vilnius Institute of Mathematics and Informatics

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1. Stylized facts of financial (daily) returns

- returns $X_t = \log(p_t / p_{t-1})$ are uncorrelated: $\mathrm{corr}(X_t, X_s) \approx 0 \, (t \neq s)$
- squared and absolute returns have long memory: $\mathrm{corr}(X_t^2, X_s^2) \neq 0$, $\mathrm{corr}(|X_t|, |X_s|) \neq 0$ $(|t-s| = 100 \div 500)$
- heavy tails: $EX_t^4 = \infty$
- conditional mean $\mu_t = E\left[X_t | F_{t-1}\right] \approx 0$, conditional variance $\sigma_t^2 = E\left[X_t^2 | F_{t-1}\right]$ "randomly varying" (conditional heteroskedasticity)
- leverage effect: past returns and future volatilities negatively correlated: $\mathrm{corr}\left(X_s, \sigma_t^2\right) < 0$ $(s < t)$
- volatility clustering

2. GARCH, ARCH(infty) and Linear ARCH (LARCH)

GARCH(p, q) :

$$X_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i X_{t-i}^2,$$

$\omega \geq 0, \alpha_i \geq 0, \beta_i \geq 0, p, q = 0, 1, \ldots, \ (\varepsilon_t)$ iid, $E\varepsilon_t = 0, \ E\varepsilon_t^2 = 1$

ARCH(∞) :

$$X_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \omega + \sum_{i=1}^{\infty} \alpha_i X_{t-i}^2,$$

- GARCH(p, q) : Engle(1982), Bollerslev(1986), Bougerol and Pickard(1992), ..., Teräsvirta(2007, review)
- ARCH(∞): Giraitis, Kokoszka and Leipus(2000), Kazakevičius and Leipus(2002,2003), ..., Giraitis et al.(2007, review)

2. GARCH, ARCH(infty) and Linear ARCH (LARCH)

- $\exists$ stationary solution of ARCH(∞) with $EX_t^2 < \infty \iff \sum_{i=1}^{\infty} \alpha_i < 1$
- ARCH(∞) does not allow for long memory in (X_t^2)
- (ε_t) symmetric $\Longrightarrow$ no leverage

Linear ARCH (LARCH)(∞) (Robinson(1991), Giraitis et al.(2000,2004), Berkes and Horváth(2003)):

$$X_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \left(\omega + \sum_{i=1}^{\infty} a_i X_{t-i} \right)^2,$$

$\sum_{i=1}^{\infty} a_i^2 < 1, \quad \omega \neq 0, \quad a_i \in \mathbf{R}, \quad (\varepsilon_t) \sim \text{ iid}(0,1)$

- $a_i \sim c i^{d-1}$ $(i \to \infty, \ \exists\ c \neq 0, \ d \in (0, 1/2)$ (e.g., FARIMA$(0, d, 0)$)
- allows for long memory in (X_t^2) and the leverage effect

2. GARCH, ARCH(infty) and Linear ARCH (LARCH)

- partial sums of (X_t^2) of LARCH(∞) may converge to fractional Brownian motion (FBM) (provided $EX_t^4 < \infty$)
- volatility $\sigma_t = \omega + \sum_{i=1}^{\infty} a_i X_{t-i}$ *not* separated from zero (bad for QMLE) and can assume negative values with positive probability
- stationary solution σ_t of LARCH(∞) admits an orthogonal Volterra expansion in $\varepsilon_s, s < t$ convergent in L^2:

$$
\begin{aligned}
\sigma_t &= 1 + \sum_{k=1}^{\infty} \sum_{s_k < \ldots < s_1 < t} a_{t-s_1} a_{s_1-s_2} \cdots a_{s_{k-1}-s_k} \varepsilon_{s_1} \ldots \varepsilon_{s_k} \\
&=: \sum_S a_t^S \varepsilon^S
\end{aligned}
$$

$S = \{s_k, \ldots, s_1\} \subset \mathbf{Z}$, $a_t^S := a_{t-s_1} a_{s_1-s_2} \cdots a_{s_{k-1}-s_k}$, $\varepsilon^S := \varepsilon_{s_1} \cdots \varepsilon_{s_k}$

3. Sentana's Quadratic ARCH (QARCH)

Sentana(1995): Generalized Quadratic ARCH(GQARCH(p, q)):

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} a_i X_{t-i} + \sum_{i,j=1}^{p} a_{ij} X_{t-i} X_{t-j} + \sum_{i=1}^{q} b_i \sigma_{t-i}^2,$$

- ω, a_i, a_{ij}, b_i real parameters
- conditions guaranteeing the existence of stationary solution (X_t, σ_t^2) with $\mu_t = E[X_t | F_{t-1}] = 0$, $\sigma_t^2 = E[X_t^2 | F_{t-1}] \geq 0$ a.s.
- sufficient condition for stationarity: $\sum_{i=1}^{p} a_{ii} + \sum_{i=1}^{q} b_i < 1$
- nests a variety of ARCH models
- can be expressed as random coefficient VAR
- no explicit solution in general
- limited to short memory models

4. LARCH+(infty)

Goal: to construct a stationary process (X_t) with

$$\begin{aligned} \mu_t &= E\left[X_t | F_{t-1}\right] = 0, \\ \sigma_t^2 &= E\left[X_t^2 | F_{t-1}\right] = \nu^2 + \left(\omega + \sum_{i=1}^{\infty} a_i X_{t-i}\right)^2, \end{aligned} \qquad (1)$$

where υ, ω, a_i are real parameters, $\sum_{i=1}^{\infty} a_i^2 < \infty$

- particular case of Sentana's QARCH
- $\nu = 0$ corresponds to LARCH(∞)
- $\nu > 0$: conditional variance separated from 0: $\sigma_t^2 \geq \nu^2 > 0$ a.s.
- the construction below can be extended to include general linear drift $\mu_t = E\left[X_t | F_{t-1}\right] = \mu + \sum_{i=1}^{\infty} c_i X_{t-i}$ (Giraitis and Surgailis(2002))

4. LARCH+(infty): definition

(η_t, ζ_t) : a sequence of iid vectors, $E\eta_t = E\zeta_t = 0$, $E\eta_t^2 = E\zeta_t^2 = 1$,

$$\rho = E\eta_t\zeta_t \tag{2}$$

Definition

LARCH$_+$ (∞) equation:

$$X_t = \kappa\eta_t + \zeta_t \sum_{i=1}^{\infty} a_i X_{t-i}, \tag{3}$$

where parameters $\kappa \in \mathbf{R}$ and $\rho \in [-1, 1]$ in (2) are related to $\nu \geq 0$ and $\omega \in \mathbf{R}$ in (1) by

$$\kappa\rho = \omega, \qquad \kappa^2 = \omega^2 + \nu^2.$$

4. LARCH₊(infty): solution

Solution

Solution of LARCH$_+(\infty)$ equation (3):

$$X_t = \kappa\left(\eta_t + \zeta_t \sum_{k=0}^{\infty} \sum_{u<s_k<\ldots<s_1<t} a_{t-s_1}\cdots a_{s_{k-1}-s_k} a_{s_k-u}\zeta_{s_1}\cdots\zeta_{s_k}\eta_u\right)$$
$$= \kappa\left(\eta_t + \zeta_t \sum_{u<t} \sum_{S\subset(u,t)} a^S_{u,t}\zeta^S\eta_u\right), \qquad (4)$$

where:

$$a^S_{u,t} := a_{t-s_1}a_{s_1-s_2}\cdots a_{s_{k-1}-s_k}a_{s_k-u}, \qquad \zeta^S := \zeta_{s_1}\cdots\zeta_{s_k},$$
$$S = \{s_k,\ldots,s_1\}, \qquad u < s_k < \ldots < s_1 < t$$

4. LARCH+(infty): existence and uniqueness in L2

Notation: $F_t := \sigma\left\{\eta_s, \zeta_s, s \le t\right\}$, $A_t := \sum_{i=1}^{\infty} a_i X_{t-i}$.

Theorem (1)

Let $\kappa \neq 0$. A L^2-bounded causal solution (X_t) of LARCH$_+(\infty)$ equation (3) exists iff

$$\|a\|^2 := \sum_{j=1}^{\infty} a_j^2 < 1.$$

In the latter case,such a solution is unique, strictly stationary and is given by the Volterra series (4) convergent in L^2. Moreover,

$$X_t = \sigma_t \varepsilon_t, \tag{5}$$

where $\sigma_t = \sqrt{\upsilon^2 + (\omega + A_t)^2}$ and where (ε_t, F_t) form a stationary martingale difference sequence with $E\left[\varepsilon_t | X_s, s < t\right] = 0$ and

$$E\left[\varepsilon_t^2 | X_s, s < t\right] = 1. \tag{6}$$

- (5)-(6) follow from $\mu_t = E[X_t | X_s, s < t] = \kappa E[\eta_t] + E[\zeta_t] A_t = 0$,

$$\begin{aligned} \sigma_t^2 &= E\left[X_t^2 | X_s, s < t\right] = \kappa^2 E\left[\eta_t^2\right] + E\left[\zeta_t^2\right] A_t^2 + 2\kappa E[\eta_t \zeta_t] A_t \\ &= \kappa^2 + A_t^2 + 2\kappa\rho A_t \\ &= \nu^2 + \varpi^2 + A_t^2 + 2\varpi A_t \\ &= \nu^2 + (\varpi + A_t)^2. \end{aligned}$$

- $EX_t^2 = \chi^2 / \left(1 - \|a\|^2\right)$
- GLARCH$_+$(1,1): $\sigma_t^2 = \nu^2 + (\varpi + A_t)^2, \quad A_t = \alpha A_{t-1} + \beta X_{t-1}, \quad \alpha^2 + \beta^2 < 1$
- GLARCH$_+(0, d, 0)$: $\sigma_t^2 = \nu^2 + (\varpi + A_t)^2$,

$$A_t = c(1-L)^{-d} X_{t-1}, \quad d \in (0, 1/2), \; c^2 < \Gamma^2(1-d)/\Gamma(1-d)$$

5. LARCH+(infty): leverage and long memory

- Everywhere below: $EX_t^4 = \infty$, $E|X_t|^3 < \infty$ (most interesting case)
- Sufficient condition (Giraitis et al.(2004)):

$$m_3^{1/3}\|a\|_3 + 3.81\|a\| < 1, \qquad m_4 = \infty, \tag{7}$$

$m_p := \max\left(E|\eta_0|^p, E|\zeta_0|^p\right)$, $\|a\|_p := \{\sum_{i=1}^{\infty}|a_i|^p\}^{1/p}$

- *Leverage effect*: past returns and future volatilities *negatively* correlated
- *Leverage function* (Giraitis et al.(2004)):

$$L_{t-s} := cov\left(X_s, \sigma_t^2\right) = EX_sX_t^2 \qquad (s < t)$$

- satisfies linear equation with Hilbert-Schmidt operator (assuming $EX_t^3 = 0$):

$$L_t = 2\omega\sigma^2 a_t + \sum_{0<s<t} a_{t-s}^2 L_s + 2a_t \sum_{s>0} a_{t+s} L_s \tag{8}$$

5. LARCH+(infty): leverage and long memory

Theorem (2)

Assume (7) and $E\eta_t^3 = E\zeta_t^3 = 0$.
(i) (Leverage property) Let $\varpi a_1 < 0, \quad \omega a_i \le 0, \ i = 1, ..., k$ *for some* $k \ge 1$. *Then* $L_i < 0, \ i = 1, ..., k$.
(ii) (Long memory) Let

$$a_i \sim c i^{d-1} \qquad (i \to \infty, \quad \exists c \neq 0, d \in (0, 1/2)) \tag{9}$$

Then

$$L_t \sim c(d) t^{d-1} \qquad (t \to \infty). \tag{10}$$

- proof uses equation (8) for the leverage function
- finite 4th moment not required
- long memory asymptotics of $\mathrm{cov}(X_0^p, X_t^p)$ $(p = 2, 3, ...)$ under suitable moment conditions

6. Limit of sums of squares: between FBM and Lévy

(X_t) : long memory $\mathrm{LARCH}_+(\infty)$ process with *infinite* 4th moment as in Thm 2

Problem

Limit distribution of partial sums process
$\sum_{t=1}^{[n\tau]}\left(X_t^2 - EX_t^2\right), \qquad \tau \in [0,1]$

Assume conditions:

$$E\left(\zeta^4 + \zeta^2\eta^2\right) < \infty \tag{11}$$

and

$$P\left(\eta^2 > x\right) \sim c_1 x^{-\alpha} \qquad \left(x \to \infty,\ \exists \alpha \in (1,2),\ c_1 > 0\right) \tag{12}$$

- $\left(\eta_t^2 - E\eta_t^2\right)$ belong to the domain of attraction of antisymmetric α-stable law
- Example of correlated (η, ζ) satisfying (11)-(12): $\zeta \sim \mathrm{N}(0,1)$, $\eta = \zeta\sqrt{\xi}, \xi \perp \zeta$, $P(\xi > x) \sim cx^{-\alpha}$, $E\xi = 1$, $E\sqrt{\xi} < 1$

6. Limit of sums of squares: between FBM and Lévy

Theorem (3)

Assume (7), (9), (11), (12) and $\rho = E\eta\zeta \neq 1$. *Then:*
(i) If $d + .5 > 1/\alpha$, *then*

$$n^{-d-.5} \sum_{t=1}^{[n\tau]} \left(X_t^2 - EX_t^2\right) \Rightarrow FBM_\tau\left(d + .5\right) \tag{13}$$

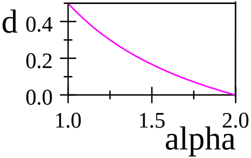
(ii) If $d + .5 < 1/\alpha$, *then*

$$n^{-1/\alpha} \sum_{t=1}^{[n\tau]} \left(X_t^2 - EX_t^2\right) \Rightarrow Lévy_\tau\left(\alpha\right) \tag{14}$$

- $\mathrm{FBM}_\tau\left(d + .5\right)$: a fractional BM with variance $c(d)\tau^{2d+1}$
- $\mathrm{Lévy}_\tau\left(\alpha\right)$: a homogeneous α-stable Lévy process with zero mean and skewness parameter $\beta = 1$

6. Limit of sums of squares: between FBM and Lévy

- $\Rightarrow$: fi-di convergence
- $d + .5 = 1/\alpha$ $(0 < d < .5,\ 1 < \alpha < 2)$: critical line



- $d = 0$: $a_i \sim ci^{d-1} = ci^{-1}$ or $\sum_{i=1}^{\infty} |a_i| < \infty$ (short memory) + (7),(11),(12): only $\alpha-$stable Lévy limit for partial sums of X_t^2 expected
- partial sums of X_t's tend to a standard BM ((X_t)=martingale differences with finite variance)

7. Idea of the proof of Thm 3

Recall $(\kappa = 1)$:

$$X_t = \eta_t + \zeta_t \sum_{u<t} \sum_{S\subset(u,t)} a_{u,t}^S \zeta^S \eta_u = \eta_t + \sum_{u<t} \eta_u \sum_{S\subset(u,t)} a_{u,t}^S \zeta^S \zeta_t,$$

The squared process $X_t^2 = Y_t^D + 2Y_t^{Off}$ splits into two parts:
$Y_t^D : \eta_{u_1} = \eta_{u_2}$: "diagonal part" with infinite variance: tends to Lévy
$Y_t^{Off} : \eta_{u_1} \neq \eta_{u_2}$: "off-diagonal part" with finite variance: tends to FBM

$$\begin{aligned} Y_t^D \quad &: \quad = (\eta_t^2 - E\eta_t^2) + \sum_{u<t} (\eta_u^2 - E\eta_t^2) \Big(\sum_{S\subset(u,t)} a_{u,t}^S \zeta^S \zeta_t \Big)^2, \\ Y_t^{Off} \quad &: \quad = \sum_{u<t} \Big(\sum_{S\subset(u,t)} a_{u,t}^S \zeta^S \zeta_t \Big)^2 + \eta_t \zeta_t \sum_{u<t} \sum_{S\subset(u,t)} a_{u,t}^S \zeta^S \eta_u \\ &\quad\quad + \zeta_t^2 \sum_{u_2<u_1<t} \eta_{u_2} \eta_{u_1} \sum_{S_1\subset(u_1,t)} \sum_{S_2\subset(u_2,t)} a_{u_1,t}^{S_1} \zeta^{S_1} a_{u_2,t}^{S_2} \zeta^{S_2} + E\eta_t^2 \end{aligned}$$

7. Proof of Thm 3: main steps:

Step 1

$n^{-d-.5}\sum_{t=1}^{[n\tau]}(Y_t^{O_{ff}} - EY_t^{O_{ff}}) \Rightarrow \mathrm{FBM}_\tau\,(d+.5)$

(similar to Giraitis et al.(2000) for LARCH(∞))

Step 2

$\sum_{t=1}^{n} Y_t^D = \sum_{u=1}^{n}(\eta_u^2 - E\eta_u^2)Z_u + o_p\left(n^{1/\alpha}\right)$, where

$$Z_u := 1 + \sum_{t>u}\left\{\sum_{S\subset(u,t)} a_{u,t}^S \zeta^S \zeta_t\right\}^2$$

is a strictly stationary process with finite variance and “anti-predictable” (recall (ζ_t) and (η_t) are *not* independent as sequences)

Step 3

$n^{-1/\alpha}\sum_{u=1}^{n}(\eta_u^2 - E\eta_u^2)Z_u \Rightarrow \text{Lévy}_\tau\,(\alpha)$

- Step 3 is based on the *principle of conditioning* due to Jakubowski (1986), which allows to replace (Z_t) by a similar process but *independent of* (η_t)
- Step 2 is technically most difficult
- Uses a bound for 4th mixed moment of Volterra series, written as a sum of *diagrams* (Giraitis et al. (2000)) over a table $I = I(k)_4$ having 4 rows $I_1, I_2, I_{3,} I_4$ of respective length $k_1, k_2, k_3, k_4 = 0, 1, \ldots;\ (k)_4 := (k_1, \ldots, k_4)$
- A diagram γ determines the *type of diagonal* in the sum

$$\Sigma_{u,0,t} \quad : \quad = \sum_{S_1}\sum_{S_2}\sum_{S_3}\sum_{S_4} a_{u,t}^{S_1} a_{u,t}^{S_2} a_{u,0}^{S_3} a_{u,0}^{S_4}\ E\zeta^{S_1\cup\{t\}}\zeta^{S_2\cup\{t\}}\zeta^{S_3\cup\{0\}}\zeta^{S_4\cup\{0\}}$$

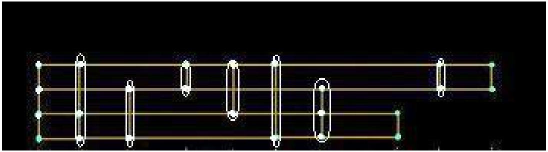
$$= \sum_{(k)_4}\sum_{\gamma\in\Gamma(k)_4} \mu_\gamma \sum_{(S)_4\sim\gamma} a_{u,(t)}^{(S)_4}$$

where $\mu_\gamma = E\zeta^{S_1\cup\{t\}}\zeta^{S_2\cup\{t\}}\zeta^{S_3\cup\{0\}}\zeta^{S_4\cup\{0\}} = 0$ unless all elements of 4 sets $S_1 \subset (u,t)$, $S_2 \subset (u,t)$, $S_3 \subset (u,0)$, $S_4 \subset (u,0)$ are "coupled"

7. Proof of Thm 3: diagrams:

- Goal: to obtain the right upper bound of $\Sigma_{u,0,t}$:for any $(k)_4 = (k_1, ..., k_4)$, $|k| := k_1 + ... + k_4$, and any diagram $\gamma \in \Gamma(k)_4$, as $-\infty \leftarrow u < 0 > t \rightarrow +\infty$

$$\sum_{(S)_4 \sim \gamma} \left| a^{(S)_4}_{u,(t)} \right| \leq C \|a\|^{|k|} |t-u|^{2d-2} |u|^{2d-2} \tag{15}$$



u s_1 s_2 s_3 ... 0 s_q t

- $s_1, ..., s_q$: coupled elements of $S_1, ..., S_4$

7. Proof of Thm 3: diagrams:

- If rows 1 and 2 (block 1) and rows 3 and 4 (block 2) are not connected, the lhs of (15) is dominated by a product of two convolutions:

$$\sum_{u<s'_{k'}<\ldots<s'_1<t} a^2_{t-s'_1}\ldots a^2_{s'_{k'}-u} \sum_{u<s''_{k''}<\ldots<s''_1<0} a^2_{-s''_1}\ldots a^2_{s''_{k''}-u} = (a^2)^{*k'}_{t-u}(a^2)^{*k''}_{-u}$$

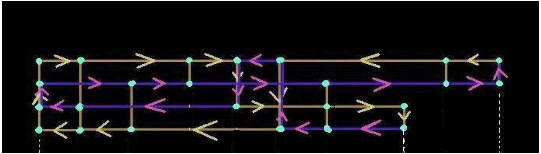
- In the above special case ("block-unconnected diagram') the bound (15) is easy
- the general case of "block-connected" diagram $\gamma \in \Gamma(k)_4$ can be reduced to the above special case by graph-theoretical argument

7. Proof of bound (15): idea: Eulerian cycle

- Idea: decoupling of diagrams. Recall the notation:
  $a^S_{u,t} := a_{t-s_1} a_{s_1-s_2} \cdots a_{s_{k-1}-s_k} a_{s_k-u}, \quad a^{(S)_4}_{u,(t)} \;:=\; a^{S_1}_{u,t} a^{S_2}_{u,t} a^{S_3}_{u,0} a^{S_4}_{u,0}.$
  The last product can be written as

$$a^{(S)_4}_{u,(t)} \equiv \prod_{all \ \text{edges}} a_e = \prod_{yellow \ \text{edges}} a_e \prod_{blue \ \text{edges}} a_e$$

$$\leq \frac{1}{2}\Bigg(\prod_{yellow \ \text{edges}} a_e^2 + \prod_{blue \ \text{edges}} a_e^2\Bigg)$$



$u \quad s_1 \quad s_2 \quad s_3 \quad \ldots \quad 0 \quad s_q \quad t$

8. A clt for martingale transform

Theorem (4)

Let

$$U_i = V_i \xi_i \qquad (i = 1, ..., n)$$

where:

(a) $(\xi_i) \in DAN(\alpha, \beta)$ *iid, zero mean,* $1 < \alpha < 2, \beta \in [-1, 1]$,

(b) (V_i) *predictable, stationary, ergodic, and* $E|V_0|^r < \infty \quad (\exists\ r > \alpha)$.

Then

$$n^{-1/\alpha} \sum_{i=1}^{[n\tau]} U_i \Rightarrow c\ Lévy_\tau\left(\alpha, \beta'\right),$$

where $c := (E|V_0|^\alpha)^{1/\alpha}, \quad \beta' := \beta E|V_0|^\alpha sign(V_0) / E|V_0|^\alpha$

- similar result as if (V_i) were iid (Breiman's lemma)
- condition (b): V_i's have lighter tails than ξ_i's
- the opposite case: Leipus et al. (2005, 2006): more difficult

8. A clt for martingale transform

- “easy” case: when (ξ_i) and (V_i) are mutually *independent*
- The Principle of Conditioning (Jakubowski (1986)): allows to replace dependent $(\xi_i)$and (V_i) by independent ones and having the same marginals
- recent developments of the conditioning approach in limit theorems: Peccati, Taqqu, Tudor, Nualart, ...

9. Open problem: what next?

Limit distribution of *sample autocovariances*:

$$\gamma_{n,X}(j) = \frac{1}{n}\sum_{i=1}^{n-j}(X_i - \overline{X_n})(X_{i+j} - \overline{X_n}),$$

$$\gamma_{n,X^2}(j) = \frac{1}{n}\sum_{i=1}^{n-j}(X_i^2 - \overline{X_n^2})\left(X_{i+j}^2 - \overline{X_n^2}\right), \quad j = 0, 1, \ldots, m$$

- Thm 3 solves the limit of $\gamma_{n,X}(0)$
- Sample autocovariances of linear processes with infinite variance: Davis and Resnick(1986)
- Sample autocovariances of short memory GARCH: Davis and Hsing(1995), Davis and Mikosch(1998), Davis and Resnick(1996)
- Appell polynomials of long memory MA: Vaičiulis(2003)

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