# Stein's method and weak convergence on Wiener space

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**Main subject**: two joint papers with **I. Nourdin** (Paris VI)

"Stein's method on Wiener chaos" (ArXiv, December 2007)

"Non-central convergence of multiple integrals" (ArXiv, September 2007)

**Framework: convergence in distribution** and explicit **Berry-Esseen type bounds** for sequences of functionals of **general Gaussian fields**.

**Principal aim of the talk:** to describe an ongoing **smooth transition** from the "**method of moments and cumulants**" (see Breuer, Major, Giraitis, Surgailis, Chambers, Slud... '70s – '80s) to "**Stein's method**" (see Stein 1972). This transition starts with some earlier papers by Nualart and Peccati (2005) and Peccati and Tudor (2005). Crucial step: connection with **Malliavin calculus** (Nualart and Ortiz-Latorre (2007), Nourdin and Peccati (2007)).

The transition is "smooth", since Stein's method allows to **recover bounds** in terms of **the same combinatorial expressions** (based, e.g., on "connected non-flat diagrams") upon which the method of cumulants is built.

## THE SETUP

- We consider a **real-valued**, **centered Gaussian field** $G = \{G(x) : x \in \mathfrak{X}\}$ ($\mathfrak{X}$ can be the real line, the sphere, a Hilbert space...).

- We denote by $L^2(G) = L^2(\sigma(G), \mathbf{P})$ the space of square-integrable functionals of $G$.

**Remark:** The most general situation is that of an **isonormal Gaussian process**. In this case, $\mathfrak{X}$ is a Hilbert space, and

$$\mathbf{E}[G(x)\,G(y)] = \langle x, y\rangle_{\mathfrak{X}} \quad (= \text{inner product on } \mathfrak{X}).$$

## WIENER CHAOS

For $q \geq 1$ we denote by $\mathcal{H}_q = \mathcal{H}_q(G)$ the $q$th Wiener chaos associated with $G$, that is, $\mathcal{H}_q$ is the $L^2$-closed space generated by r.v.'s of the type

$$\mathbb{H}_q\left(G(x_1), ..., G(x_m)\right), \quad (x_1, ..., x_m) \in \mathfrak{X}^m,$$

where $\mathbb{H}_q$ is a **generalized Hermite polynomial** of degree $q$, in $m$ variables.

- One has $L^2(G) = \bigoplus \mathcal{H}_q$.

- For every $q \geq 1$, one has that $Y \in \mathcal{H}_q$ if, and only if,

  $$Y = I_q(f),$$

  where $f$ is some (unique) symmetric kernel, and $I_q$ is a **multiple Wiener-Itô integral** of order $q$.

## PROBLEMS (One-dimensional Gaussian Approximations)

Let $N \sim \mathbf{N}(0,1)$ be a standard Gaussian random variable. Let $\{F_n : n \geq 1\} \in L^2(G)$ be a **centered** sequence such that $\mathbf{E}\left(F_n^2\right) \to 1$.

**Problem I:** Find **conditions** to have that, as $n \to \infty$,

$$F_n \xrightarrow{\mathbf{LAW}} N.$$

**Problem II:** Estimate explicitly the **distance** between the laws of $F_n$ and $N$. For instance, find conditions for the existence of some $\varphi(n) \searrow 0$ such that

$$\sup_z |\mathbf{P}[F_n \leq z] - \mathbf{P}[N \leq z]| \leq \varphi(n)$$

(i.e., find effective bounds on the **Kolmogorov distance** in CLTs).

**THE METHOD OF MOMENTS AND CUMULANTS** (Breuer and Major (1983), Giraitis and Surgailis (1985), Chambers and Slud (1989),...)

- Write $F_n$ in its "chaotic form", i.e. $F_n = \sum_{q \geq 1} I_q \left( f_q^n \right)$.

- For every $q$, prove that $\mathbf{E}[I_q \left( f_q^n \right)^2] \to \sigma_q^2 > 0$, and show by the method of moments that $I_q \left( f_q^n \right) \overset{\mathbf{LAW}}{\to} N \left( 0, \sigma_q^2 \right)$.

- Prove asymptotic independence of $I_q \left( f_q^n \right)$, $I_{q'} \left( f_{q'}^n \right)$, $q \neq q'$.

- Use $L^2$-approximation arguments to deduce that $F_n \overset{\mathbf{LAW}}{\to} N(0,1)$ (NB : $\sum_q \sigma_q^2 = 1$).

**Crucial part**: prove that $I_q\left(f_q^n\right) \stackrel{\mathbf{LAW}}{\rightarrow} N\left(0,\sigma_q^2\right)$ by showing that

$$\chi_k\left(I_q\left(f_q^n\right)\right) \rightarrow 0, \quad \text{for every } k \geq 3 \quad (\chi_k = k\text{th cumulant}).$$

The quantity $\chi_k\left(I_q\left(f_q^n\right)\right)$ is assessed by means of **diagram formulae**. Idea: (1) **Isomorphism** between $\mathcal{H}_q$ and a space of symmetric functions, (2) Use **multiplication formulae** and **Leonov-Shyryaev** representation of cumulants.

**No information** on upper bounds for the Kolmogorov distance.

**Recent uses of this or related methods:** D. Marinucci (2007), about convergence of the angular bispectrum for spherical Gaussian fields; Ginovyan and Sahakyan (2007), on quadratic functionals of stationary processes.

## THEOREM: NUALART AND PECCATI (AoP, 2005)

Let $F_n = I_q(f_n)$, $n \geq 1$, be a sequence of multiple integrals such that $\mathbf{E}\left(F_n^2\right) \to 1$. Then, the following are equivalent:

1. $F_n \stackrel{\mathbf{LAW}}{\longrightarrow} N \sim \mathbf{N}(0,1)$

2. $\mathbf{E}\left(F_n^4\right) \longrightarrow 3$

3. for every $r = 1, ..., q-1$, one has $\|f_n \otimes_r f_n\| \to 0$ ($f_n \otimes_r f_n$ is the $r$th contraction of the kernel $f_n$).

**Comments:**

- For instance if $q = 2$, $r = 1$ and $\mathfrak{X} = L^2([0,1])$

$$f \otimes_1 f(x,y) = \int_0^1 f(x,z) f(y,z) \, dz.$$

- Drastic simplification of the method of moments.

- The connection between moments and contractions comes from the formula

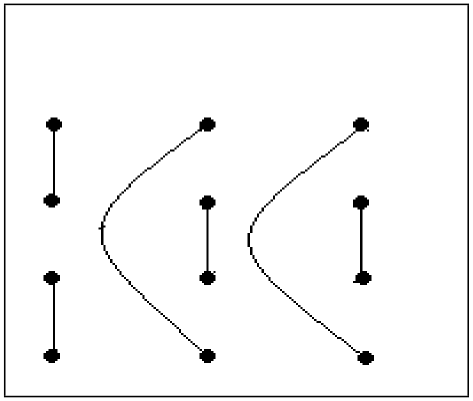
$$\chi_4(I_q(f)) = \sum_{r=1}^{q-1} \left\{ A_{q,r} \| f \otimes_r f \|^2 + B_{q,r} \left\| \widetilde{f \otimes_r f} \right\|^2 \right\},$$

$\sim$ = symmetrization; $A_{q,r}, B_{q,r}$ = universal combinatorial coefficients.

- One can prove

$$\sum_{r=1}^{q-1} A_{q,r} \, \|f \otimes_r f\|^2 = \sum \{\textit{circular diagrams with 4 levels}\}\,.$$

- A typical **circular diagram with 4 levels** (for $q = 3$) has the form



- Still **no information on bounds.**

**THEOREM: PECCATI AND TUDOR (Séminaire, 2005)**

If

$$I_q(f_n) \xrightarrow{\mathbf{LAW}} N\left(0, \sigma_1^2\right), \text{ and } \quad I_p(g_n) \xrightarrow{\mathbf{LAW}} N\left(0, \sigma_2^2\right),$$

and $\mathbf{E} I_q(f_n) I_p(g_n) \to R$, then

$$(I_q(f_n), I_p(g_n)) \xrightarrow{\mathbf{LAW}} \mathbf{N}_2\left(\mathbf{0}, \begin{bmatrix} \sigma_1^2 & R \\ R & \sigma_2^2 \end{bmatrix}\right).$$

**Comment:** automatic **asymptotic independence** of multiple integrals of different orders.

Several applications, e.g.: **Fractional linear differential equations** (Neuenkirch and Nourdin, 2007); **High-resolution limit theorems on homogeneous spaces** (Marinucci and Peccati, 2007ab); **Self-intersection local times of fractional Brownian motion** (Hu and Nualart, 2006); **Power variations of iterated Brownian motion** (Nourdin and Peccati, 2007c), **Estimation of selfsimilarity orders** (Tudor and Viens, 2007).....

Extension to **stable convergence**: Peccati and Taqqu, 2007; Nourdin and Nualart, 2008.

## ENTERS MALLIAVIN CALCULUS

$G = \{G(h) : h \in \mathfrak{X}\}$, $\mathfrak{X}$ is a Hilbert space.

Recall that the **derivative operator** $D$ is defined on smooth functionals $F = f(G(h_1), ..., G(h_m))$ as follows

$$Df(G(h_1), ..., G(h_m)) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} f(G(h_1), ..., G(h_m)) h_i \in L^2(\mathfrak{X}, \mathbf{P}).$$

Write $\mathbb{D}^{1,2}$ for the **domain** of $D$. Write $\delta$ for the **Skorohod integral** operator.

Recall the **integration by parts formula**: $\forall F \in \mathbb{D}^{1,2}$, $\forall u \in \mathbf{dom}(\delta)$

$$\mathbf{E}(\delta(u) F) = \mathbf{E}(\langle u, DF \rangle_{\mathfrak{X}}).$$

## THEOREM: NUALART AND ORTIZ-LATORRE (SPA, 2008)

Let $F_n = I_q(f_n)$, $n \geq 1$, be a sequence of multiple integrals such that $\mathbf{E}\left(F_n^2\right) \to 1$. Then, the following are equivalent:

1. $F_n \overset{\mathbf{LAW}}{\longrightarrow} N \sim \mathbf{N}(0,1)$
2. $\frac{1}{q}\|DF_n\|^2_{\mathfrak{X}} \longrightarrow 1$ in $L^2$.

**Comments:**

- The proof is based on the fact that the characteristic function $\lambda \mapsto \psi(\lambda) = \mathbf{E}e^{i\lambda N}$ verifies the equation

$$\lambda\psi(\lambda) + \psi'(\lambda) = 0.$$

- A crucial tool is the integration by parts formula, giving: for every $\lambda$

$$\mathbf{E}\left[I_q(f_n) \exp\{i\lambda I_q(f_n)\}\right] = i\lambda \mathbf{E}\left(\frac{1}{q}\|DI_q(f_n)\|_{\mathfrak{X}}^2 \exp\{i\lambda I_q(f_n)\}\right).$$

- One can prove that

$$\begin{aligned} \mathbf{E}\left(\frac{1}{q}\left\|DI_q\left(f_n\right)\right\|_{\mathfrak{X}}^2-1\right)^2 &\leq cst.\times\sum_{r=1}^{q-1}\left\|f_n\otimes_r f_n\right\|^2 \\ &\approx \sum\left\{\textit{circular diagrams with 4 levels}\right\} \\ &\approx \chi_4\left(I_q\left(f_n\right)\right). \end{aligned}$$

- **No bounds**, e.g. on the Kolmogorov distance between the laws of $I_q\left(f_n\right)$ and $N$.

## THEOREM: NOURDIN AND PECCATI (Preprint, 2007)

(Indeed, for every Gamma law)

Let $F_n = I_q(f_n)$, $n \geq 1$, be a sequence of multiple integrals such that $\mathbf{E}\left(F_n^2\right) \to 2$. Then, the following are equivalent:

1. $F_n \overset{\mathbf{LAW}}{\longrightarrow} N^2 - 1 \sim$ **Chi-squared (centered)**
2. $\frac{1}{q}\|DF_n\|^2_{\mathfrak{X}} - 2(1 + F_n) \longrightarrow 0$ in $L^2$.
3. $\mathbf{E}\left(F_n^4\right) - 12\mathbf{E}\left(F_n^3\right) \to -36 = \mathbf{E}\left(N^2 - 1\right)^4 - 12\mathbf{E}\left(N^2 - 1\right)^3$.

## Comments

- Another drastic simplification of the method of moments, but no information on bounds.

- Also: multi-dimensional results.

- The techniques are based on Malliavin calculus and on a differential operator characterizing the Fourier transform of the law of $N^2 - 1$.

- The use of characterizing differential operators is at the very heart of Stein's method.

**STEIN'S METHOD IN A NUTSHELL (Gaussian approximations in the Kolmogorov distance; Stein 1972, 1986)**

- Stein Lemma: a random variable $Z$ has a standard Gaussian $\mathbf{N}(0,1)$ distribution if, and only if, for every smooth $f$

$$\mathbf{E}\left[f'(Z) - Zf(Z)\right] = 0.$$

- Heuristically, for every random variable $X$, one expects that, if

$$\mathbf{E}\left[f'(X) - Xf(X)\right]$$

is **close to zero** for 'many' functions $f$, then $X$ has a distribution which is **close to Gaussian**.

- For every fixed $y \in \mathbb{R}$, select a solution $f_y$ to the Stein's equation ($N \sim \mathbf{N}(0,1)$)

$$\mathbf{1}(x \le y) - \mathbf{P}(N \le y) = f'(x) - xf(x), \quad x \in \mathbb{R},$$

which is bounded by 1 and such that $\left|f_y'\right| \le 1$.

- Deduce that, for every random variable $X$,

$$\sup_{y \in \mathbb{R}} |\mathbf{P}(X \le y) - \mathbf{P}(N \le y)| \le \sup_{f} \left|\mathbf{E}\left[f'(X) - Xf(X)\right]\right|,$$

where the $f$'s are bounded by 1, piecewise differentiable and such that $|f'| \le 1$.

## CRUCIAL IDEA (NOURDIN AND PECCATI, Preprint 2007)

For every centered $F \in \mathbb{D}^{1,2}$, one can estimate expressions such as

$$\mathbf{E}\left[f'(F) - Ff(F)\right],$$

by integrating by parts, yielding

$$\mathbf{E}\left[f'(F) - Ff(F)\right] = \mathbf{E}\left[f'(F) - \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{X}} \times f'(F)\right],$$

where $L^{-1}$ is the **inverse of the generator of the Ornstein-Uhlenbeck semigroup**. Therefore,

$$\left|\mathbf{E}\left[f'(F) - Ff(F)\right]\right| \leq \sqrt{\mathbf{E}\left[f'(F)^2\right]} \times \sqrt{\mathbf{E}\left[\left(1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{X}}\right)^2\right]}$$

The operator $L^{-1}$ acts on centered random variables of the type

$$F = \sum_{q \geq 1} I_q\left(f_q\right),$$

as follows

$$L^{-1}F = \sum_{q \geq 1} -\frac{I_q\left(f_q\right)}{q}.$$

## THE CASE OF MULTIPLE INTEGRALS

- When applied to $\boxed{F = I_q(f)}$, one has

$$\left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{X}} = \frac{1}{q} \|DF\|_{\mathfrak{X}}^2 ,$$

and therefore

$$\left| \mathbf{E}\left[ f'(F) - Ff(F) \right] \right| \leq \sqrt{\mathbf{E}\left[ f'(F)^2 \right]} \times \sqrt{\mathbf{E}\left[ \left( 1 - \frac{1}{q} \|DF\|_{\mathfrak{X}}^2 \right)^2 \right]}.$$

- Thanks to Stein's method one gets ($N$ standard Gaussian)

$$\sup_{y\in\mathbb{R}} \left|\mathbf{P}\left(I_q\left(f\right) \le y\right) - \mathbf{P}\left(N \le y\right)\right| \le \sqrt{\mathbf{E}\left[\left(1 - \frac{1}{q}\left\|DI_q\left(f\right)\right\|_{\mathfrak{X}}^2\right)^2\right]},$$

which gives explicit bounds and an alternate proof of Nualart and Ortiz-Latorre's crucial implication.

- One can explicitly represent $\mathbf{E}\left[\left(1 - \frac{1}{q}\left\|DI_q\left(f\right)\right\|_{\mathfrak{X}}^2\right)^2\right]$ by means of isometry and multiplication formulae.

- One can prove that

$$\mathbf{E}\left[\left(1-\frac{1}{q}\left\|DI_q\left(f\right)\right\|_{\mathfrak{X}}^2\right)^2\right] \leq \left\{1-\mathbf{E}\left(I_q\left(f\right)^2\right)\right\}^2 + \sum_{r=1}^{q-1} A_{r,q}\left\|f\otimes_r f\right\|^2.$$

This yields:

$$\begin{aligned}
&\sup_{y\in\mathbb{R}}\left|\mathbf{P}\left(I_q\left(f\right)\leq y\right)-\mathbf{P}\left(N\leq y\right)\right| \\
\leq &\sqrt{\left\{1-\mathbf{E}\left(I_q\left(f\right)^2\right)\right\}^2+\sum_{r=1}^{q-1} A_{r,q}\left\|f\otimes_r f\right\|^2} \\
\approx &\sqrt{\left\{1-\mathbf{E}\left(I_q\left(f\right)^2\right)\right\}^2+\chi_4\left(I_q\left(f\right)\right)}.
\end{aligned}$$

- In the Gamma case, one uses the fact that $Z \overset{\mathbf{LAW}}{=} N^2 - 1$ if, and only if,

$$\mathbf{E}\left[Zf(Z) - 2(1+Z)f'(Z)\right] = 0.$$

  One therefore deduces bounds for Gamma approximations, but with more **ad hoc distances**. This is due to the irregularity of the solutions to the Stein equation in the Gamma case.

- The previous result implies that Kolmogorov distances on Wiener space **can be estimated by the method of moments** (even better than that: they basically depend on the **first two even moments** of multiple integrals).

- Same results for: **Total variation distance**, **Wasserstein distance**, **bounded Wasserstein distance**. Consequence: all these distances metrize the convergence to Gaussian on a fixed Wiener chaos.

- One can use these computations to study the approximations of r.v.'s with a possibly infinite chaotic decomposition.

**Application: Berry-Esseen bounds in the Breuer-Major-Giraitis-Surgailis CLT.**

Recall the classic **Berry-Esseen theorem**: *Let* $\{X_i : i \geq 1\}$ *be a sequence of i.i.d. random variables such that* $\mathbf{E}X_i = 0$, $\mathbf{E}|X_i|^3 = \rho < \infty$, $\mathbf{E}X_i^2 = 1$. *Define* $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$. *Then,*

$$\sup_{z \in \mathbb{R}} \left| \mathbf{P}(S_n \leq z) - \mathbf{P}(N \leq z) \right| \leq \frac{3\rho}{\sqrt{n}},$$

*where* $N \sim N(0,1)$.

Let $\left\{B_t^H : t \geq 0\right\}$ be a fractional Brownian motion of order $H \in (0,1)$. In particular,

$$\mathbf{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

Recall the classic result (Breuer, Major, Giraitis, Surgailis): *For $q \geq 2$, let $\mathbf{H}_q$ be the $q$th Hermite polynomial. Then, for every $H < \frac{2q-1}{2q}$ there exists an explicit constant $\sigma_H > 0$ such that*

$$S_n^H = \frac{1}{\sigma_H \sqrt{n}} \sum_{i=1}^{n} \mathbf{H}_q\left(B_{i+1}^H - B_i^H\right) \stackrel{\mathbf{LAW}}{\longrightarrow} N(0,1).$$

*Extension to functions with arbitrary* **Hermite rank**.

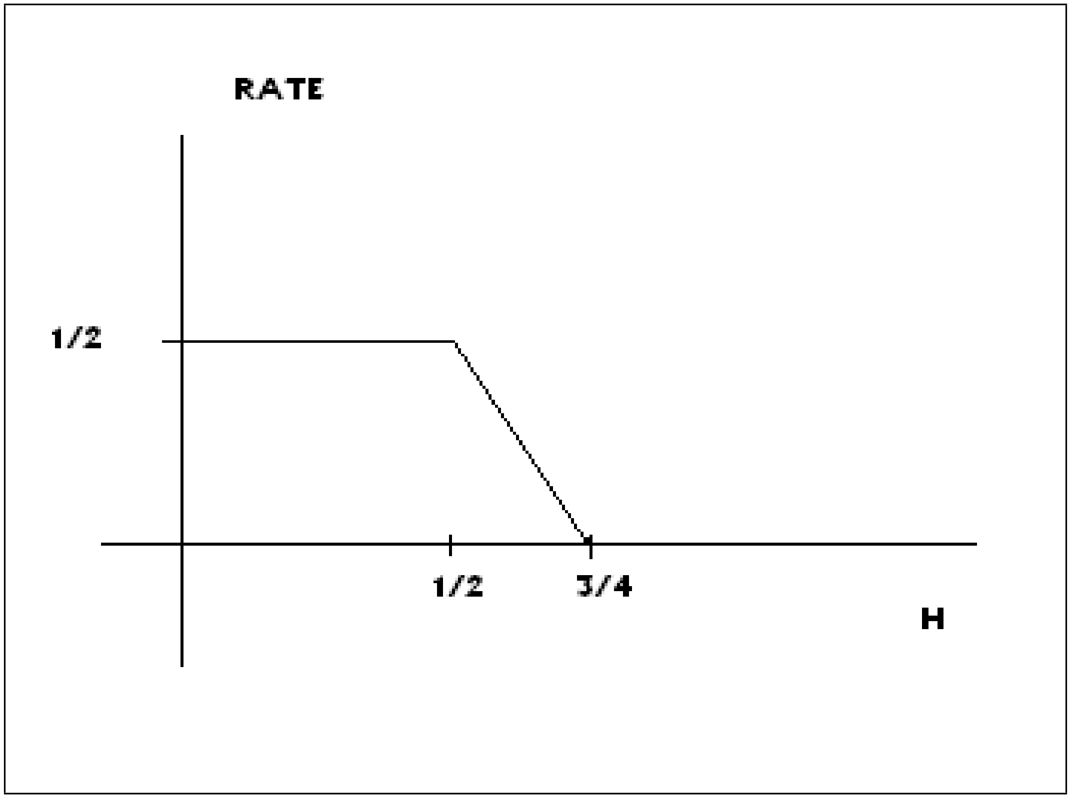
**THEOREM (NOURDIN AND PECCATI, Preprint 2007)**

By embedding $B^H$ into an isonormal process (see e.g. Pipiras and Taqqu (2003)) and by applying the previous theory, one obtains that

$$\sup_{z\in\mathbb{R}}\left|\mathbf{P}\left(S_n^H \leq z\right) - \mathbf{P}\left(N \leq z\right)\right| \leq c_H \times \begin{cases} n^{-1/2}, & \text{if } H \leq 1/2 \\ n^{H-1}, & \text{if } H \in (\frac{1}{2}, \frac{2q-3}{2q-2}] \\ n^{qH-q+\frac{1}{2}}, & \text{if } H \in (\frac{2q-3}{2q-2}, \frac{2q-1}{2q}) \end{cases}$$

**NB:** for $H = 1/2$ (Brownian motion), the speed is always $n^{-1/2}$ ( $=$ Berry-Esseen).

For instance, for $q = 2$,



Although there exist mixing-type characterizations of fBm (Picard, 2007), the **use of mixing techniques to obtain our bounds seems mostly unfeasible**. This is due to the fact that, for values of $H$ and $q$ outside the range of our theorem, a non-CLT holds (e.g., towards Rosenblatt laws).

**Other applications**: generalizations of results by Chatterjee (2007), connected to fluctuations of eigenvalues of random matrices and Poincaré inequalities.

**Related recent works**:

Privault and Reveillac, 2007;

Hsu, 2003;

Decreusefond and Savy, 2007 (Poisson)