On square-integrability of an AR process with Markov switching

J. YAO* SAMOS - MATISSE, CNRS UMR 8595

Abstract

For an autoregressive process with Markov switching, we give a condition ensuring the existence of a square-integrable stationary solution. Unlike conditions based on top Lyapounov exponents, our condition is directly expressed in terms of the parameters of the model. Specific examples are also provided to give more details on this condition.

Keywords : Markov switching ; AR process ; Stationary solution.

AMS classification code: 60J20, 62M10.

1 Introduction

Let $\mathbf{X} := (X_n)_{n\in\mathbb{Z}}$ be a stationary positive recurrent Markov chain on a finite set $E = \{1, \ldots, m\}$, with transition probability matrix P and invariant probability measure (hereafter i.p.m.) μ. We consider a d-dimensional multivariate *AR process with Markov switching* $\mathbf{Y} = (Y_n)$ (abbreviated as ARMS) defined, for $n \in \mathbb{Z}$, by

$$Y_n = A_{X_n} Y_{n-1} + \varepsilon_n \,, \qquad Y_n \in \mathbb{R}^d \,. \tag{1}$$

Here the noise process $\varepsilon := (\varepsilon_n)_{n\geq 0}$ is a $\mathbb{R}^d$-valued stationary sequence of random variables and $(A_k), k = 1, \ldots, m$ a family of $d \times d$ autoregressive matrices.

Therefore, the process can switch between m different AR processes (*regimes*), the switching being controlled by a Markov chain. The use of the Markov switching offers

*SAMOS, Université Paris 1, 90 rue de Tolbiac, F-75634 Paris Cedex 13, France (yao@univ-paris1.fr)

new possibilities for modeling time series "*subject to discrete shifts in regime - episodes across which the dynamic behaviour of the series is markedly different*", as noted by (Hamilton, 1989) who first introduced such a model to analyse the US annual GNP (gross national product) series. These models have since then attracted a considerable interest in the statistical community, especially for econometric series modeling, see (Hamilton, 1990), (Hamilton, 1996), (Holst et al., 1994), (McCulloch and Tsay, 1994) and (Francq and Roussignol, 1996).

Criterion for the existence of a stationary solution for the equation (1) is already known. Let us fix a vector norm $||x||$ on $\mathbb{R}^d$ and let $||A||$ be the underlying operator norm for a $d \times d$ real matrix A. Assume that $\mathbb{E}\log^+ ||\varepsilon_0|| < \infty$. Then following (Brandt, 1986), the equation (1) has a stationary and ergodic solution if the following top Lyapounov exponent γ of the sequence of random matrices (A_{X_n}) is negative, i.e.

$$\gamma = \inf_{p\geq 1} \frac{1}{p}\, \mathbb{E}\log ||A_{X_p} A_{X_{p-1}} \cdots A_{X_1}|| < 0. \tag{2}$$

This condition is also necessary if in addition, the noise ε is an i.i.d. sequence (Bougerol and Picard, 1992).

This deep criterion has however two drawbacks. First, it does not rely directly on the model parameters, namely the transition matrix P, the noise process ε and the autoregressive matrices (A_k). It is usually impossible to know if a given model has a negative Lyapounov exponent without simulations. This fact heavily limits the interests of the criterion in statistical applications. The criterion has another serious handicap in applications, for we often need to know more about the stationary solution. For instance, the solution need to have some moments to make an estimation theory possible and the condition (2) do not guarantee such moments. Therefore, we have to search for conditions ensuring moments for the stationary solution. But usually under such a moment condition, the Lyapounov exponent γ will be automatically negative.

In this note, we give a condition ensuring the existence of a square-integrable stationary and ergodic solution for the equation (1). The main result is stated in Section 2, followed by several applications to specific examples. The proof is given in Section 3.

2 Main theorem

The spectral radius of a real matrix $A = (a_{ij})$ is denoted by $\rho(A)$ and its transpose by A^{T}. If B is another matrix, we denote by $A \otimes B$ their Kronecker tensor product, that is the matrix $(a_{ij}B)$ in the bloc form. Let $P = (p_{ij})$ be the transition probability matrix of the Markov chain X, with $p_{ij} = \mathbb{P}(X_{t+1} = j \,|\, X_t = i)$. The following matrix will play a central role throughout the paper:

$$M = \begin{pmatrix} p_{11}\left(A_1^{\mathrm{T}} \otimes A_1^{\mathrm{T}}\right) & \cdots & p_{m1}\left(A_m^{\mathrm{T}} \otimes A_m^{\mathrm{T}}\right) \\ \vdots & p_{ji}\left(A_j^{\mathrm{T}} \otimes A_j^{\mathrm{T}}\right) & \vdots \\ p_{1m}\left(A_1^{\mathrm{T}} \otimes A_1^{\mathrm{T}}\right) & \cdots & p_{mm}\left(A_m^{\mathrm{T}} \otimes A_m^{\mathrm{T}}\right) \end{pmatrix} . \tag{3}$$

Our main result is the following

Theorem 1 *Assume that the noise process* $\varepsilon := (\varepsilon_n)_{n\geq 0}$ *is stationary and ergodic such that for any* $n \in \mathbb{Z}$, ε_n *is independent of the past* $\{X_m \,,\, m < n\}$ *of the Markov chain* X. *Assume also* $\mathbb{E}||\varepsilon_1||^2 < \infty$. *Then, the equation (1) has an unique square-integrable, stationary and ergodic solution if*

$$\rho(M) < 1 \ . \tag{4}$$

The proof is given in Section 3. It is worth noting that since we are seeking for a square-integrable solution, the assumption $\mathbb{E}||\varepsilon_1||^2 < \infty$ is necessary. Note that the condition (4) has been conjectured in (Holst et al., 1994). We now consider some specific examples to detail this result.

Example 1 - Case without switching:

This is just for illustration purpose. If there is no switching, i.e. $m = 1$,we have $M = A_1$: the condition (4) is the classical stability condition for a multivariate AR process.

Example 2 - The univariate case :

Let us consider an univariate ARMS model (i.e. $d = 1$), the matrix M becomes

$$M = \begin{pmatrix} p_{11}a_1^2 & \cdots & p_{m1}a_m^2 \\ \vdots & p_{ji}a_j^2 & \vdots \\ p_{1m}a_1^2 & \cdots & p_{mm}a_m^2 \end{pmatrix} ,$$

where we have written $a_k = A_k$. Assume that in each regime k, the underlying AR process is stable, that is $|a_k| < 1$. Since all $p_{ij} \in [0,1]$, we have $\rho(M) < 1$ and the resulting ARMS model is also stationary and square integrable. However, this simultaneous stability of all the m regimes is not necessary. To be specific, let us fix more by taking $m = 2$ (switching between 2 univariate AR processes) and denote the transition probability matrix by

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} ,$$

for some $p \in (0,1)$. Then

$$M = \begin{pmatrix} (1-p)\, a_1^2 & p\, a_2^2 \\ p\, a_1^2 & (1-p)\, a_2^2 \end{pmatrix} .$$

By straightforward computations, we know that $\rho(M) < 1$ is equivalent to the following two conditions :

$$\begin{cases} (2p-1)\, a_1^2 a_2^2 + (1-p)(a_1^2 + a_2^2) & < \ 1 , \\ (1-p)(a_1^2 + a_2^2) & \leq \ 2 . \end{cases} \tag{5}$$

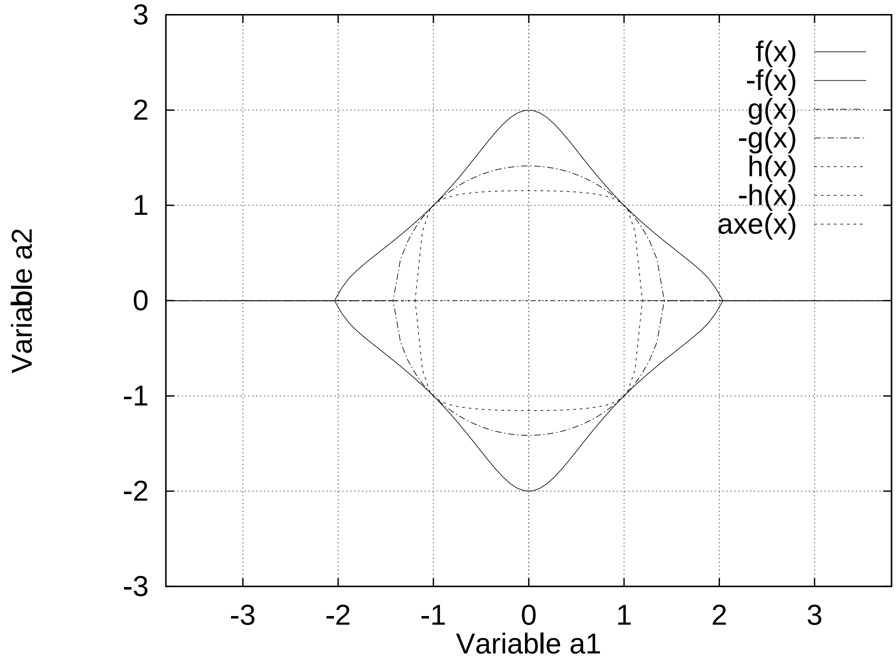

Figure 1: Plots of the boundary curves $\rho(M) = 1$ in the case of switching between two univariate AR processes. The switching probability p is respectively $\frac{3}{4}$ (function $\pm f(x)$, solid line), $\frac{2}{4}$ (function $\pm g(x)$, broken line) and $\frac{1}{4}$ (function $\pm h(x)$, dotted line) . The square-integrability regions are the domains enclosed within these boundary curves.

Figure 1 depicts these conditions in three situations $p = \frac{3}{4}$, $p = \frac{2}{4}$ and $p = \frac{1}{4}$, respectively. All the three square-integrability regions include the unit square $(-1, +1)^2$ that corresponds to the simultaneous stability of both regimes. However, one of the AR regimes can be heavily explosive,(e.g. $|a_1| \gg 1$), provided that the other one is sufficiently stable (e.g. $|a_2| \ll 1$). The difference between these two regimes can be as big as the switching activity is high, i.e. with p close to 1 (see the case $p = \frac{3}{4}$). The resulting ARMS model remains well-defined and square integrable.

Example 3 - Generalization to switching AR(p) processes

Consider a Markov switching between m different univariate AR(p) models. That is in each regime $k = 1, \dots m$, there is an associated AR(p) model

$$Y_n = a_k(1)Y_{n-1} + \cdots + a_k(p)Y_{n-p} + \varepsilon_n \ , \quad n \in \mathbb{Z} \ .$$

And the considered switching model is

$$Y_n = a_{X_n}(1)Y_{n-1} + \cdots + a_{X_n}(p)Y_{n-p} + \varepsilon_n \ , \quad n \in \mathbb{Z} \ .$$

By setting $Z_n = (Y_n, \dots, Y_{n-p+1})^{\mathrm{T}}$, this model can be rewritten in the form of (1) with the companion matrix

$$A_{X_n} = \begin{pmatrix} a_{X_n}(1) & a_{X_n}(2) & \cdots & a_{X_n}(p) \\ 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ .$$

Theorem 1 applies as well.

3 Proof of Theorem 1

In this section, C_j will be a generic notation for positive constants. To prove the main theorem, we need to consider the following product of random matrices, defined for $k > 0$,

$$\Pi_{n,k} = A_{X_n} A_{X_{n-1}} \cdots A_{X_{n-k+1}} \quad .$$

The main idea of the proof is to establish that under the condition (4), the square norm $\mathbb{E}||\Pi_{n,k}||^2$ of the above product vanishes exponentially fast (Proposition 1). Two important consequences then follow. First, the top Lyapounov exponent γ of the sequence (A_{X_n}) is negative. Hence the equation (1) has an unique stationary and ergodic solution. Moreover, this solution can be expressed as

$$Y_n = \sum_{k=1}^{\infty} \left(A_{X_n} A_{X_{n-1}} \cdots A_{X_{n-k+1}}\right) \varepsilon_{n-k} = \sum_{k=1}^{\infty} \Pi_{n,k} \varepsilon_{n-k} \quad , \tag{6}$$

where the series converges almost surely.

As a second important consequence of the exponential decay of $\mathbb{E}\,||\Pi_{n,k}||^2$, we will prove that the above series also converges in the space L^2 of square-integrable variables. Hence this solution (Y_n) is also square-integrable.

Proposition 1 **(Exponential decay of $\mathbb{E}\,||\Pi_{n,k}||^2$)** *Under the condition (4), there is a positive constant* $r \in [0, 1)$ *such that for each* n*,*

$$\mathbb{E}\,||\Pi_{n,k}||^2 \leq C_1\, r^k \ , \quad k > 0\,. \tag{7}$$

Proof. Since for fixed k, the sequence $(\Pi_{n,k})_n$ is stationary with respect to n, it will be enough to prove that

$$\mathbb{E}\,||\Pi_{k,k}||^2 = \mathbb{E}\,||A_{X_k} A_{X_{k-1}} \cdots A_{X_1}||^2 \leq C_1 r^k \ , \quad k > 0. \tag{8}$$

To this end, let us consider the following recursive equation

$$Z_{k+1} = A_{X_k} Z_k \ , \ k > 0\,,$$

with an arbitrary starting (non random) point $Z_0 \in \mathbb{R}^d$. Let us set for each regime $i \in \{1, \ldots, m\}$, the $d \times d$ matrix

$$F_k(i) = \mathbb{E}\,\left[Z_k Z_k^{\mathrm{T}}\, \mathbb{1}_{\{X_k = i\}}\right] \ ,$$

where $\mathbb{1}_B$ denotes the indicator function of a set B. We have by definition,

$$F_{k+1}(i) = \sum_{j=1}^{m} \mathbb{E}\,\left[A_{X_k} Z_k Z_k^{\mathrm{T}} A_{X_k}^{\mathrm{T}}\, \mathbb{1}_{\{X_{k+1}=i,\, X_k=j\}}\ .\right]$$

Therefore,

$$F_{k+1}(i) = \sum_{j=1}^{m} A_j F_k(j) A_j^{\mathrm{T}} P(j,i) \ . \tag{9}$$

Let us consider the (non-random) vector

$$\mathbf{F}_k = (F_k(1),\, F_k(2),\, \dots,\, F_k(m)) \ ,$$

of (high) dimension md^2. Then the recursive equation (9) defines a map from $\mathbf{F}_k$ to $\mathbf{F}_{k+1}$. This map, say T, is linear and non random. Note that $\mathbb{E}||A_{X_k}A_{X_{k-1}}\cdots A_{X_1}||^2$ vanishes exponentially fast if and only if $||\mathbf{F}_k||$ does for any initial vector Z_0. This is again equivalent to that the spectral radius of the map T is smaller than 1.

To evaluate this spectral radius, we now write down the associated matrix of the map T in a (canonical) basis. This basis is constructed as follows. First we take the canonical basis of the space of $d \times d$ matrices $\mathcal{B}_1$ built with the following d^2 elementary matrices (Γ_{ij}), $1 \le i,j \le d$ where all elements of the matrix Γ_{ij} are null except that its (i,j)-th element is 1. In other words,

$$\Gamma_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{a single non null value 1 at position } (i,j) \quad .$$

Let us define for $\ell = 1, \dots, m$, the md^2-dimensional vector

$$E_{\ell ij} = (\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \Gamma_{ij}, \mathbf{0}, \dots, \mathbf{0}) \ ,$$

where $\mathbf{0}$ is the $d \times d$ null matrix and Γ_{ij} being at position ℓ. The collection of these md^2 vectors

$$\mathcal{B} = \{E_{111}, \dots,\, E_{11d},\, E_{121}, \dots,\, E_{12d}, \dots,\, E_{1dd},\, E_{211}, \dots,\, E_{2dd}, \dots,\, E_{mdd}\}$$

forms a basis of the space of $\mathbf{F}_k$-vectors. It then can be checked that the associated matrix of the linear map T in the basis $\mathcal{B}$ is exactly the matrix M defined in (3). Therefore, the condition we are looking for is $\rho(M) < 1$. ■

End of the proof of Theorem 1:

Now we prove that the top Lyapounov exponent γ is negative. Since for fixed k, the sequence $(\Pi_{n,k})_n$ is stationary in n, we have by (7)

$$\begin{aligned}\mathbb{E}\log ||A_{X_p}A_{X_{p-1}}\cdots A_{X_1}|| &= \mathbb{E}\log ||\Pi_{p,p}|| = \mathbb{E}\log ||\Pi_{0,p}|| \\ &= \frac{1}{2}\,\mathbb{E}\log ||\Pi_{0,p}||^2 \leq \frac{1}{2}\log\ \mathbb{E}||\Pi_{0,p}||^2 \\ &\leq C_2 + \frac{1}{2}p\log r \quad .\end{aligned}$$

Hence,

$$\gamma = \inf_{p\geq 1}\frac{1}{p}\,\mathbb{E}\log ||A_{X_p}A_{X_{p-1}}\cdots A_{X_1}|| \leq \frac{1}{2}\log r < 0 \quad .$$

By Theorem 1 of (Brandt, 1986), see also Theorem 2.5 of (Bougerol and Picard, 1992), the series in (6) converges a.s. for each n which gives the unique stationary and ergodic solution for the ARMS model (1).

Next, we prove that this series also converges absolutely in the space of square-integrable random variables, i.e. we have for each n,

$$\sum_{k=1}^{\infty}\left[\,\mathbb{E}||\Pi_{n,k}\varepsilon_{n-k}||^2\right]^{\frac{1}{2}} < \infty \quad . \tag{10}$$

As ε_{n-k} is independent of $\Pi_{n,k}$, this is an easy consequence of (7), since we have

$$\begin{aligned}\mathbb{E}||\Pi_{n,k}\varepsilon_{n-k}||^2 &\leq \mathbb{E}\left(||\Pi_{n,k}||^2||\varepsilon_{n-k}||^2\right) \\ &= \mathbb{E}||\Pi_{n,k}||^2\ \mathbb{E}||\varepsilon_{n-k}||^2 \\ &\leq C_3\,r^k\,,\end{aligned}$$

where ε_{n-k} is independent of $\Pi_{n,k}$. This ends the proof of Theorem 1.

ACKNOWLEDGMENT The author is grateful to Philippe BOUGEROL for helpful comments on this work.

References

Bougerol, P. and Picard, N. (1992). Strict stationarity of generalized autoregressive processes. *Ann. Proba.*, 20:1714–1730.

Brandt, A. (1986). The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients. *Adv. Appl. Prob.*, 18:211–220.

Francq, C. and Roussignol, M. (1996). Consistency of MLE for AR models with Markov switching. *Prépublication No. 10/96*. Université de Marne-La-Vallée, to appear in *Statistics*.

Hamilton, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57(2):357–384.

Hamilton, J. D. (1990). Analysis of time series subject to changes in regime. *J. Economet.*, 45:39–70.

Hamilton, J. D. (1996). Specification testing in Markov-switching time-series models. *J. Economet.*, 70:127–157.

Holst, U., Lingren, G., Holst, J., and Thuvesholmen, M. (1994). Recursive estimation in switching autoregressions with a Markov regime. *J. Times Series Anal.*, 15(5):489–506.

McCulloch, R. E. and Tsay, R. S. (1994). Statistical analysis of econometric time series via Markov switching models. *J. Time Series Anal.*, 15(5):523–539.