Chaos expansions and level crossings

by

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Abstract

In this paper we show that the number of crossings $N_t(x)$ at any level x by a smooth stationary Gaussian process X on the interval $[0,t]$ can be defined as $N_t(x) = \int_0^t \delta_x(X_s) \mid \dot{X}_s \mid ds$ in the Sobolev space $\mathbb{D}^{2,\alpha}$ for any $\alpha < 1/4$.

1 Introduction

Studies on level-crossings by stationary Gaussian processes have begun about fifty years ago. Different approachs have been proposed, in particular Slud (1991) introduced the powerful tool of MWI's to study counts of crossings. More recently Kratz and León (1997) used a direct method to give a MWI expansion of the number of crossings by a smooth stationary Gaussian process. The idea in their paper was to define an analytical formula involving the Dirac function:

$$N_t(x) = \int_0^t \delta_x(X_s) \mid \dot{X}_s \mid ds \tag{1.1}$$

to approximate the number of crossings. This formula is of the same type as the one involving the Dirac function for the definition of local times. So the purpose of this paper is to 'validate mathematically' this formula for crossings-counts by choosing the right space, as it has been done for Brownian local time (see Nualart & Vives (1992), Imkeller et al. (1995)). An appropriate framework is the Sobolev spaces $\mathbb{D}^{2,\alpha}$ over canonical Wiener space obtained via completion of the set of polynomials F with respect to the norms $||F||_{2,\alpha} = ||(I-L)^{\alpha/2}F||_2$, $\alpha \in \mathbb{R}$.
In this paper, we show that, whereas the distribution $\delta_x(X_s)$ and the random

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variable $\mid \dot{X}_s \mid$ belong to $\mathbb{D}^{2,\alpha}$ for any $\alpha < -1/2$ and for any $\alpha \in \mathbb{R}$ respectively, the integral in the definition of $N_t(x)$ has a smoothing effect, and $N_t(x)$ is a random variable which belongs to $\mathbb{D}^{2,\alpha}$ for any $\alpha < 1/4$. For that, we use the Malliavin calculus, quite powerful tool when dealing with such non-smooth functionals.

2 Preliminaries and notation

Let X be a real stationary Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$, with mean zero, variance one, and let W be the standard Brownian Motion (or Wiener process).
Let H_n denotes the n-th Hermite polynomial defined as

$$e^{tx-\frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \qquad \text{or} \qquad H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}).$$

Let $H(X)$ denotes the space of real square integrable functionals of the process X.
Recall that $H(X) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n$ being the Wiener Chaos, i.e. the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(W(h)), h \in L^2(\mathbb{R}, dx), ||h|| = 1\}$.
We can make use as well of the multiple Wiener-Itô integral I_n defined as in Major (1981), since we have

$$H_n(W(h)) = I_n(h^{\otimes n}) \qquad \forall h \in L^2(\mathbb{R}, dx),\ s.t. ||h|| = 1.$$

We can introduce the Sobolev spaces $\mathbb{D}^{2,\alpha}$ for $\alpha \in \mathbb{R}$ as in Watanabe (1984). A functional $f \in H(X)$ with the development

$$f = E[f] + \sum_{n=1}^{\infty} I_n(f_n)$$

belongs to $\mathbb{D}^{2,\alpha}$ if and only if

$$||f||_{2,\alpha}^2 = \sum_{n=1}^{\infty} (1+n)^{\alpha} ||f_n||_2^2 = ||(I-L)^{\alpha/2} f||_2^2 < \infty,$$

where L is the operator defined on $H(X)$ by $Lf = \sum_{n=0}^{\infty} -nI_n(f_n)$.
L coincides with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T_u, u \geq 0\}$ of contraction operators on $H(X)$ defined by $T_u(f) = \sum_{n=1}^{\infty} e^{-nu} I_n(f_n)$, for any $f = \sum_{n=0}^{\infty} I_n(f_n)$.
In particular, we have, for $\beta > 0$,

$$(I-L)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} e^{-u} u^{\beta-1} T_u du. \tag{2.2}$$

(cf. Watanabe (1984, p.24), or Nualart (1995, §1.4), Nualart & Vives (1992b))

3 Chaos expansions

Let $X = (X_t; t \geq 0)$ be a real stationary Gaussian process with mean zero, variance one. Assume that its covariance function r is twice differentiable and that $-r''(0) = 1$. Note that this last condition is made with no loss of generality since X_t can be replaced by $X_{t/\sqrt{-r''(0)}}$.
$r(t)$ has the well-known representation

$$r(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dF(\lambda)\ ,\ -\infty < t < \infty,$$

where F is the spectral distribution.
Suppose that F is absolutely continuous, then $r(t)$ has also the representation

$$r(t) = \int_{-\infty}^{\infty} b(t+s)b(s)ds$$

for some $b \in L^2$, which can be chosen as the Fourier transform in L^2 of $(F'(\lambda))^{1/2}$, F' being the Radon-Nikodym derivative of F with respect to $d\lambda$ (see Berman, p.149),
and X_t has the stochastic integral representation

$$X_t = \int_{-\infty}^{\infty} b(t+s)W(ds)$$

where $W(s)$ is the standard Brownian Motion (see Berman, p.157).
We have

$$X_t = W(h_t)\ ,\ \text{with } h_t(\cdot) = b(t+\cdot) \in L^2(\mathbf{R}).$$

Note that $\forall t$, $||h_t||_2 = 1$ (since we suppose $r(0) = 1$).

Let δ be the Dirac's δ-distribution and p_ε be the centered Gaussian density with variance ε.
First we determine the order of Sobolev spaces in which $\delta_x(X_s)$, ($x \in \mathbf{R}$, $s \geq 0$), takes place, after having adapted the formula of Imkeller et al. (1995) to get a representation of this distribution by a series of multiple Wiener-Itô integrals.

Under the conditions above, we have

Proposition 3.1 *$\delta_x(X_s) = \lim_{\epsilon\to 0} p_\varepsilon(X_s - x)$ exists in $\mathbb{D}^{2,\alpha}$ for any $\alpha < -1/2$ and is given by the series*

$$\begin{aligned} \delta_x(X_s) &= \sum_{n=0}^{\infty} c_n(x) H_n(X_s) = \sum_{n=0}^{\infty} c_n(x) H_n(W(h_s)) \\ &= \sum_{n=0}^{\infty} c_n(x) I_n(h_s^{\otimes n}), \end{aligned}$$

$$\textit{with } c_n(x) = \frac{1}{n!} H_n(x) p_1(x), \forall n \geq 0. \tag{3.3}$$

Proof. We only need to compute $\| \delta_x(X_s) \|^2_{2,\alpha}$, since the representation of $\delta_x(X_s)$ has already been proved (see for instance Kratz & León (1997)).
We have

$$\| \delta_x(X_s) \|^2_{2,\alpha} = \sum_{n=0}^{\infty} (1+n)^{\alpha} \frac{1}{(n!)^2} H_n^2(x)(p_1(x))^2 \| H_n(X_s) \|^2 .$$

But Mehler formula and $r(0)=1$ provide $\| H_n(X_s) \|^2 = E[(H_n(X_s))^2] = n!$,
so

$$\| \delta_x(X_s) \|^2_{2,\alpha} = \sum_{n=0}^{\infty} \frac{(1+n)^{\alpha}}{n!} (H_n(x)p_1(x))^2.$$

We apply Proposition 3 in Imkeller et al. (1995) (with $\alpha = 1/2$) to conclude to the convergence of the series whenever $\alpha < -1/2$. □
Note that we get back the Theorem 2.1 in Nualart & Vives (1992b).

Suppose further that the process X_s is differentiable and let $\dot{X}_s$ denote its derivative. We have

$$\dot{X}_s = W(\dot{h}_s), \text{ with } \dot{h}_s(t) = \frac{d}{ds} b(s+t) \in L^2(\mathbb{R}).$$

Note that $\forall t$, $||\dot{h}_t||_2 = 1$ (since we suppose $-r''(0) = 1$).

The Hermite expansion of the function $|.|$ being well known (cf. for instance Kratz & León (1997)), we can easily prove that

Proposition 3.2 *$| \dot{X}_s |$ exists in $\mathbb{D}^{2,\alpha}$ for any α and is given by the series*

$$\begin{aligned} | \dot{X}_s | &= \sum_{n=0}^{\infty} a_{2n} H_{2n}(\dot{X}_s) = \sum_{n=0}^{\infty} a_{2n} H_{2n}(W(\dot{h}_s)) \\ &= \sum_{n=0}^{\infty} a_{2n} I_{2n}(\dot{h}_s^{\otimes 2n}), \end{aligned}$$

$$\textit{with } a_{2n} = \sqrt{\frac{2}{\pi}} \frac{(-1)^{n+1}}{2^n n!(2n-1)}, \forall n \geq 0. \tag{3.4}$$

We are now able to enunciate the main result:

Theorem 3.1 *Let $X = (X_t, t \geq 0)$ be a real differentiable stationary Gaussian process with mean zero, variance one, and correlation function r twice differentiable. Assume that $-r''(0) = 1$ and that r satisfies the Geman condition*

$$\frac{r''(t) - r''(0)}{t} = L(t) \in L^1([0, \delta], dx), \text{ for some } \delta > 0.$$

Suppose moreover that the spectral distribution F is absolutely continuous. So $r(t)$ has the representation $r(t) = \int_{-\infty}^{\infty} b(t+s)b(s)ds$, where $b \in L^2(\mathbb{R})$ is chosen as the Fourier transform in $L^2(\mathbb{R})$ of $(F'(\lambda))^{1/2}$.
Let $N_t(x)$ be the number of crossings at any level x by the process on the interval $[0, t]$. Then

$$N_t(x) = \int_0^t \delta_x(X_s) \mid \dot{X}_s \mid ds \quad \text{exists in } \mathbb{D}^{2,\alpha} \text{ for any } \alpha < 1/4$$

and is given by the series

$$\begin{aligned} N_t(x) &= \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} a_{2m} c_{n-2m}(x) \int_0^t H_{n-2m}(X_s) H_{2m}(\dot{X}_s) ds \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} a_{2m} c_{n-2m}(x) \int_0^t I_{n-2m}(h_s^{\otimes n-2m}) I_{2m}(\dot{h}_s^{\otimes 2m}) ds, \end{aligned}$$

where $h_s(.) = b(s+.)$, a_{2m} and $c_{n-2m}(x)$ are given respectively in (3.4) and (3.3).

Remarks:

i) Recall that the Geman condition is a necesary and sufficient condition to have $E[(N_t(0))^2] < \infty$ for each $t < \infty$. In the paper of Kratz & León (1997), the authors gave the condition $r^{(iv)}(0) < \infty$ (which implies the Geman condition) to prove formally that $N_t(x) = \int_0^t \delta_x(X_s) \mid \dot{X}_s \mid ds$. To prove it mathematically, the Geman condition will be enough.

ii) As direct applications, this theorem allows to prove easily some classical formulas (as noticed formally in Kratz & León (1997)), such that
the Rice formula:

$$E[N_t(x)] = \int_0^t E[\delta_x(X_s)]E \mid \dot{X}_s \mid ds = \frac{t}{\sqrt{2\pi}} e^{-x^2/2} \sqrt{-r''(0)} \ ;$$

the Kac formula:
since $\hat{\delta}_0(t) = 1$, by applying formally the Fourier inversion formula, we have $\delta_0(u) = \frac{1}{2\pi} \int_{\mathbb{R}} cos(tu) dt$,

$$N_t(0) = \int_0^t \delta_0(X_s) \mid \dot{X}_s \mid ds = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t cos(\xi X_s) \mid \dot{X}_s \mid d\xi ds;$$

more generally if g is a positive function on $\mathbb{R}$

$$\int_{\mathbb{R}} N_t(x)g(x)dx = \int_0^t \left[\int_{\mathbb{R}} \delta_{X_s}(x)g(x)dx\right] \mid \dot{X}_s \mid ds = \int_0^t g(X_s) \mid \dot{X}_s \mid ds.$$

Proof.

The representation of $N_t(x)$ is given in Slud (1994) or in Kratz & León (1997). To define the framework, the main tool will be the Malliavin calculus. It is a quite powerful tool, since it already allowed Nualart & Vives (1992b) proving that the local time of the Brownian motion belongs to the Sobolev space $\mathbb{D}^{p,\alpha}$, for any $p \geq 2$ and $0 < \alpha < 1/p$ and since it reveals itself quite adapted when dealing with functionals a bit more complicated as in this paper.

Let $p_{\varepsilon,x}(y) := p_\varepsilon(y-x)$.
We will need the following technical results:

Lemma 3.1 *For $\beta > 0$, we have*

$$E\left[p_{\varepsilon,x}(X_\tau)(I-L)^{-\beta} p_{\varepsilon,x}(X_v)\right] = \frac{1}{2\pi\Gamma(\beta)} \int_0^\infty e^{-u} u^{\beta-1} \varphi_{u,\varepsilon}(\tau, v) du,$$

where $\varphi_{u,\varepsilon}(\tau,v) = \dfrac{e^{-\frac{x^2}{\varepsilon+1+r(\tau-v)e^{-u}}}}{\sqrt{(\varepsilon+1)^2 - r^2(\tau-v)e^{-2u}}}.$

Proof: First note that

$$p_\beta(ay+bx)p_\delta(cy+dx) = p_{\delta a^2+\beta c^2}(x(ad-bc))p_{\frac{\beta\delta}{\delta a^2+\beta c^2}}\left(y + x\frac{ab\delta + cd\beta}{\delta a^2 + \beta c^2}\right). \qquad (3.5)$$

By using Lemma 1.1 in Nualart & Vives (1992b) with $||h_s||_2^2 = 1$ (hypothesis), we have

$$T_u\left(p_{\varepsilon,x}(X_s)\right) = p_{\varepsilon+1-e^{-2u},x}(e^{-u}X_s), \qquad (3.6)$$

which, combined with (3.5), gives

$$\begin{aligned} E\left[p_{\varepsilon,x}(X_\tau)T_u\left(p_{\varepsilon,x}(X_v)\right)\right] &= E\left[p_{\varepsilon+1-e^{-2u}}(e^{-u}X_0 - x)p_{1+\varepsilon-r^2(\tau-v)}\left(r(\tau-v)X_0 - x\right)\right] \\ &= \frac{1}{\sqrt{2\pi}}\, p_{(\varepsilon+1)^2 - r^2(\tau-v)e^{-2u}}\left(x\sqrt{\varepsilon+1-r(\tau-v)e^{-u}}\right). \end{aligned}$$

Then (2.2) provides the result. □

Lemma 3.2 *Let $\beta > 0$. Then* $\dfrac{d}{ds}\left((I-L)^{-\beta}\left(p_{\varepsilon,x}(X_s)\right)\right) = (I-L)^{-\beta}\left(\dfrac{d}{ds}p_{\varepsilon,x}(X_s)\right).$

Proof: First we show that

$$T_u\left(\frac{d}{ds}p_{\varepsilon,x}(X_s)\right) = \frac{d}{ds}\left(T_u(p_{\varepsilon,x}(X_s))\right). \qquad (3.7)$$

On one hand, because of (3.6) we have

$$\frac{d}{ds}\left(T_u(p_{\varepsilon,x}(X_s))\right) = e^{-u}\dot{X}_s \frac{x - e^{-u}X_s}{\varepsilon + 1 - e^{-2u}} p_{\varepsilon+1-e^{-2u},x}(e^{-u}X_s).$$

On the other hand,

$$\begin{aligned}
&T_u\left(\frac{d}{ds}p_{\varepsilon,x}(X_s)\right) = T_u\left(\dot{X}_s\frac{x - X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right) := T_u\left(f_1(\dot{X}_s)f_2(X_s)\right) \\
&= T_u\left(f_1(\dot{X}_s)\right)T_u\left(f_2(X_s)\right) \qquad (3.8)\\
&= \int_{\mathbb{R}} f_1(e^{-u}\dot{X}_s + y)p_{1-e^{-2u}}(y)dy \int_{\mathbb{R}} f_2(e^{-u}X_s + z)p_{1-e^{-2u}}(z)dz \\
&= \int_{\mathbb{R}} (e^{-u}\dot{X}_s + y)p_{1-e^{-2u}}(y)dy \int_{\mathbb{R}} \frac{x - (e^{-u}X_s + z)}{\varepsilon}p_{\varepsilon,x}(e^{-u}X_s + z)p_{1-e^{-2u}}(z)dz,
\end{aligned}$$

by using the independence of $X_s = W(h_s)$ and $\dot{X}_s = W(\dot{h}_s)$ and Melher's formula to $F := f(W(h_s, \dot{h}_s)) = f_1(W(h_s))f_2(W(\dot{h}_s))$ (note that it is a generalization to the dimension 2 of Lemma 1.1 in Nualart & Vives (1992b)).
But $\int_{\mathbb{R}}(e^{-u}\dot{X}_s + y)p_{1-e^{-2u}}(y)dy = e^{-u}\dot{X}_s$
and, by using (3.5) for the computation of the integral in z, we have

$$\int_{\mathbb{R}} \frac{x - (e^{-u}X_s + z)}{\varepsilon}p_{\varepsilon,x}(e^{-u}X_s + z)p_{1-e^{-2u}}(z)dz = \frac{x - e^{-u}X_s}{\varepsilon + 1 - e^{-2u}}p_{\varepsilon+1-e^{-2u},x}(e^{-u}X_s). \qquad (3.9)$$

Hence the result (3.7) follows.
Then (2.2) and Lebesgue theorem allow us to conclude. □

Let $N_t^\varepsilon(x) := \int_0^t p_{\varepsilon,x}(X_s) \mid \dot{X}_s \mid ds$.
We want to show that $N_t^\varepsilon(x)$ converges to $N_t(x)$ in $\mathbb{D}^{2,\alpha}$, $\forall \alpha < 1/4$.
Fix $0 < \alpha < 1$.
Let us prove that $(I-L)^{\frac{\alpha}{2}}N_t^\varepsilon(x)$ is bounded in $L^2(\Omega)$, uniformly in $\varepsilon > 0$, or equivalently that

$$(I-L)^{\frac{\alpha-1}{2}}D_\theta N_t^\varepsilon(x)$$

is bounded in $L^2(\Omega, L^2(\mathbb{R}, dx))$, uniformly in $\varepsilon > 0$, where D_θ is the derivative operator, $\theta \in \mathbb{R}$.
Let us compute

$$\begin{aligned}
||(I-L)^{\frac{\alpha-1}{2}}D_\theta N_t^\varepsilon(x)||^2_{L^2(\Omega,L^2(\mathbb{R},dx))} &= E\left[||(I-L)^{\frac{\alpha-1}{2}}D_\theta N_t^\varepsilon(x)||_2^2\right] \\
&= E\left|\int_0^t \left|(I-L)^{\frac{\alpha-1}{2}}D_\theta N_t^\varepsilon(x)\right|^2 d\theta\right|.
\end{aligned}$$

We have

$$\begin{aligned}
D_\theta N_t^\varepsilon(x) &= \int_0^t D_\theta\left(\mid \dot{X}_s \mid p_{\varepsilon,x}(X_s)\right)ds \\
&= \int_0^t D_\theta(\mid \dot{X}_s \mid)p_{\varepsilon,x}(X_s)ds + \int_0^t \mid \dot{X}_s \mid D_\theta\left(p_{\varepsilon,x}(X_s)\right)ds,
\end{aligned}$$

therefore

$$E\left|(I-L)^{\frac{\alpha-1}{2}}D_\theta N_t^\varepsilon(x)\right|^2 \le 2(E_1(\theta)+E_2(\theta)),$$

with $E_1(\theta)=E_{1,\alpha,\varepsilon}(\theta):=E\left|\int_0^t (I-L)^{\frac{\alpha-1}{2}}\left(D_\theta(\mid \dot{X}_s\mid)p_{\varepsilon,x}(X_s)\right)ds\right|^2$

and $E_2(\theta)=E_{2,\alpha,\varepsilon}(\theta):=E\left|\int_0^t (I-L)^{\frac{\alpha-1}{2}}\left(\mid \dot{X}_s\mid D_\theta(p_{\varepsilon,x}(X_s))\right)ds\right|^2$,

hence

$$||(I-L)^{\frac{\alpha-1}{2}}D_\theta N_t^\varepsilon(x)||^2_{L^2(\Omega,L^2(\mathbb{R},dx))} \le 2\int_{\mathbb{R}}(E_1(\theta)+E_2(\theta))d\theta.$$

First let us consider $E_1(\theta)$.

We have $E_1(\theta)\le \int_0^t E\left|(I-L)^{\frac{\alpha-1}{2}}\left(D_\theta(\mid \dot{X}_s\mid)p_{\varepsilon,x}(X_s)\right)\right|^2 ds.$

But $D_\theta(|\dot{X}_s|)=sg(\dot{X}_s)\dot{h}_s(\theta)$, using Proposition 1.2.3 in Nualart (1995) and the fact that $\dot{X}_s=W(\dot{h}_s)$.

Therefore

$$\begin{aligned}E\left|(I-L)^{\frac{\alpha-1}{2}}\left(D_\theta(\mid \dot{X}_s\mid)p_{\varepsilon,x}(X_s)\right)\right|^2 &= E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_s)\dot{h}_s(\theta)p_{\varepsilon,x}(X_s)\right)\right|^2\\ &= |\dot{h}_s(\theta)|^2E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_s)p_{\varepsilon,x}(X_s)\right)\right|^2.\end{aligned}$$

Moreover, the process X being stationary, we obtain

$$E_1(\theta)\le E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2\int_0^t|\dot{h}_s(\theta)|^2ds,$$

and then

$$\int_{\mathbb{R}}E_1(\theta)d\theta\le E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2\int_{\mathbb{R}}\int_0^t|\dot{h}_s(\theta)|^2dsd\theta.$$

Noticing that $\int_{\mathbb{R}}\int_0^t|\dot{h}_s(\theta)|^2dsd\theta=t$, we have

$$\int_{\mathbb{R}}E_1(\theta)d\theta\le t\ E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2.$$

Let us compute $E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2$.

Since

$$\begin{aligned}(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right) &= (I-L)^{\frac{\alpha-1}{2}}\left(1_{(\dot{X}_0>0)}p_{\varepsilon,x}(X_0)\right)-(I-L)^{\frac{\alpha-1}{2}}\left(1_{(\dot{X}_0\le 0)}p_{\varepsilon,x}(X_0)\right)\\ &= 2(I-L)^{\frac{\alpha-1}{2}}\left(1_{(\dot{X}_0>0)}p_{\varepsilon,x}(X_0)\right)-(I-L)^{\frac{\alpha-1}{2}}\left(p_{\varepsilon,x}(X_0)\right),\end{aligned}$$

we have

$E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2\le$

$8E\left[(I-L)^{\frac{\alpha-1}{2}}\left(1_{(\dot{X}_0>0)}p_{\varepsilon,x}(X_0)\right)\right]^2+2E\left[(I-L)^{\frac{\alpha-1}{2}}\left(p_{\varepsilon,x}(X_0)\right)\right]^2.$

By using (2.2), then the independence of X_s and $\dot{X}_s$, and (3.8), we have
$E\left|(I-L)^{\frac{\alpha-1}{2}}\left(1_{(\dot{X}_0>0)}p_{\varepsilon,x}(X_0)\right)\right|^2 =$

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-u}u^{-\alpha}E\left[1_{(\dot{X}_0>0)}T_u\left(1_{(\dot{X}_0>0)}\right)\right]E\left[p_{\varepsilon,x}(X_0)T_u\left(p_{\varepsilon,x}(X_0)\right)\right]du.$$

Hence we obtain
$E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2 \leq$

$$\frac{2}{\Gamma(1-\alpha)}\int_0^\infty e^{-u}u^{-\alpha}E\left[p_{\varepsilon,x}(X_0)T_u\left(p_{\varepsilon,x}(X_0)\right)\right]\left(1+4E\left[1_{(\dot{X}_0>0)}T_u\left(1_{(\dot{X}_0>0)}\right)\right]\right)du.$$

On one hand, Lemma 3.1 provides

$$E\left[p_{\varepsilon,x}(X_0)T_u\left(p_{\varepsilon,x}(X_0)\right)\right] = \frac{1}{2\pi}\varphi_\varepsilon(u) \qquad \text{with} \qquad \varphi_\varepsilon(u) := \frac{e^{-\frac{x^2}{1+\varepsilon+e^{-u}}}}{\sqrt{(\varepsilon+1)^2-e^{-2u}}}.$$

On the other hand, Lemma 1.1 in Nualart & Vives (1992b)) gives

$$E\left[1_{(\dot{X}_0>0)}T_u\left(1_{(\dot{X}_0>0)}\right)\right] = \int_0^\infty p_1(z)\int_{-ze^{-u}}^\infty p_{1-e^{-2u}}(y)dydz \leq \frac{1}{2}.$$

Therefore

$$E\left|(I-L)^{\frac{\alpha-1}{2}}\left(sg(\dot{X}_0)p_{\varepsilon,x}(X_0)\right)\right|^2 \leq \frac{3}{\pi\Gamma(1-\alpha)}\int_0^\infty e^{-u}u^{-\alpha}\varphi_\varepsilon(u)du,$$

which implies

$$\int_{\mathbb{R}} E_1(\theta)d\theta \leq \frac{3t}{\pi\Gamma(1-\alpha)}\int_0^\infty e^{-u}u^{-\alpha}\varphi_\varepsilon(u)du.$$

But we have

$$\sup_{\varepsilon>0}\int_0^\infty e^{-u}u^{-\alpha}\varphi_\varepsilon(u)du < \infty, \text{ whenever } \alpha < 1/2.$$

Indeed, as $\varepsilon \to 0$, the dominated convergence theorem provides $\displaystyle\int_0^\infty e^{-u}u^{-\alpha}\varphi(u)du =$ $\displaystyle\int_0^\infty \frac{e^{-u}u^{-\alpha}}{\sqrt{1-e^{-2u}}}e^{-\frac{x^2}{1+e^{-u}}}du$, which converges for $\alpha < 1/2$.
It allows us to conclude that $\forall \alpha < 1/2$,

$$\int_{\mathbb{R}} E_{1,\alpha,\varepsilon}(\theta)d\theta < \infty \qquad \text{uniformly in } \varepsilon > 0.$$

Now let us consider $E_2(\theta)$.
Since $D_\theta(p_{\varepsilon,x}(X_s)) = \dfrac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)h_s(\theta)$, we can write

$$E_2(\theta) = E\left|\int_0^t h_s(\theta)(I-L)^{\frac{\alpha-1}{2}}\left(F(s)\right)ds\right|^2,$$

with $F(s) := \mid \dot{X}_s \mid \dfrac{x - X_s}{\varepsilon} p_{\varepsilon,x}(X_s)$.
We have by stationarity

$$\begin{aligned} E_2(\theta) &= E\left[\int_0^t \int_0^t h_s(\theta)(I-L)^{\frac{\alpha-1}{2}}(F(s))h_v(\theta)(I-L)^{\frac{\alpha-1}{2}}(F(v))dsdv\right] \\ &= 2\int_0^t (t-s)E\left[h_0(\theta)(I-L)^{\frac{\alpha-1}{2}}(F(0))h_s(\theta)(I-L)^{\frac{\alpha-1}{2}}(F(s))\right]ds \\ &= 2\int_0^t (t-s)h_0(\theta)h_s(\theta)E\left[F(s)(I-L)^{\alpha-1}(F(0))\right]ds. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{R}} E_2(\theta)d\theta &= 2\int_0^t (t-s)E\left[F(s)(I-L)^{\alpha-1}(F(0))\right]\left(\int_{\mathbb{R}} h_0(\theta)h_s(\theta)d\theta\right)ds \\ &= 2\int_0^t (t-s)r(s)E\left[F(s)(I-L)^{\alpha-1}(F(0))\right]ds \\ &= 2E\left[((I-L)^{\alpha-1}F(0))\int_0^t (t-s)r(s)F(s)ds\right] \\ &\leq 2E^{1/2}\left[\left((I-L)^{\alpha-1}F(0)\right)^2\right]E^{1/2}\left[\left(\int_0^t (t-s)r(s)F(s)ds\right)^2\right], \end{aligned}$$

by using the property of self-adjointness of L and the Cauchy-Schwartz inequality.
Let $E_{21} := E\left[\left((I-L)^{\alpha-1}F(0)\right)^2\right]$ $\left(= E\left[\left((I-L)^{\alpha-1}F(s)\right)^2\right]\right.$, by stationarity),
and $E_{22} := E\left[\left(\int_0^t (t-s)r(s)F(s)ds\right)^2\right]$.
Let us begin to study E_{21}.
We have via (2.2)

$$E_{21} = E[F(s)(I-L)^{2(\alpha-1)}(F(s))] = \frac{1}{\Gamma(2(1-\alpha))}\int_0^\infty e^{-u}u^{1-2\alpha}E[F(s)T_u(F(s))]du.$$

But

$$E[F(s)T_u(F(s))] = E\left[\mid \dot{X}_s \mid T_u\left(\mid \dot{X}_s \mid\right)\right]E\left[\left(\frac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right)T_u\left(\frac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right)\right]$$

and

$$E\left[\frac{d}{ds}(p_{\varepsilon,x}(X_s))T_u\left(\frac{d}{ds}(p_{\varepsilon,x}(X_s))\right)\right] = E\left[\dot{X}_sT_u\left(\dot{X}_s\right)\right]E\left[\left(\frac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right)T_u\left(\frac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right)\right],$$

by using again in both cases the independence of X_s and $\dot{X}_s$, and (3.8).
Since $T_u\left(\dot{X}_s\right) = \int_{\mathbb{R}}(e^{-u}\dot{X}_s + y)p_{1-e^{-2u}}(y)dy = e^{-u}\dot{X}_s$, which implies
$E\left[\dot{X}_sT_u\left(\dot{X}_s\right)\right] = e^{-u}E[\dot{X}_s^2] = e^{-u}$, we obtain

$$E\left[\left(\frac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right)T_u\left(\frac{x-X_s}{\varepsilon}p_{\varepsilon,x}(X_s)\right)\right] = e^uE\left[\frac{d}{ds}(p_{\varepsilon,x}(X_s))T_u\left(\frac{d}{ds}(p_{\varepsilon,x}(X_s))\right)\right],$$

and then

$$E[F(s)T_u(F(s))] = e^u E\left[\mid \dot{X}_s \mid T_u\left(\mid \dot{X}_s \mid\right)\right] E\left[\frac{d}{ds}(p_{\varepsilon,x}(X_s))T_u\left(\frac{d}{ds}(p_{\varepsilon,x}(X_s))\right)\right].$$

Some direct computations give

$$T_u\left(\mid \dot{X}_s \mid\right) = 2(1-e^{-2u})p_{1-e^{-2u}}(e^{-u}\dot{X}_s)+e^{-u}\dot{X}_s\left(\Phi_{1-e^{-2u}}(e^{-u}\dot{X}_s) - \Phi_{1-e^{-2u}}(-e^{-u}\dot{X}_s)\right),$$

and

$$E\left[\mid \dot{X}_s \mid T_u\left(\mid \dot{X}_s \mid\right)\right] = \frac{2}{\pi}\sqrt{1-e^{-2u}} + 4e^{-u}\int_0^\infty p_{1-e^{-2u}}(z)\Phi_1(-e^u z)dz,$$

where $\Phi_a(z) = \int_{-\infty}^z p_a(y)dy$.
Moreover Lemmas 3.1 and 3.2 (in particular (3.7)) provide

$$E\left[\frac{d}{ds}(p_{\varepsilon,x}(X_s))T_u\left(\frac{d}{ds}(p_{\varepsilon,x}(X_s))\right)\right] = \frac{1}{2\pi}\left[\frac{d^2}{d\tau dv}\varphi_{u,\varepsilon}(\tau,v)\right]_{\tau=v=s}.$$

Therefore after differenciation (and with $r'(0)=0$), we obtain

$$E\left[\frac{d}{ds}(p_{\varepsilon,x}(X_s))T_u\left(\frac{d}{ds}(p_{\varepsilon,x}(X_s))\right)\right] = \frac{e^{-u}}{2\pi}\frac{e^{-\frac{x^2}{\varepsilon+1+e^{-u}}}}{\sqrt{(\varepsilon+1)^2-e^{-2u}}}\left(\frac{x^2}{(\varepsilon+1+e^{-u})^2}+\frac{e^{-u}}{(\varepsilon+1)^2-e^{-2u}}\right).$$

Consequently

$$\begin{aligned} E[F(s)T_u(F(s))] &= \frac{1}{2\pi}\left(\frac{2}{\pi}\sqrt{1-e^{-2u}} + 4e^{-u}\int_0^\infty p_{1-e^{-2u}}(z)\Phi_1(-e^u z)dz\right)\frac{e^{-\frac{x^2}{\varepsilon+1+e^{-u}}}}{\sqrt{(\varepsilon+1)^2-e^{-2u}}} \\ &\quad \times\left(\frac{x^2}{(\varepsilon+1+e^{-u})^2}+\frac{e^{-u}}{(\varepsilon+1)^2-e^{-2u}}\right) \end{aligned}$$

and so $\sup\limits_{\varepsilon\to 0} E_{21} < \infty$ whenever $\alpha < 1/4$.

Now let us study E_{22}.
Using again the stationarity (and $r(0)=1$), we have

$$\begin{aligned} E_{22} &= 2tE\left[\int_0^t (t-s)^2 r(s)F(0)F(s)ds\right] \\ &= 2t\int_0^t (t-s)^2 r(s)E\left[F(0)F(s)\right]ds. \end{aligned}$$

Let $\delta > 0$ such that $\forall s > \delta$, $r(s) \neq 1$. We can write

$$E_{22} = 2t(I_\delta + I_t),$$

where I_δ and I_t are the terms corresponding to the integral over $[0,\delta]$ and $[\delta,t]$ respectively.

Let us consider I_δ.
As in Wschebor (1985), we can introduce the following decomposition

$$\begin{cases} \dot X_0 &= \xi + \alpha_1 X_0 + \alpha_2 X_s \\ \dot X_s &= \xi^* - \alpha_2 X_0 - \alpha_1 X_s \end{cases}$$

where ξ and ξ^* are centered Gaussian random variables, each independent of (X_0, X_s), with joint normal distribution, and
$\alpha_1 = -\dfrac{r(s)r'(s)}{(1-r^2(s))}$ and $\alpha_2 = \dfrac{r'(s)}{(1-r^2(s))}$.
Let $\phi_s(y,z)$ be the density of (X_0, X_s) and $f_\varepsilon(y) := \dfrac{x-y}{\varepsilon} p_{\varepsilon,x}(y)$.
Then

$$\begin{aligned} I_\delta &= \int_0^\delta (t-s)^2 r(s) \int_{\mathbb{R}^2} f_\varepsilon(y) f_\varepsilon(z) \phi_s(y,z) E\left[|\dot X_0 \dot X_s| \mid X_0 = y, X_s = z\right] dydzds \\ &= \int_0^\delta (t-s)^2 r(s) \int_{\mathbb{R}^2} f_\varepsilon(y) f_\varepsilon(z) \phi_s(y,z) E\left[|\xi + \alpha_1 y + \alpha_2 z||\xi^* - \alpha_2 y - \alpha_1 z|\right] dydzds. \end{aligned}$$

By writting

$$I_\delta = \int_{\mathbb{R}^2} f_\varepsilon(y) f_\varepsilon(z) \int_0^\delta (t-s)^2 r(s) \phi_s(y,z) E\left[|\xi + \alpha_1 y + \alpha_2 z||\xi^* - \alpha_2 y - \alpha_1 z|\right] dsdydz,$$

and by using the Geman condition,
we can conclude that $I_\delta < \infty$ uniformly in $\varepsilon > 0$.
Consider now I_t.

$$\begin{aligned} I_t &= E\left[\int_\delta^t (t-s)^2 r(s) F(0) F(s)\right] ds \\ &= \int_\delta^t (t-s)^2 r(s) \int_{\mathbb{R}^4} |\dot x_1| f_\varepsilon(x_1) |\dot x_2| f_\varepsilon(x_2) \phi_s(x_1, \dot x_1, x_2, \dot x_2) dx_1 d\dot x_1 dx_2 d\dot x_2 ds, \end{aligned}$$

where $\phi_s(y_i, 1 \le i \le 4)$ denotes the density of $(X_0, \dot X_0, X_s, \dot X_s)$.
With the change of variables $u_i = \dfrac{x - x_i}{\sqrt{\varepsilon}}$ for $i = 1, 2$, we get

$$\begin{aligned} I_t &= \int_\delta^t (t-s)^2 \frac{r(s)}{\varepsilon} \int_{\mathbb{R}^4} |\dot x_1| u_1 p_1(u_1) |\dot x_2| u_2 p_1(u_2) \phi_s(x - u_1\sqrt{\varepsilon}, \dot x_1, x - u_2\sqrt{\varepsilon}, \dot x_2) du_1 d\dot x_1 du_2 d\dot x_2 ds \\ &= \int_\delta^t (t-s)^2 r(s) \int_{\mathbb{R}^4} \psi_{s,\varepsilon}(u_1, \dot x_1, u_2, \dot x_2) du_1 d\dot x_1 du_2 d\dot x_2 ds, \end{aligned}$$

with

$$\psi_{s,\varepsilon}(u_1, \dot x_1, u_2, \dot x_2) := |\dot x_1| u_1 p_1(u_1) |\dot x_2| u_2 p_1(u_2) \frac{\phi_s(x - u_1\sqrt{\varepsilon}, \dot x_1, x - u_2\sqrt{\varepsilon}, \dot x_2) - \phi_s(x, \dot x_1, x, \dot x_2)}{\varepsilon}.$$

Applying Taylor formula to ϕ_s, there exists $\theta \in]0,1[$ such that

$$\begin{aligned} &\frac{\phi_s(x - u_1\sqrt{\varepsilon}, \dot x_1, x - u_2\sqrt{\varepsilon}, \dot x_2) - \phi_s(x, \dot x_1, x, \dot x_2)}{\varepsilon} = \\ &u_1 u_2 \frac{\partial^2 \phi_s}{\partial y_1 \partial y_3}(x - \theta u_1 \sqrt{\varepsilon}, \dot x_1, x - \theta u_2 \sqrt{\varepsilon}, \dot x_2) - \frac{1}{\sqrt{\varepsilon}} \left(u_1 \frac{\partial \phi_s}{\partial y_1}(x, \dot x_1, x, \dot x_2) + u_2 \frac{\partial \phi_s}{\partial y_3}(x, \dot x_1, x, \dot x_2) \right). \end{aligned}$$

But

$$\int_{\mathbb{R}^4} |\dot{x}_1|u_1p_1(u_1)|\dot{x}_2|u_2p_1(u_2)\left(u_1\frac{\partial\phi_s}{\partial y_1}(x,\dot{x}_1,x,\dot{x}_2)+u_2\frac{\partial\phi_s}{\partial y_3}(x,\dot{x}_1,x,\dot{x}_2)\right)du_1d\dot{x}_1du_2d\dot{x}_2$$
$$=\int_{\mathbb{R}^2}|\dot{x}_1||\dot{x}_2|\frac{\partial\phi_s}{\partial y_1}(x,\dot{x}_1,x,\dot{x}_2)\int_{\mathbb{R}}u_1^2p_1(u_1)du_1\int_{\mathbb{R}}u_2p_1(u_2)du_2$$
$$+\int_{\mathbb{R}^2}|\dot{x}_1||\dot{x}_2|\frac{\partial\phi_s}{\partial y_3}(x,\dot{x}_1,x,\dot{x}_2)\int_{\mathbb{R}}u_1p_1(u_1)du_1\int_{\mathbb{R}}u_2^2p_1(u_2)du_2=0,$$

therefore

$$I_t=\int_\delta^t(t-s)^2r(s)\int_{\mathbb{R}^4}|\dot{x}_1|u_1^2p_1(u_1)|\dot{x}_2|u_2^2p_1(u_2)\frac{\partial^2\phi_s}{\partial y_1\partial y_3}(x-\theta u_1\sqrt{\varepsilon},\dot{x}_1,x-\theta u_2\sqrt{\varepsilon},\dot{x}_2)du_1d\dot{x}_1du_2d\dot{x}_2ds.$$

The integrand is a continuous function in $\varepsilon\in]0,1]$.
Moreover

$$\lim_{\varepsilon\to 0}I_t=\int_\delta^t(t-s)^2r(s)\int_{\mathbb{R}^2}|\dot{x}_1||\dot{x}_2|\frac{\partial^2\phi_s}{\partial y_1\partial y_3}(x,\dot{x}_1,x,\dot{x}_2)d\dot{x}_1d\dot{x}_2ds<\infty.$$

Hence we can extend the integrand into a continuous function in $\varepsilon\in[0,1]$, which means that it reaches its supremum on $[0,1]$. So we can conclude that $I_t<\infty$ uniformly in $\varepsilon>0$.
It implies that $E_{22}<\infty$ uniformly in $\varepsilon>0$, and consequently that $\forall\alpha<1/4$,

$$\int_{\mathbb{R}}E_{2,\alpha,\varepsilon}(\theta)d\theta<\infty\qquad\text{uniformly in }\varepsilon>0,$$

which provides the final result. □

4 Conclusion

This new result should allow to consider problems related to crossings in a more direct way.
Moreover the interest of this new approach consists in generalizing the results in an easier way, on one hand to the number of crossings $N_t(\psi(t))$ of a C^1-curve ψ by the process X on the interval $[0,t]$, on the other hand to any dimension, as it has been done for the Brownian local time (cf. Imkeller et al. (1995)).
Another classes of processes could also be investigated, using a similar methodology.
It will be the object of further developments.

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