On implicit and explicit discretization schemes for parabolic SPDEs in any dimension

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Abstract

We study the speed of convergence of the explicit and implicit space-time discretization schemes of the solution $u(t,x)$ to a parabolic partial differential equation in any dimension perturbed by a space-correlated Gaussian noise. The coefficients only depend on $u(t,x)$ and the influence of the correlation on the speed is observed; in the limit case, corresponding to the space-time white noise in dimension 1, we recover the speeds obtained by I. Gyöngy.

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1 Introduction

Discretization schemes for parabolic SPDEs driven by the space-time white noise have been considered by several authors. I. Gyöngy and D. Nualart [8] and [9], have studied implicit time discretization schemes for the heat equation in dimension 1. J. Printems [13] has studied several time discretization schemes (implicit and explicit Euler schemes as well as the Crank-Nicholson one) for Hilbert-valued parabolic SPDEs, such as the Burgers equation on $[0,1]$, introduced several notions of order of convergence in order to deal with coefficients with polynomial growth and proved convergence in the Hilbert space norm. This work has been completed by E. Hausenblas [10], who studied several schemes for quasi-linear equations driven by a space-time white noise or a nuclear noise, and taking values in a Hilbert or a Banach space X. Several approximation procedures (such as the Galerkin approximation, finite difference methods or wavelets approximations) were considered, but the coefficients of the SPDE were supposed to depend on the whole function $u(t,.)$ in X, and not only on the evaluation of the process u at (t,x). Notice that, unlike [10], the coefficients considered in this paper do not depend on the whole function $u(s,.)$.

I. Gyöngy [6] has studied the strong speed of convergence in the norm of uniform convergence over the space variable for a space finite-difference scheme u^n with mesh $1/n$ for the parabolic SPDE with homogeneous Dirichlet's boundary conditions. He has also studied the speed of convergence of an implicit (resp. explicit) finite-difference discretization scheme $u^{n,m}$ (resp.

u^n_m) with time mesh T/m and space mesh $1/n$ for the solution u to the following parabolic SPDE in dimension 1 driven by the space-time white noise W:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x} + b(t,x,u(t,x))\,, \\ u(t,0) = u(t,1) = 0\,, \end{cases} \tag{1.1}$$

with the initial condition u_0. He has proved that, if the coefficients $\sigma(t,x,.)$ and $b(t,x,.)$ satisfy the usual Lipschitz property uniformly in (t,x) and if the functions $\sigma(t,x,y)$ and $b(t,x,y)$ are 1/4-Hölder continuous in t and 1/2-Hölder continuous in x uniformly with respect to the other variables, then for $t \in]0,T]$, $p \in [1,+\infty[$, $0 < \beta < \frac{1}{4}$ and $0 < \gamma < \frac{1}{2}$ one has:

$$\sup_{x\in[0,1]} \mathbb{E}\left(|u^{n,m}(t,x) - u(t,x)|^p\right) \le K(t)\left(m^{-\beta p} + n^{-\gamma p}\right)\,. \tag{1.2}$$

Furthermore, if $u_0 \in \mathcal{C}^3([0,1])$, then (1.2) holds on $[0,T]$ with $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ and with a constant K which does not depend on t. A similar result holds for the explicit scheme u^n_m if $\frac{n^2 T}{m} \le q < \frac{1}{2}$.

J. Printems [5] has studied a discretization scheme for the KDV equation, and C. Cardon-Weber [2] has studied explicit and implicit discretization schemes for the solution to the stochastic Cahn-Hilliard equation. Since the bi-Laplacian Δ^2 is more regularizing than Δ, a function-valued solution to this equation can be obtained in dimension $d \le 3$ when the driving noise is the space-time white noise. The polynomial growth of the drift term made her require the diffusion coefficient σ to be bounded; furthermore, she proved convergence in probability of the scheme (respectively in L^p with a given rate of a localized version of the scheme).

In the present paper, we deal with a d-dimensional version of (1.1). As it is well-known, we can no longer use the space-time white noise for the perturbation; indeed, in dimension $d \ge 2$, the Green function associated with $\frac{\partial}{\partial t} - \Delta$ with the homogeneous Dirichlet boundary conditions on $[0,1]^d$ is not square integrable. Thus, we replace W by some Gaussian process F which is white in time and has a space correlation given by a Riesz potential f, i.e., we require that for some $\alpha \in]0, 2\wedge d[$:

$$E[F(t,x)\,F(t,y)] = (s\wedge t)\,|x-y|^{-\alpha}\,.$$

See e.g. [11], [4], [12] and [3] for more general results concerning necessary and sufficient conditions on the covariance of the Gaussian noise F ensuring the existence of a function-valued solution to (1.1) with F instead of W.

The aim of this paper is threefold. We at first study the speed of convergence of space and space-time finite discretization implicit (resp. explicit) schemes in dimension $d \ge 1$, i.e., on the grid $(\frac{iT}{m}, (\frac{j_k}{n}, 1 \le k \le d))$, $0 \le i \le m$, $0 \le j_k \le n$ and extended to $[0,T] \times [0,1]^d$ by linear interpolation. As in [6] and [7], the processes u^n and $u^{n,m}$ (resp. u^n_m) have an evolution formulation written in terms of approximations $(G_d)^n$, $(G_d)^{n,m}$ and $(G_d)^n_m$ of the Green function G_d, while u is solution of an evolution equation defined in terms of G_d. These evolution equations involve stochastic integrals with respect to the worthy martingale-measure defined by F (see e.g. [15] and [4]).

As usual, the speed of convergence is given by the norm of the differences of stochastic integrals; more precisely, the optimal speed of convergence for the implicit scheme is the norm of the difference $G_d(.,x,.) - (G_d)^{n,m}(.,x,.)$ in $L^2([0,T],\mathcal{H}_d)$, where $\mathcal{H}_d$ is the Reproducing Kernel Hilbert Space defined by the covariance function. More precisely, if φ and ψ are continuous functions on $Q = [0,1]^d$, set

$$<\varphi,\psi>_{\mathcal{H}_d} = \int_Q\int_Q \varphi(x)\,f(|x-y|)\,\psi(y)dx\,dy\,. \tag{1.3}$$

We denote by $\mathcal{H}_d$ the completion of this pre-Hilbert space; note that $\mathcal{H}_d$ elements which are not functions and that a function φ belongs to $\mathcal{H}_d$ if and only if $\int_Q \int_Q |\varphi(y)|\, f(|y-z|)\, |\varphi(z)|\, dy\, dz < +\infty$. However, unlike in [6] and [7], the functions

$$\varphi_j(x) = \sqrt{2}\, \sin(j\pi x), \quad j \geq 1 \quad \text{and} \quad \varphi_j(\kappa_n(x)), \quad 1 \leq j \leq n, \quad \text{where} \quad \kappa_n(y) = [ny]\, n^{-1},$$

are not an orthonormal family of $\mathcal{H}_1$. Thus, even in dimension $d = 1$, the use of the Parseval identity has to be replaced by more technical computations based on Abel's transforms. Similar results could be obtained for more general covariance functions, but the speed would depend on integrals including f and would be less transparent than that stated in the case of Riesz potentials. The key technical lemmas, giving upper estimates of $\|G_d(.,x,.) - (G_d)^n(.,x,.)\|_{L^2([0,\infty[,\mathcal{H}_d)}$ and $\|(G_d)^n(.,x,.) - (G_d)^{n,m}(.,x,.)\|_{L^2([0,T],\mathcal{H}_d)}$ (resp. $\|(G_d)^n(.,x,.) - (G_d)^n_m(.,x,.)\|_{L^2([0,T],\mathcal{H}_d)}$), are proved in section 4.

We describe the discretization schemes in any dimension $d \geq 1$ and introduce some notations in section 2. In section 3, an argument similar to that in [6] shows that for $0 < \alpha < d \wedge 2$, and $p \in [1, +\infty[$, if u_0 is regular enough, then

$$\sup_{(t,x)\in[0,+\infty[\times Q} \mathbb{E}(\|u(t,x) - u^n(t,x)\|^{2p}) \leq C_{p,\alpha}\, n^{-(2-\alpha)p}\,, \tag{1.4}$$

and extending [7] we prove in section 4 that

$$\sup_{(t,x)\in[0,T]\times Q} \mathbb{E}(\|u^n(t,x) - u^{n,m}(t,x)\|^{2p}) \leq C_p\, m^{-(1-\frac{\alpha}{2})p}\,. \tag{1.5}$$

In the "limit case" $d = 1$ and $\alpha = 1$, which corresponds to the space-time white noise, we recover the speed of convergence proved by Gyöngy.

In dimension $d \geq 2$, the proof depends on the product form of the Green function and its approximations, as well as of upper estimates of $|x-y|^{-\alpha}$ in terms of $\prod_{i=1}^d |x_i - y_i|^{-\alpha_i}$ for some well-chosen α_i. Thus, estimates of the $\mathcal{H}_d$-norm of the differences of $G_d(s,x,.) - (G_d)^n(s,x,.)$, $(G_d)^n(s,x,.) - (G_d)^{n,m}(s,x,.)$ and $(G_d)^n(s,x,.) - (G_d)^n_m(s,x,.)$ in dimension $d \geq 2$ depend on bounds of the $\mathcal{H}_1$-norm of similar differences as well as of $\mathcal{H}_r$-norms of $G(s,x,.)$, $G^n(s,x,.)$ and $G^{n,m}(s,x,.)$ for $r < d$.

Section 5 contains some numerical results. For $T = 1$, we have implemented in C the (more stable) implicit discretization scheme for affine coefficients $\sigma(t,y,u) = \sigma_1\, u + \sigma_2$ and $b(t,x,u) = b_1\, u + b_2$ and for $\sigma(t,y,u) = b(t,y,u) = \cos(u)$. We have studied the "experimental" speed of convergence with respect to one mesh, when the other one is fixed and gives a "much smaller" theoretical error. The second moments are computed by Monte-Carlo approximations. These implementations have been done in dimension $d = 1$ for the space-time white noise W and the colored noise F. As expected, the observed speeds are better than the theoretical ones, and decrease with α. For example, choosing N and M "large" with $M \geq N^2$ and considering "small" divisors n of N, we have computed the observed linear regression coefficient and drawn the curves of $\sup_{x\in[0,1]} \ln(E(|u^{n,M}(1,x) - u^{N,M}|^2(1,x)|^2))$ as a function of $\ln(n)$ for various values of α.

Note that all the results of this paper remain true if in (1.1) we replace the homogeneous Dirichlet boundary conditions $u(t,x) = 0$ for $x \in \partial Q$ by the homogeneous Neumann ones $\frac{\partial u}{\partial x}(t,x) = 0$ for $x \in \partial Q$. In this last case, the eigenfunctions of $\frac{\partial}{\partial t} - \Delta$ in dimension one is $\varphi_0(x) = 1$ and for $j \geq 1$, $\varphi_j(x) = \sqrt{2}\, \cos(j\pi x)$. Since the upper estimates of the partial sums $\sum_{j=1}^K \varphi_j(x)$ used in the Abel transforms still hold in the case of Neumann conditions, the crucial result is proved in a similar way in this case, and the speed of convergence is preserved.

2 Formulation of the problem

We denote by $x = (x_1, \cdots, x_d)$ an element of $\mathbb{R}^d$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $Q = [0,1]^d$ for some integer $d \geq 1$ and let $F = \big(F(\varphi)\,,\, \varphi \in \mathcal{D}(\mathbb{R}_+ \times Q)\big)$ be an $L^2(P)$-valued centered Gaussian process, which is white in time but has a space correlation defined as follows: given φ and ψ in $\mathcal{D}(\mathbb{R}_+ \times Q)$, the covariance functional of $F(\varphi)$ and $F(\psi)$ is

$$J(\varphi, \psi) = E\big(F(\varphi)\, F(\psi)\big) = \int_0^{+\infty} dt \int\int_{(Q-Q)^*} \varphi(t,y)\, f(y-z)\, \psi(t,z)\, dy\, dz\,, \qquad (2.1)$$

where $(Q-Q)^* = \{y - z \,:\, y, z \in Q,\, y \neq z\}$ and $f : (Q-Q)^* \to [0, +\infty[$ is a continuous function. The bilinear form J defined by (2.1)is non-negative definite if and only if f is the Fourier transform of a non-negative tempered distribution μ on Q. Then F defines a martingale-measure (still denoted by F), which allows to use stochastic integrals (see [15]). In the sequel, we suppose that for $z \in \mathbb{R}^d$, $z \neq 0$, $f(z) = |z|^{-\alpha}$, where $|z|$ denotes the Euclidean norm of the vector z. Since $x^2 + y^2 \geq 2xy$, if $\alpha_j = \alpha 2^{-j}$ for $1 \leq j < d$ and $\alpha_d = \alpha 2^{-d+1}$, there exists a positive constant C such that for any $z = (z_1, \cdots, z_d) \in \mathbb{R}^d$,

$$f(z) \leq C \prod_{j=1}^d f_{\alpha_j}(z_j), \qquad (2.2)$$

where $f_\alpha(\zeta) = |\zeta|^{-\alpha}$ for any $\zeta \in \mathbb{R}$, $\zeta \neq 0$. To lighten the notations, for this choice of f and $\varphi \in \mathcal{H}_d$ set

$$\|\varphi\|^2_{(\alpha)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(y)|\, |y - z|^{-\alpha}\, |\varphi(z)|\, dy dz\,. \qquad (2.3)$$

For any $t \geq 0$, we denote by $\mathcal{F}_t$ the sigma-algebra generated by $\{F([0,s] \times A) : 0 \leq s \leq t\,,\, A \subset Q\}$. Let $\sigma : [0, +\infty[\times Q \times \mathbb{R} \to \mathbb{R}$ and $b : [0, +\infty[\times Q \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

There exists a positive constant C such that for every $s, t \in [0, \infty[$, $x, y \in Q$, $r, v \in \mathbb{R}$, the linear growth condition

$$|\sigma(t,x,r)| + |b(t,x,r)| \leq C\,(1 + |r|)\,, \qquad (2.4)$$

and either Lipschitz condition

$$|\sigma(t,x,r) - \sigma(t,x,v)| + |b(t,x,r) - b(t,x,v)| \leq C|r - v|\,, \qquad (2.5)$$

$$|\sigma(t,x,r) - \sigma(t,y,v)| + |b(t,x,r) - b(t,y,v)| \leq C(|x-y|^{1-\frac{\alpha}{2}} + |r - v|)\,, \qquad (2.6)$$

$$|\sigma(s,x,r) - \sigma(t,y,v)| + |b(s,x,r) - b(t,y,v)| \leq C(|t-s|^{\frac{1}{2}-\frac{\alpha}{4}} + |x-y|^{1-\frac{\alpha}{2}} + |r - v|)\,, \qquad (2.7)$$

hold. For any function u_0 which vanishes on the boundary of Q, let $u(t,x)$ denote the solution to the parabolic SPDE, which is similar to (1.1)

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \sigma(t,x,u(t,x))\frac{\partial^2 F}{\partial t \partial x} + b(t,x,u(t,x))\,, \\ u(t,x) = 0 \quad \text{for} \quad x \in \partial Q\,, \end{cases} \qquad (2.8)$$

with initial condition $u(0,x) = u_0(x)$. Let $\mathbb{N}^*$ denote the set of strictly positive integers; for any $j \in \mathbb{N}^*$ and $\xi \in \mathbb{R}$, set $\varphi_j(\xi) = \sqrt{2}\sin(j\pi\xi)$ and for $\underline{k} = (k_1, \cdots, k_d) \in \mathbb{N}^{*d}$, set

$$|\underline{k}| = \sum_{j=1}^d k_j, \quad \varphi_{\underline{k}}(x) = \prod_{j=1}^d \varphi_{k_j}(x_{k_j}) \quad \text{for} \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Let $G_d(t,x,y)$ denote the Green function associated with the operator $\frac{\partial}{\partial t} - \Delta$ on Q with homogeneous Dirichlet boundary conditions; then for $t>0$, $x,y\in Q$,

$$G_d(t,x,y) = \sum_{\underline{k}\in\mathbb{N}^{*d}} \exp(-|\underline{k}|^2\,\pi^2\,t)\,\varphi_{\underline{k}}(x)\,\varphi_{\underline{k}}(y)\,;$$

when $d=1$, set $G_1=G$. Then there exist positive constants c and C such that for every $t>0$, $x,y\in\mathbb{R}^d$, $|G_d(t,x,y)|\le C\,t^{-\frac{d}{2}}\exp\Big(-c\frac{|x-y|^2}{t^d}\Big)$. The equation (2.8) makes sense in a weak form which is equivalent to the following evolution formulation:

$$\begin{aligned} u(t,x) &= \int_Q G_d(t,x,y)\,u_0(y)\,dy + \int_0^t\int_Q G_d(t-s,x,y) \\ &\quad \times\big[\sigma(s,y,u(s,y))\,F(ds,dy) + b(s,y,u(s,y))\,dsdy\big]\,. \end{aligned} \tag{2.9}$$

We also consider the parabolic SPDE with the homogeneous boundary conditions $\frac{\partial u}{\partial x}(t,x)=0$ for $x\in\partial Q$. In that case, the functions $(\varphi_j\,;\,j\ge 1)$ are replaced by $\varphi_0(\xi)=1$ and $\varphi_j(\xi)=\sqrt{2}\,\cos(j\pi\xi)$ for $\xi\in\mathbb{R}$ and $j\ge 1$. All the other formulations remain true with $\underline{k}\in\mathbb{N}^d$ instead on $\mathbb{N}^{*d}$.

2.1 Space discretization scheme

As in [6], we at first consider a finite space discretization scheme, replacing the Laplacian by its discretization on the grid $\frac{k}{n}=(\frac{k_1}{n},\cdots,\frac{k_d}{n})$, where $k_j\in\{0,\cdots,n\}$, $1\le j\le d$. In dimension 1, we proceed as in [6], and consider the $(n-1)\times(n-1)$-matrix D_n associated with the homogeneous Dirichlet boundary conditions and defined by $D_n(i,i)=-2$, $D_n(i,j)=1$ if $|i-j|=1$ and $D_n(i,j)=0$ for $|i-j|\ge 2$; then $\frac{\partial^2 u(t,x)}{\partial x^2}$ is replaced by $n^2\,D_n\vec{u}_n(t,.)$, where $\vec{u}^n(t)$ denotes the $(n-1)$-dimensional vector of an approximate solution defined on the grid $j/n, 1\le j\le n$. In arbitrary dimension, we proceed as in [2] and define $D_n^{(d)}$ by induction. Let $D_n^{(1)}=D_n$ and suppose that $D_n^{(d-1)}$ has been defined as a $(n-1)^{d-1}\times(n-1)^{d-1}$ matrix. Let Id_k denotes the $k\times k$ identity matrix and given a $(n-1)^{d-1}\times(n-1)^{d-1}$ matrix A, let $diag(A)$ denote the $(n-1)^d\times(n-1)^d$ matrix with $d-1$ diagonal blocs equal to A; let $D_n^{(d)}$ denote the $(n-1)^d\times(n-1)^d$-matrix $D_n^{(d)}$ defined by

$$D_n^{(d)} = diag(D_n^{(d-1)}) + \begin{pmatrix} -2Id_{n^{d-1}} & Id_{n^{d-1}} & 0 & \cdots & 0 \\ Id_{n^{d-1}} & -2Id_{n^{d-1}} & Id_{n^{d-1}} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & Id_{n^{d-1}} & -2Id_{n^{d-1}} & Id_{n^{d-1}} \\ 0 & \cdots & 0 & Id_{n^{d-1}} & -2Id_{n^{d-1}} \end{pmatrix}.$$

Let $\vec{u}^n(t)$ denote the $(n-1)^d$-dimensional vector defined by $\vec{u}^n(t)_{\underline{\mathbf{k}}}=u_n(t,\underline{\mathbf{x}}_{\underline{\mathbf{k}}})$, with

$$\underline{\mathbf{x}}_{\underline{\mathbf{k}}}=(x_{k_1},\cdots,x_{k_d}),\ \text{where}\ \ k_j\quad\text{is the unique integer such that}\ x_{k_j}=\frac{k_j-1}{n}$$

$$\text{and}\ k_j\in\{1,\cdots,n-1\}\ \text{is such that}\ \underline{\mathbf{k}}=(k_d-1)(n-1)^{d-1}+\cdots+(k_2-1)(n-1)+k_1.$$

Let $\mathcal{L}=\{\underline{\mathbf{x}}_{\underline{\mathbf{k}}}:\underline{\mathbf{k}}\in\{1,\cdots,(n-1)^d\}$, $\Box\underline{\mathbf{x}}_{\underline{\mathbf{k}}}$ be the lattice parallepiped of diagonal $\underline{\mathbf{x}}_{\underline{\mathbf{k}}}=(x_{k_1},\cdots,x_{k_d})$ and $(x_{k_1}+\frac{1}{n},\cdots,x_{k_d}+\frac{1}{n})$, and set $F^n(t,\underline{\mathbf{x}}_{\underline{\mathbf{k}}})=\int_{\Box\underline{\mathbf{x}}_{\underline{\mathbf{k}}}}dF(t,x)$. Given a function

$h : [0,+\infty[\times Q\times \mathbb{R}\to\mathbb{R}$, and $\vec{u}\in\mathbb{R}^r$, let $h(t,x,\vec{u}) = (h(t,x,u_1),\cdots,h(t,x,u_r))$. Then $\vec{u}^n(t)$ is solution to the following equation

$$d\vec{u}^n(t) = n^2\, D_n^{(d)}\vec{u}^n(t)\,dt + n\,\sigma(t,x,\vec{u}^n(t))\,dF(t,.) + b(t,x,\vec{u}^n(t)) \quad , \quad 1\le j\le (n-1)^d\,, \tag{2.10}$$

$\vec{u}^n(0) = \left(u_0\left(\frac{j}{n}\right)\right), 1\le j\le (n-1)^d)$. We then complete $u^n(t,.)$ from the lattice $\mathcal{L}$ to Q as follows. If $d=1$, set $u^n(t,0)=u^n(t,1)=0$, $u^n(t,\frac{j}{n}) = \vec{u}^n(t)_j$, $\kappa_n(y)=[ny]/n$, $\varphi_j^n(i/n)=\varphi_j(i/n)$ for $0\le i\le n$, and for $x\in]i/n,(i+1)/n[$, $0\le i<n$, let

$$\varphi_j^n(x) = \varphi_j\left(\frac{i}{n}\right) + (nx-i)\left[\varphi_j\left(\frac{i+1}{n}\right) - \varphi_j\left(\frac{i}{n}\right)\right]$$

and let

$$\lambda_j^n = -4\sin^2\left(\frac{j\pi}{2n}\right)n^2 = -j^2\pi^2 c_n^j \quad \text{with} \quad c_n^j = \sin^2\left(\frac{j\pi}{2n}\right)\left(\frac{j\pi}{2n}\right)^{-2}\in\left[\frac{4}{\pi^2},1\right],\ 1\le j\le n-1\,,$$

denote the eigenvalues of $n^2\,D_n = n^2\,D_n^{(1)}$; then for $t>0$, $x,y\in[0,1]$,

$$G_n(t,x,y) = \sum_{j=1}^{n-1}\exp(\lambda_j^n\,t)\varphi_j^n(x)\,\varphi_j(\kappa_n(y))\,. \tag{2.11}$$

In dimension $d\ge 2$, we also complete the solution $u^n(t,x)$ from $x\in\mathcal{L}$, defined as $u^n(t,\mathbf{x}_{\underline{\mathbf{k}}}) = \vec{u}^n(t)$ to $x\in Q$ by linear interpolation, interpolating inductively on the points (x,y) for $x\in\mathbb{R}^i$ and $y=(k_{i+1}/n,\cdots,k_d/n)$. The eigenvalues and eigenvectors of $n^2\,D_n^{(d)}$ are

$$\lambda_{\mathbf{k}}^n = \sum_{j=1}^d \lambda_{k_j}^n \quad \text{and} \quad \varphi_{\mathbf{k}}\left(\frac{k_1\pi}{n},\cdots,\frac{k_d\pi}{n}\right)\,,$$

and for $t>0$, x and $y\in Q$ if $\kappa_n(y)=(\kappa_n(y_1),\cdots,\kappa_n(y_d))$, let

$$(G_d)^n(t,x,y) = \sum_{\underline{\mathbf{k}}\in\{1,\cdots,n-1\}^d}\exp(\lambda_{\underline{\mathbf{k}}}^n t)\varphi_{\underline{\mathbf{k}}}^n(x)\varphi_{\underline{\mathbf{k}}}^n(\kappa_n(y))\,; \tag{2.12}$$

when $d=1$, simply set $G_1=G$ and $(G_1)^n=G^n$. Then the linear interpolation of $u^n(t,.)$ from the lattice $\mathcal{L}$ to $Q=[0,1]^d$ is solution to the evolution equation

$$\begin{aligned} u^n(t,x) \;&=\; \int_Q (G_d)^n(t,x,y)\,u_0(\kappa_n(y))\,dy + \int_0^t\int_Q (G_d)^n(t-s,x,y) \\ &\quad\times\Big[\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))F(ds,dy) + b(s,\kappa_n(y),u^n(s,\kappa_n(y))\,dsdy\Big] \end{aligned} \tag{2.13}$$

The $n\times n$ matrix $D_n = D_n^{(1)}$ associated with the homogeneous Neumann boundary conditions is defined by $D_n(1,1)=D_n(n,n)=-1$, $D_n(1,2)=D_n(n,n-1)=1$ and for $2\le i\le n-1$ and $1\le j\le n$, $D_n(i,i)=-2$, $D_n(i,j)=1$ if $|j-i|=1$ and $D_n(i,j)=0$ for $|j-i|\ge 2$. The inductive procedure used to construct $D_n^{(d)}$ is similar to the previous one; one replaces 1 by Id_{n^d}. Then the eigenvalues of $n^2\,D_n$ are $\lambda_j^n = -4\,n^2\,\sin^2(\frac{j\pi}{2n}) = -j^2\,\pi^2\,\tilde{c}_n^j$ with $\tilde{c}_n^j\in[\frac{2}{\pi^2},1]$. The corresponding normed eigenvectors $(e_j, 0\le j\le n-1)$ are again evaluations of φ_j. More precisely, $e_j(k) = \frac{1}{\sqrt{n}}\,\varphi_j(\frac{2k-1}{2n})$ for $0\le j\le n-1$ and $1\le k\le n$. The eigenvalues $\lambda_{\frac{\mathbf{k}}{n}}$ and the eigenfunctions $\varphi_{\underline{\mathbf{k}}}$ of $n^2\,D_n^{(d)}$ are defined in a way similar to the Dirichlet case, taking sums over $\underline{\mathbf{k}}\in\{1,\cdots,n\}^d$; formulas similar to (2.11) and (2.12) still hold and (2.13) is unchanged.

2.2 Implicit space-time discretization scheme

We now introduce a space-time discretization scheme. Given $T > 0$, $n, m \geq 1$ we use the space mesh $1/n$ and the time mesh T/m, set $t_i = iTm^{-1}$ for $0 \leq i \leq m$ and replace the time derivative by a backward difference. Thus for $d = 1$, in the case of Dirichlet's homogeneous boundary conditions, set $\vec{u}_0 = (u_0(j/n), 1 \leq j \leq n-1)$ and for $i \leq m$, set $\vec{u}_i = (u^{n,m}(iTm^{-1}, jn^{-1}), 1 \leq j \leq n-1)$, and for $g = \sigma$ and $g = b$ let $g(t_i, ., \vec{u}_i) = g((t_i, jn^{-1}, (u^{n,m}(t_i, jn^{-1}))\,, 1 \leq j \leq n-1)$. Let $\square_{n,m}F(t_i, .)$ denote the $(n-1)$-dimensional Gaussian vector of space-time increments of F on the space-time grid, i.e., for $1 \leq j \leq n-1$, set

$$\square_{n,m}F(t_i, j) = nmT^{-1}[F(t_{i+1}, (j+1)n^{-1}) - F(t_i, (j+1)n^{-1}) - F(t_{i+1}, jn^{-1}) + F(t_i, jn^{-1})]\,;$$

then for every $0 \leq i < m$

$$\vec{u}_{i+1} = \vec{u}_i + n^2\, Tm^{-1}\, D_n \vec{u}_{i+1} + Tm^{-1}\left[\sigma(t_i, ., \vec{u}_i)\square_{n,m}F(t_i, .) + b(t_i, ., \vec{u}_i)\right]\,. \tag{2.14}$$

Since $Id - Tm^{-1}D_n$ is invertible,

$$\vec{u}_{i+1} = (Id - Tm^{-1}D_n)^{-(i+1)}\vec{u}_0 + \sum_{k=0}^{i}(Id - Tm^{-1})^{-(i-k-1)}\left[\sigma(t_k, ., \vec{u}_k)\square_{n,m}F(t_k, .) + b(t_k, ., \vec{u}_k)\right]\,. \tag{2.15}$$

If $d \geq 2$, we set $\square_{n,m}F(t_i, \underline{\mathbf{x}_\mathbf{k}}) = n^d mT^{-1}\int_{t_i}^{t_{i+1}}\int_{\square \underline{\mathbf{x}_\mathbf{k}}} dF(t, x)$, and for homogeneous Dirichlet (resp. Neumann) boundary conditions, define similarly $\vec{u}_{i+1}$ as the $(n-1)^d$-dimensional (resp. n^d-dimensional) vector such that (2.15) holds with $D_n^{(d)}$ instead of D_n. We only describe the scheme in the case of Dirichlet's conditions; obvious changes will give it in the case of Neumann's conditions. The process $u^{n,m}$ is defined on the space-time lattice $\mathcal{L}_T = \{(t_i, \underline{\mathbf{x}_\mathbf{k}}) : 0 \leq i \leq m\,, \underline{\mathbf{k}} \in \{1, \cdots, n-1\}^d\}$ as $(u^{n,m}(t_i, \underline{\mathbf{x}_\mathbf{k}})\,, 0 \leq i \leq m\,, \underline{\mathbf{k}} \in \{1, \cdots, n-1\}^d\}) = \vec{u}_i$; it is then extended to the time lattice (t_i, x), $0 \leq i \leq m$, $x \in Q$ as in the previous subsection, and then extended to $[0, T] \times Q$ by time linear interpolation. Since $\lambda_{\underline{\mathbf{k}}} = \sum_{i=1}^d \lambda_{k_i}^n$ and $\varphi_{\underline{\mathbf{k}}}(\underline{\mathbf{x}_\mathbf{k}})$ are the eigenvalues and eigenvectors of $D_n^{(d)}$, if

$$(G_d)^{n,m}(t, x, y) = \sum_{\underline{\mathbf{k}} \in \{1, \cdots, n-1\}^d} \left(1 - Tm^{-1}\lambda_{\underline{\mathbf{k}}}\right)^{-[mtT^{-1}]}\ \varphi_{\underline{\mathbf{k}}}^n(x)\ \varphi_{\underline{\mathbf{k}}}(\kappa_n(y))\,, \tag{2.16}$$

then for $t = iTm^{-1}$, $1 \leq i \leq m$, if for $s \in [0, T]$, one sets $\Lambda_m(s) = [msT^{-1}]m^{-1}$ one has:

$$\begin{aligned} u^{n,m}(t, x) \;&=\; \int_Q (G_d)^{n,m}(t, x, y)u_0(\kappa_n(y))\, dy + \int_0^t \int_Q (G_d)^{n,m}(t - s + Tm^{-1}, x, y) \\ &\quad \times \Big[\sigma(\Lambda_m(s), \kappa_n(y), u^{n,m}(\Lambda_m(s), \kappa_n(y)))\ F(ds, dy) \\ &\quad\quad + b(\Lambda_m(s), \kappa_n(y), u^{n,m}(\Lambda_m(s), \kappa_n(y)))dyds\Big]\,. \end{aligned} \tag{2.17}$$

Again for $d = 1$, let $G^{n,m} = (G_1)^{n,m}$.

2.3 Explicit schemes

For $T > 0$, a space mesh n^{-1} and a time mesh Tm^{-1}, we now replace the time derivative by a forward difference. Thus if u_m^n denotes the approximating process defined for $t = t_i = iTm^{-1}$ and $x_{k_j} \in \{1, \cdots, n-1\}$, setting $\vec{u}_i = u_m^n(t_i, .)$, we have

$$\vec{u}_{i+1} = \vec{u}_i + n^2\, Tm^{-1}\, D_n^{(d)} \vec{u}_i + Tm^{-1}\left[\sigma(t_i, ., \vec{u}_i)\square_{n,m}F(t_i, .) + b(t_i, ., \vec{u}_i)\right]\,. \tag{2.18}$$

In the case of homogeneous Dirichlet boundary conditions, let $(G_d)^n_m(t,x,y)$ denote the corresponding approximation of the Green function G_d defined by

$$(G_d)^n_m(t,x,y) = \sum_{\underline{\mathbf{k}}\in\{1,\cdots,n-1\}^d} \big(1+Tm^{-1}\lambda_{\underline{\mathbf{k}}}\big)^{[mtT^{-1}]}\ \varphi^n_{\underline{\mathbf{k}}}(x)\ \varphi_{\underline{\mathbf{k}}}(\kappa_n(y))\,; \tag{2.19}$$

again for $d=1$, let $G^n_m = (G_1)^n_m$. Then for $t=t_i=iTm^{-1}$, when completing the solution $u^n_m(t_i,.)$ from the space lattice $\mathcal{L}$ to Q, we obtain the solution to the following equation

$$\begin{aligned} u^n_m(t,x) &= \int_Q (G_d)^n_m(t,x,y)u_0(\kappa_n(y))\,dy + \int_0^t\int_Q (G_d)^n_m(t-s+Tm^{-1},x,y) \\ &\quad\times\Big[\sigma(\Lambda_m(s),\kappa_n(y),u^n_m(\Lambda_m(s),\kappa_n(y)))\ F(ds,dy) \\ &\qquad + b(\Lambda_m(s),\kappa_n(y),u^n_m(\Lambda_m(s),\kappa_n(y)))dyds\Big]\,. \end{aligned} \tag{2.20}$$

We then complete the process $u^n_m(.,x)$ by time linear interpolation and obvious changes yield the explicit scheme for homogeneous Neumann boundary conditions.

3 Convergence Results for the discretization schemes

In this section, we study the speed of convergence for the d-dimensional space scheme and then of the d-dimensional implicit and explicit space-time schemes.
We first prove moment estimates of the solutions u, u^n, $u^{n,m}$ and u^n_m uniformly in n,m.

Proposition 3.1 *Let $u_0\in\mathcal{C}(Q)$ satisfy the homogeneous Neumann or Dirichlet boundary conditions, and suppose that the coefficients σ and b satisfy the conditions (2.4) and (2.5); then the equation (2.9) (resp. (2.13), (2.17) and (2.20)) has a unique solution u (resp. u^n, $u^{n,m}$ and u^n_m) such that for every $p\in[1,+\infty[$ and $T>0$:*

$$\sup_{n\ge1}\sup_{m\ge1}\sup_{0\le t\le T}\sup_{x\in Q}\mathbb{E}\big(|u(t,x)|^{2p}+|u^n(t,x)|^{2p}+|u^{n,m}(t,x)|^{2p})+|u^n_m(t,x)|^{2p}\big)<+\infty\,. \tag{3.1}$$

Proof: The existence and uniqueness of the solution to (2.9) and the control of moments stated in (3.1) for $u(t,x)$ have been proved in [11] and [4]. The proofs of the existence of u^n to (2.13), $u^{n,m}$ to (2.17) and u^n_m to (2.20) are similar; we briefly sketch the argument for u^n. Consider the Picard iteration scheme $u^n(0)(t,x)=\int_Q (G_d)^n(t,x,y)\,u_0(\kappa_n(y))\,dy$ and for $k\ge0$,

$$\begin{aligned} u^n(k+1)(t,x) &= u^n(0)(t,x)+\int_0^t\int_Q (G_d)^n(t-s,x,y)\,\big[\sigma(s,\kappa_n(y),u^n(k)(s,\kappa_n(y)))\,F(ds,dy) \\ &\quad + b(s,\kappa_n(y),u^n(k)(s,\kappa_n(y)))\,dyds\,\big]\,. \end{aligned}$$

Then Burkholder's and Hölder's inequalities, the Lipschitz property (2.5) on the coefficients σ and b, (A.19) and (A.20) imply the existence of $\lambda\in]0,1[$ such that for any $p\in[1,+\infty[$ there exists a constant $C_p>0$ such that for every $t\in[0,T]$, $n\ge1$ and $k\ge1$:

$$\sup_{x\in Q}\mathbb{E}(|u^n(k+1)(t,x)-u^n(k)(t,x)|^{2p}) \le C_p\Big(\int_0^t \sup_{x\in Q}\|\,|(G_d)^n(t-s,x,.)|\,\|_{(\alpha)}^2\,ds\Big)^{p-1}$$
$$\times\int_0^t \sup_{x\in Q}\|\,|(G_d)^n(t-s,x,.)|\,\|_{(\alpha)}^2 \sup_{y\in Q}\mathbb{E}(|u^n(k)(s,y)-u^n(k-1)(s,y)|^{2p})\,ds$$
$$+\Big(\int_0^t \sup_{x\in Q}\|(G_d)^n(t-s,x,.)\|_1^2\,ds\Big)^{2p-1}$$
$$\times\int_0^t \sup_{x\in Q}\|(G_d)^n(t-s,x,.)\|_1 \sup_{y\in Q}\mathbb{E}(|u^n(k)(s,y)-u^n(k-1)(s,y)|^{2p})\,ds$$
$$\le C_p\int_0^t (t-s)^{-\lambda}\sup_{y\in Q}\mathbb{E}(|u^n(k)(s,y)-u^n(k-1)(s,y)|^{2p})\,ds\,.$$

A similar argument using (2.4) instead of (2.5) shows that

$$\sup_{n\ge 1}\sup_{x\in Q}\mathbb{E}(|u^n(1)(t,x)|^{2p}) \le C_p\big(1+\int_0^t (t-s)^{-\lambda}\,\|u_0\|_\infty\,ds\big) < +\infty\,,$$

while $\sup_{n\ge1}\sup_{t\in[0,T]}\sup_{x\in Q} u^n(0)(t,x)\le \|u_0\|_\infty$. Thus Lemma 3.3 in [15] and standard arguments show that the sequence $u^n(k)(.), k\ge 0$ converges in $L^{2p}(\Omega\times[0,T]\times Q)$ to the solution to (2.13), and that this solution is unique. Finally, using again Burkholder's and Hölder's inequalities, (2.4), (A.19) and (A.20), we deduce that for some $\lambda\in]0,1[$ and $p\in[1,+\infty[$, there exists a constant $C>0$ such that for every $t\in[0,T]$ and $n\ge 1$:

$$\begin{aligned}\sup_{x\in Q}\sup_{n\ge1}\mathbb{E}(|u^n(t,x)|^{2p}) &\le C_p\Big[\|u_0\|_\infty^{2p}+\int_0^t\big[\|\,|(G_d)^n(t-s,x,.)|\,\|_{(\alpha)}^2+\|(G_d)^n(t-s,x,.)\|_1\big]\\ &\qquad\times\big[1+\sup_{y\in Q}\sup_{n\ge1}\mathbb{E}(|u^n(s,y)|^{2p})\big]\,ds\\ &\le C_p+C_p\int_0^t(t-s)^{-\lambda}\sup_{y\in Q}\sup_{n\ge1}\mathbb{E}(|u^n(s,y)|^{2p})\,ds\end{aligned}$$

and Gronwall's lemma shows that (3.1) holds for u^n. A similar argument based on the version of Gronwall's lemma stated in [7] Lemma 3.4, (A.25) and (A.26) proves that (3.1) also holds for $u^{n,m}$ or u^n_m; this concludes the proof of the proposition. □

We now prove Hölder regularity properties of the trajectories of u and u^n. Note that for u, a similar result has been proved in [14] for the heat equation with free boundary with a perturbation driven by a Gaussian process with a more general space covariance; see also [3] for a related result in the case of a more general even order differential operator.

Proposition 3.2 *Suppose that the coefficients b and σ satisfy the Lipschitz property (2.5), that the initial condition u_0 satisfies the homogeneous Dirichlet or Neumann boundary condition.*

(i) Suppose furthermore that $u_0\in\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$ and fix $T>0$. Then, for every $p\in[1,+\infty[$, there exists a constant C such that for $x,x'\in Q$ and $0\le t<t'\le T$,

$$\sup_{0\le t\le T}\mathbb{E}\left(|u(t,x)-u(t,x')|^{2p}\right)\le C|x'-x|^{p(2-\alpha)}\,, \tag{3.2}$$

$$\sup_{x\in Q}\mathbb{E}\left(|u(t',x)-u(t,x)|^{2p}\right)\le C|t'-t|^{p(1-\frac{\alpha}{2})}\,. \tag{3.3}$$

(ii) Suppose furthermore that $u_0 \in \mathcal{C}^2(Q)$; then for every $p \in [1, +\infty[$, there exists a constant C such that for $x, x' \in Q$ and $0 \le t < t' \le T$,

$$\sup_{n\ge 1}\sup_{x\in Q} \mathbb{E}\left(|u^n(t',x) - u^n(t,x)|^{2p}\right) \le C|t'-t|^{p(1-\frac{\alpha}{2})}. \tag{3.4}$$

Proof: (i) We sketch the proof of (3.2) and (3.3) for the sake of completeness. For every $t > 0$ and $x \in Q$, set $v(t,x) = \int_Q G_d(t,x,y)u_0(y)dy$ and let $w(t,x) = u(t,x) - v(t,x)$. We at first prove the corresponding regularity for v. As in Lemma A.2 of [1], the semi-group property of G_d and (A.5) imply that

$$\begin{aligned}
|v(t',x) - v(t,x)| &= \left| \int_Q G_d(t,x,y) \int_Q G_d(t'-t,y,z)[u_0(z) - u_0(y)]dzdy \right| \\
&\le C \int_Q |G_d(t,x,y)| \int_Q |G_d(t'-t,y,z)||y-z|^{1-\frac{\alpha}{2}} dz\, dy \\
&\le C \int_Q |G_d(t,x,y)| \int_{\mathbb{R}^d} |t'-t|^{-\frac{d}{2}}\, e^{-c\frac{|y-z|^2}{t'-t}}\, |y-z|^{1-\frac{\alpha}{2}}\, dz\, dy \qquad (3.5)\\
&\le C(t'-t)^{\frac{1}{2}-\frac{\alpha}{4}}. \qquad (3.6)
\end{aligned}$$

A similar computation shows that for $0 = t < t'$,

$$\begin{aligned}
|v(t',x) - u_0(x)| &\le C \int_Q |G_d(t',x,y)|\, |u_0(y) - u_0(z)|\, dy \\
&\le C\, t'^{\frac{1}{2}-\frac{\alpha}{4}}.
\end{aligned}$$

On the other hand, for Dirichlet's boundary conditions, $G(t,x,y) = \phi_t(x+y) - \phi_t(x-y)$, where $\phi_t(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{(x-2n)^2}{4t}\right)$; we remark that, since $u_0(y) = 0$ if $y_i \in \{0,1\}$ for some $i \in \{1, \cdots, d\}$, setting $\eta_i = x'_i - x_i$, $1 \le i \le d$, $\hat{z}_i = (z_j, 1 \le j \le d, j \ne i)$, denoting by e_i the *ith* unit vector of the canonical basis and assuming without loss of generality that $\eta_i > 0$, we deduce:

$$\begin{aligned}
|v(t,x) - v(t,x')| \le{}& \sum_{i=1}^{d} \int_{Q^{d-1}} d\hat{z}_i \left(\prod_{j=1}^{i-1} |G(t,x_j,z_j)| \right) \left(\prod_{j=i+1}^{d} |G(t,x'_j,z_j)| \right) \\
&\times \Bigg[\left| \int_{\eta_i}^{1} \phi_t(x_i+z_i)[u_0(z) - u_0(z-\eta_i e_i)]dz_i \right| + \left| \int_0^{1-\eta_i} \phi_t(x_i-z_i)[u_0(z) - u_0(z+\eta_i e_i)]dz_i \right| \\
&\quad + \left| \int_0^{\eta_i} \phi_t(x_i+z_i)u_0(z)dz_i \right| + \left| \int_{1-\eta_i}^{1} \phi_t(x_i+z_i)u_0(z)dz_i \right| \Bigg] \\
\le{}& C\, |\eta|^{1-\frac{\alpha}{2}}. \qquad (3.7)
\end{aligned}$$

The case of Neumann's boundary conditions is treated in [1], Lemma A2.

We now prove (3.2); since σ and b satisfy (2.5), Burkholder's inequality, then Hölder's inequality with respect to $\|\, |G_d(t-s,x,\cdot) - G_d(t-s,x'\cdot)|\, \|_{(\alpha)}^2\, ds$ and $|G_d(t-s,x,y) - G_d(t-s,x'y)|)\, dy\, ds$, Schwarz's inequality with respect to P along with inequalities (A.2), (A.12) and

(3.1) yield

$$\mathbb{E}\left(|u(t,x)-u(t,x')|^{2p}\right) \le C_p\Big\{ |v(t,x)-v(t,x')|^{2p}$$

$$+\Big(\int_0^t \|\,|G_d(t-s,x,\cdot)-G_d(t-s,x',\cdot)|\,\|_{(\alpha)}^2\,ds\Big)^{p-1}$$

$$\times\int_0^t ds\iint_{Q^2}\mathbb{E}\left(|G_d(t-s,x,y)-G_d(t-s,x',y)|\;|\sigma(s,y,u(s,y))|^p\right)|y-z|^{-\alpha}$$

$$\times|G_d(t-s,x,z)-G_d(t-s,x',z)||\sigma(s,z,u(s,z))|^p))\;dy\,dz$$

$$+\Big(\int_0^t\int_Q |G_d(t-s,x,y)-G_d(t-s,x',y)|dyds\Big)^{2p-1}$$

$$\times\int_0^t\int_Q |G_d(t-s,x,y)-G_d(t-s,x',y)|\;|b(s,y,u(s,y))|^{2p})\;dy\,ds\Big\}$$

$$\le\; C_p\left\{|v(t,x)-v(t,x')|^{2p}+\left(|x-x'|^{(2-\alpha)p}+|x-x'|^{2p}\right)\sup_{(s,y)\in[0,t]\times Q}\mathbb{E}(|u(s,y)|^{2p})\right\}$$

$$\le\; C_p\,|x-x'|^{(2-\alpha)p}\,. \tag{3.8}$$

Finally, in order to prove (3.3), we write

$$u(t',x)-u(t,x)=v(t',x)-v(t,x)+u_1(t,t',x)+u_2(t,t',x)$$

where

$$u_1(t,t',x) \;=\; \int_0^t\int_Q [G_d(t'-s,x,y)-G_d(t-s,x,y)]$$

$$\times\{b(s,y,u(s,y))\,dy\,ds+\sigma(s,y,u(s,y))\,F(dy,ds)\},$$

$$u_2(t,t',x) \;=\; \int_t^{t'}\int_Q G_d(t'-s,x,y)\;\{b(s,y,u(s,y))\,dy\,ds+\sigma(s,y,u(s,y))\,F(dy,ds)\}\,.$$

The inequalities (A.3) and (A.13) (with $\mu=1-\frac{\alpha}{2}$), and a computation similar to that used to prove (3.2) show that

$$\sup_{x\in Q}\mathbb{E}(|u_1(t,t',x)|^{2p})\le C_p(t'-t)^{p(1-\frac{\alpha}{2})}\,. \tag{3.9}$$

Furthermore, Burkholder's, Hölder's and Schwarz's inequalities, together with (3.1), (A.4) and (A.14) yield

$$\sup_{x\in Q}\mathbb{E}(|u_1(t,t',x)|^{2p})$$

$$\le\; C_p\Big\{\Big(\int_t^{t'}\|\,|G_d(t'-s,x,\cdot)|\,\|_{(\alpha)}^2\,ds\Big)^p+\Big(\int_t^{t'}|G_d(t'-s,x,y|\,dyds\Big)^{2p}\Big\}$$

$$\times\Big[1+\sup_{s\le t'}\sup_{y\in Q}\mathbb{E}(|u(s,y)|^{2p})\Big]\le C_p\,(t'-t)^{p(1-\frac{\alpha}{2})}\,. \tag{3.10}$$

The inequalities (3.6), (3.9) and (3.10) yield (3.3).

(ii) The argument is similar ; for every $t>0$, let $v^n(t,x)=\int_Q(G_d)(t,x,y)\,u_0(\kappa_n(y)\,dy$ and $w^n(t,x)=u^n(t,x)-v^n(t,x)$. Since $(G_d)^n$ is the fundamental solution $\frac{\partial}{\partial t}-\Delta_n=0$, where

$$\Delta_nU(y)=n^2\sum_{i=1}^d\Big[U\Big(\sum_{j\ne i}\frac{[ny_j]}{n}\,e_j+\frac{[ny_i]+1}{n}\,e_i\Big)-2\,U\Big(\frac{[ny]}{n}\Big)+U\Big(\sum_{j\ne i}\frac{[ny_j]}{n}\,e_j+\frac{[ny_i]-1}{n}\,e_i\Big)\Big]\,, \tag{3.11}$$

and $(e_i, 1 \le i \le d)$ denotes the canonical basis of $\mathbb{R}^d$, then if $u_0 \in \mathcal{C}^2(Q)$,

$$v^n(t,x) = u_0(x) + \int_0^t \int_Q (G_d)^n(t,x,y)\, \Delta_n u_0(y)\, dy\,. \tag{3.12}$$

Thus, using the fact that $\Delta_n u_0$ is bounded if $u_0 \in \mathcal{C}^2(Q)$, and (A.40), we deduce that for any $\lambda > 0$:

$$\sup_{n\ge 1} \sup_{x\in Q} |v_n(t,x) - v_n(t',x)| \le \int_t^{t'} \int_Q |(G_d)^n(s,x,y)|\, |\Delta_n u_0(y)|\, dy \le C\, |t'-t|^{1-\lambda}\,. \tag{3.13}$$

Furthermore, for any $p \in [1,+\infty[$, there exists $C_p > 0$ such that for any $t, t' \in [0,T]$ with $t < t'$, $\sup_{n\ge1} \sup_{x\in Q} \mathbb{E}(|w^n(t,x) - w^n(t',x)|^{2p}) \le C_p \sum_{i=1}^2 T_i(t,t')$, where (with the convention that $(G_d)^n(r,x,y) = 0$ for $r < 0$,

$$\begin{aligned} T_1(t,t') &= \sup_{n\ge1}\sup_{x\in Q} \Big(\int_0^{t'} \| \,|(G_d)^n(t'-s,x,.) - (G_d)^n(t,x,.)|\, \|_{(\alpha)}^2\, ds\Big)^{p-1} \\ &\quad \times \sup_{n\ge1}\sup_{x\in Q} \int_0^{t'} \| \,|(G_d)^n(t'-s,x,.) - (G_d)^n(t,x,.)|\, \|_{(\alpha)}^2 \sup_{z\in Q} \big[1 + \mathbb{E}(|u^n(t,z)|^{2p})\big]\, ds \\ T_2(t,t') &= \sup_{n\ge1}\sup_{x\in Q} \Big(\int_0^{t'} \|(G_d)^n(t'-s,x,.) - (G_d)^n(t,x,.)\|_1\, ds\Big)^{2p-1} \\ &\quad \times \sup_{n\ge1}\sup_{x\in Q} \int_0^{t'} \||(G_d)^n(t'-s,x,.) - (G_d)^n(t,x,.)|\|_1 \sup_{z\in Q} \big[1 + \mathbb{E}(|u^n(t,z)|^{2p})\big]\, ds\,. \end{aligned}$$

The inequalities (A.39)-(A.42) and (3.1) show the existence of $C_p > 0$ such that for any $0 \le t < t' \le T$:

$$\sup_{n\ge1}\sup_{x\in Q} \mathbb{E}(|w^n(t,x) - w^n(t',x)|^{2p}) \le C_p\, |t'-t|^{p(1-\frac{\alpha}{2})}\,. \tag{3.14}$$

The inequalities (3.13) and (3.14) conclude the proof of (3.4). □

The first convergence result of this section is that of u^n to u.

Theorem 3.3 *Let σ and b satisfy the conditions (2.4) and (2.6), u and u^n be the solutions to (2.9) and (2.13) respectively, where the Green functions G_d and $(G_d)^n$ are defined with the homogeneous Neumann or Dirichlet boundary conditions on Q.*

(i) If the initial condition u_0 belongs to $\mathcal{C}^3(Q)$, then for every $T > 0$ and $p \in [1+\infty[$, there exists a constant $C_p(T) > 0$ such that:

$$\sup_{(t,x)\in[0,T]\times Q} \mathbb{E}\left(|u(t,x) - u^n(t,x)|^{2p}|\right) \le C_p(T)\, n^{-(2-\alpha)\,p}\,. \tag{3.15}$$

(ii) If the initial condition u_0 belongs to $\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$, then there exists $\nu > 0$ such that given any $p \in [1,+\infty[$, there exists a constant $C_p > 0$ such that, for every $t > 0$:

$$\sup_{x\in Q} \mathbb{E}\left(|u(t,x) - u^n(t,x)|^{2p}|\right) \le C_p\, t^{-\nu}\, n^{-(2-\alpha)\,p}\,. \tag{3.16}$$

(iii) Finally, if u_0 belongs to $\mathcal{C}_0(Q)$, then for all $p \in [1,+\infty[$, $\sup_{(t,x)\in[0,T]\times Q} \mathbb{E}\left(|u(t,x) - u^n(t,x)|^{2p}|\right)$ converges to 0 an $n \to +\infty$, and the sequence $u^n(t,x)$ converges a.s. to $u(t,x)$ uniformly on $[0,T]\times Q$.

Proof: For the sake of simplicity, we suppose that the boundary conditions are the homogeneous Dirichlet ones; an easy modification yields similar result for the homogeneous Neumann boundary conditions. As in [6], set $u(t,x)=v(t,x)+w(t,x)$, $u^n(t,x)=v^n(t,x)+w^n(t,x)$, where $v(t,x)=\int_Q G_d(t,x,y)\,u_0(y)\,dy$ and $v^n(t,x)=\int_Q (G_d)^n(t,x,y)\,u_0(\kappa_n(y))\,dy$. If $u_0\in\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$ (and hence is bounded), using (4.1) and (A.19), we deduce that for any $\lambda\in]0,1[$, there exists $\mu>0$, $C>0$ such that for $t>0$, $\nu=\lambda\vee\mu$,

$$\begin{aligned}\sup_{x\in Q}|v(t,x)-v^n(t,x)| &\le \int_Q \Big[\,|G_d t,x,y)-(G_d)^n(t,x,y)|\,|u_0(y)| \\ &\qquad +|(G_d)^n t,x,y)|\,|u_0(y)-u_0(\kappa_n(y))|\,\Big]\,dy \\ &\le C\,n^{-(1-\frac{\alpha}{2})}\,(1+t^{-\mu}+t^{-\lambda})\,e^{-ct}\le C\,(1+t^{-\nu})\,e^{-ct}\,n^{-(1-\frac{\alpha}{2})} \quad (3.17)\end{aligned}$$

If $u_0\in\mathcal{C}^3(Q)$, then since G_d (resp. $(G_d)^n$) is the fundamental solution of $\frac{\partial}{\partial t}-\Delta=0$ (resp $\frac{\partial}{\partial t}-\Delta_n=0$), where Δ_n is defined by (3.11), integrating by parts we deduce that $v(t,x)=u_0(x)+\int_0^t\int_Q G_d(s,x,y)\,\Delta u_0(y)\,dy$ and $v^n(t,x)=u_0(x)+\int_0^t\int_Q (G_d)^n(s,x,y)\,\Delta_n u_0(y)\,dy$. Hence $|v(t,x)-v^n(t,x)|\le\sum_{i=1}^3 A_i(t,x)$, where

$$\begin{aligned}A_1(t,x) &= |u_0(t,x)-u_0(\kappa_n(x))|\,, \\ A_2(t,x) &= \Big|\int_0^t\int_Q [G_d(s,x,y)-(G_d)^n(s,x,y)]\,\Delta u_0(y)\,dy\,ds\Big|\,, \\ A_3(t,x) &= \Big|\int_0^t\int_Q (G_d)^n(s,x,y)\,[\Delta u_0(y)-\Delta_n u_0(\kappa_n(y))]\,dy\,ds\Big|\,.\end{aligned}$$

Since Δu_0 is bounded and $\|u_0(.)-u_0(\kappa_n(.))\|_\infty+\|\Delta u_0(.)-\Delta_n u_0(\kappa_n(.))\|_\infty\le C\,n^{-1}$, the inequalities (A.19) and (4.1) imply

$$\sup_{(t,x)\in[0,+\infty[\times Q}|v(t,x)-v^n(t,x)|\le C\,n^{-1}\,. \quad (3.18)$$

Furthermore, for every $0<t<T$, $\sup_{x\in Q}\mathbb{E}(|w(t,x)-w^n(t,x)|^{2p})\le C\sum_{i=1}^6 B_i(t)$, where

$$\begin{aligned}B_1(t) &= \sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y)\,[\sigma(s,y,u(s,y))-\sigma(s,\kappa_n(y),u(s,\kappa_n(y))]\,F(ds,dy)\Big|^{2p}\Big)\,, \\ B_2(t) &= \sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y) \\ &\qquad\times[\sigma(s,\kappa_n(y),u(s,\kappa_n(y)))-\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))]\,F(ds,dy)\Big|^{2p}\Big)\,, \\ B_3(t) &= \sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q [G_d(t-s,x,y)-(G_d)^n(t-s,x,y)] \\ &\qquad\times\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))]\,F(ds,dy)\Big|^{2p}\Big)\,, \\ B_4(t) &= \sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y)\,[b(s,y,u(s,y))-b(s,\kappa_n(y),u(s,\kappa_n(y))]\,dy\,ds\Big|^{2p}\Big)\,,\end{aligned}$$

$$
\begin{aligned}
B_5(t) &= \sup_{x\in Q} \mathbb{E}\Big(\Big| \int_0^t \int_Q G_d(t-s,x,y) \\
&\qquad \times [b(s,\kappa_n(y),u(s,\kappa_n(y))) - b(s,\kappa_n(y),u^n(s,\kappa_n(y))]\, dy\, ds\Big|^{2p}\Big)\,, \\
B_6(t) &= \sup_{x\in Q} \mathbb{E}\Big(\Big| \int_0^t \int_Q [G_d(t-s,x,y) - (G_d)^n(t-s,x,y)]\, b(s,\kappa_n(y),u^n(s,\kappa_n(y))]\, dy\, ds\,\Big|^{2p}\Big)\,.
\end{aligned}
$$

Burkholder's inequality, (A.1), Hölder's inequality with respect to the measure $|G_d(t-s,x,y)|$ $|y-z|^{-\alpha}\,|G_d(t-s,s,z)|\,ds\,dy\,dz$, Fubini's theorem, (2.6), Schwarz's inequalities and (3.2) imply that

$$
\begin{aligned}
B_1(t) &\le C_p \sup_{x\in Q} \mathbb{E}\Big(\Big| \int_0^t \int_Q \int_Q |G_d(t-s,x,y)|\, |y-z|^{-\alpha}\, |G_d(t-s,x,z)| \\
&\qquad \times \Big(n^{-\frac{2-\alpha}{2}} + |u(s,y) - u(s,\kappa_n(y)|\Big)\Big(n^{-\frac{2-\alpha}{2}} + |u(s,z)-u(s,\kappa_n(z)|\Big)\, dy\, dz\, ds\,\Big|^p\Big) \\
&\le C_p \Big(\int_0^t \sup_{x\in Q} \|G_d(t-s,x,.)\|^2_{(\alpha)}\, ds\Big)^{p-1} \int_0^t \sup_{x\in Q} \int_Q \int_Q |G_d(t-s,x,y)|\, |y-z|^{-\alpha} \\
&\qquad \times |G_d(t-s,x,z)|\,\Big[n^{-p\,(2-\alpha)} + \sup_{(s,\xi)\in[0,t]\times Q} \mathbb{E}(|u(s,\xi) - u(s,\kappa_n(\xi))|^{2p})\Big]\, dy\, dz\, ds \\
&\le C_p\, n^{-p(2-\alpha)}\,.
\end{aligned} \tag{3.19}
$$

A similar argument based on (A.1) and (2.6) implies that

$$
\begin{aligned}
B_2(t) &\le C_p \Big(\int_0^t \sup_{x\in Q} \|\,|G_d(t-s,x,.)|\,\|^2_{(\alpha)}\, ds\Big)^{p-1} \int_0^t \int_Q \int_Q |G_d(t-s,x,y)|\, |y-z|^{-\alpha} \\
&\qquad \times |G_d(t-s,x,z)| \sup_{x\in Q} \mathbb{E}\Big(|u(s,x) - u^n(s,x)|^{2p}\Big)\, dy\, dz\, ds \\
&\le C_p \int_0^t (t-s)^{-\frac{\alpha}{2}} \sup_{x\in Q} |v(s,x) - v^n(s,x)|^{2p}\, ds \\
&\quad + C_p \int_0^t (t-s)^{-\frac{\alpha}{2}} \sup_{x\in Q} \mathbb{E}(|w(s,x) - w^n(s,x)|^{2p})\Big]\, ds\,.
\end{aligned} \tag{3.20}
$$

Again, a similar argument based on (4.2), (2.4) and (3.1) implies that for $t\in[0,T]$,

$$
\begin{aligned}
B_3(t) \le C_p \Big(\int_0^t \sup_{x\in Q} \|\,|G_d(t-s,x,.) - (G_d)^n(t-s,x,.)|\,\|^2_{(\alpha)}\, ds\Big)^p \\
\times \Big(1 + \sup_{y\in Q}\sup_{s\le T} \mathbb{E}(|u^n(s,y)|^{2p}\Big)\, ds \le C_p\, n^{-(2-\alpha)\,p}\,.
\end{aligned} \tag{3.21}
$$

The deterministic integrals are easier to deal with; using Hölder's inequality with respect to the measure $|G(t-s,x,y)|\,dy\,ds$, (A.5), (2.6) and (3.2) we deduce that

$$
\begin{aligned}
B_4(t) &\le C_p \sup_{x\in Q} \mathbb{E}\Big(\Big| \int_0^t \int_Q |G_d(t-s,x,y)|\, dy\, ds\Big)^{2p-1} \\
&\qquad \times \int_0^t \int_Q |G_d(t-s,x,y)|\,\Big(n^{-(2-\alpha)\,p} + \mathbb{E}(|u(s,y) - u(s,\kappa_n(y))|^{2p})\Big)\, dy\, dz\, ds \\
&\le C\, n^{-p(2-\alpha)}\,.
\end{aligned} \tag{3.22}
$$

Similarly, the inequalities (2.6) and (A.5) imply

$$\begin{aligned} B_5(t) &\leq C\Big(\int_0^t\int_Q |G_d(t-s,x,.)|\,\mathbb{E}(|u(s,y)-u^n(s,y)|^{2p})\,dy\,ds \\ &\leq C\int_0^t \sup_{x\in Q}|v(s,x)-v^n(s,x)|^{2p}\,ds + C\int_0^t \sup_{x\in Q}\mathbb{E}(|w(s,x)-w^n(s,x)|^{2p}\,ds\,, \end{aligned} \tag{3.23}$$

while (2.4), (3.1) and (4.1) yield

$$\begin{aligned} B_6(t) &\leq C_p \sup_{x\in Q}\Big(\int_0^t\int_Q \big|G_d(t-s,x,y)-(G_d)^n(t-s,x,y)\big|\,dy\,ds\Big)^{2p-1} \\ &\quad\times \int_0^t\int_Q \big|G_d(t-s,x,y)-(G_d)^n(t-s,x,y)\big|\,\Big(1+\sup_{x\in Q}\sup_{s\leq T}\mathbb{E}(|u^n(s,x)|^{2p}\Big)\,dy\,ds \\ &\leq C\,n^{-2}\,. \end{aligned} \tag{3.24}$$

The inequalities (3.19)-(3.24) imply that for any $T>0$ and $p\in[1,+\infty[$, there exists a constant $C>0$ such that for $0\leq t\leq T$,

$$\begin{aligned} \sup_{x\in Q}\mathbb{E}(|w(t,x)-w^n(t,x)|^{2p}) &\leq C\Big[n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}}\sup_{x\in Q}|v(s,x)-v^n(s,x)|^{2p}\,ds \\ &\quad + \int_0^t (t-s)^{-\frac{\alpha}{2}}\sup_{x\in Q}\mathbb{E}(|w(s,x)-w^n(s,x)|^{2p})\,ds\,. \end{aligned} \tag{3.25}$$

Thus, (3.18) and Gronwall's lemma (see e.g. [7], lemma 3.4) imply that if $u_0\in\mathcal{C}^3(Q)$,

$$\begin{aligned} \sup_{x\in Q}\mathbb{E}(|w(t,x)-w^n(t,x)|^{2p}) &\leq C_p\Big[n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}}\sup_{x\in Q}\mathbb{E}(|w(s,x)-w^n(s,x)|^{2p})\;ds\Big] \\ &\leq C_p\,n^{-p(2-\alpha)}\,; \end{aligned}$$

this inequality together with (3.18) yield (3.15). If $u\in\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$, using again Gronwall's lemma and (3.17), we deduce that for some $\lambda\in]0,1[$, one has

$$\begin{aligned} \sup_{x\in Q}\mathbb{E}(|w(t,x)-w^n(t,x)|^{2p}) &\leq C_p\Big[n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}}\,s^{-\lambda}\,n^{-p(2-\alpha)}\,ds \\ &\quad + \int_0^t (t-s)^{-\frac{\alpha}{2}}\sup_{x\in Q}\mathbb{E}(|w(s,x)-w^n(s,x)|^{2p})\;ds\Big] \leq C_p\,n^{-p(2-\alpha)}\,; \end{aligned}$$

This inequality and (3.17) imply (3.16).

Finally, let $u_0\in\mathcal{C}^0(Q)$; then for any $\varepsilon>0$, let $u_{0,\varepsilon}$ denote a function in $\mathcal{C}^3(Q)$ such that $\|u_0-u_{0,\varepsilon}\|_\infty\leq\varepsilon$. Let $u_\varepsilon=v_\varepsilon+w_\varepsilon$ and $u_\varepsilon^n=v_\varepsilon^n+w_\varepsilon^n$ denote the previous decompositions of the solution u_ε and its space discretization u_ε^n with the initial condition $u_{0,\varepsilon}$; then

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times Q}|v(t,x)-v^n(t,x)| &\leq \sup_{(t,x)\in[0,T]\times Q}|v_\varepsilon(t,x)-v_\varepsilon^n(t,x)| \\ &\quad + \Big|\int_Q G_d(t,x,y)\,[u_0(y)-u_{0,\varepsilon}(y)]\,dy\Big| + \Big|\int_Q (G_d)^n(t,x,y)\,[u_0(\kappa_n(y))-u_{0,\varepsilon}(\kappa_n(y))]\,dy\Big| \\ &\leq C\varepsilon + \sup_{(t,x)\in[0,T]\times Q}|v_\varepsilon(t,x)-v_\varepsilon^n(t,x)|\,. \end{aligned} \tag{3.26}$$

Hence (3.25) and (3.26) imply that

$$\sup_{x\in Q}\mathbb{E}(|w(t,x)-w^n(t,x)|^{2p}) \le C\left[\varepsilon + n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}} \sup_{x\in Q}\mathbb{E}(|w(s,x)-w^n(t,x)|^{2p})\,ds\right];$$

Gronwall's lemma concludes the proof of the theorem. □

We now prove the convergence of $u^{n,m}$ and of u^n_m to u^n as $m\to+\infty$.

Theorem 3.4 *Let σ and b satisfy the conditions (2.4) and (2.7). Then*

(i) If $u_0\in\mathcal{C}^2(Q)$, then for every $T>0$ and $p\in[1,+\infty[$, there exists a constant $C_p(T)>0$ such that

$$\sup_{n\ge1}\sup_{t\in[0,T]}\sup_{x\in Q}\mathbb{E}(|u^n(t,x)-u^{n,m}(t,x)|^{2p}) \le C_p(T)\,m^{-p(1-\frac{\alpha}{2})}. \tag{3.27}$$

(ii) If $u_0\in\mathcal{C}(Q)$, then $\sup_{n\ge1}\sup_{t\in[0,T]}\sup_{x\in Q}|u^n(t,x)-u^{n,m}(t,x)|$ converges to 0 as $m\to+\infty$ and for every $t>0$ and $p\in[1,+\infty[$ there exists a constant $C_p(t)$ such that

$$\sup_{n\ge1}\sup_{x\in Q}\mathbb{E}(|u^n(t,x)-u^{n,m}(t,x)|^{2p}) \le C_p(t)\,m^{-p(1-\frac{\alpha}{2})}.$$

(iii) The results of (i) and (ii) hold with u^n_m instead of $u^{n,m}$ if one requires that $\frac{n^2T}{m}\le q<\frac12$.

Proof: Again, we only prove parts (i) and (ii) of the theorem under the homogeneous Dirichlet boundary conditions; the proof of the other cases, which is similar, is omitted. Let $v^n(t,x)=\int_Q (G_d)^n(t,x,y)\,u_0(\kappa_n(y))\,dy$ and $v^{n,m}(t,x)=\int_Q (G_d)^{n,m}(t,x,y)\,u_0(\kappa_n(y))\,dy$. Suppose at first that $u_0\in\mathcal{C}^2(Q)$ and as in the proof of (3.23) in [7], set for $d=1$: $I=\sup_{t\in[0,T]}\sup_{x\in[0,1]}|v^{n,m}(t,x)-v^n(t,x)|\le\sum_{i=1}^3 I_i$, where

$$\begin{aligned}
I_1 &= \sup_{t\in[0,T]}\sup_{x\in[0,1]}\left|v^{n,m}\big([mtT^{-1}]\,Tm^{-1},x\big)-v^n\big([mtT^{-1}]\,Tm^{-1},x\big)\right|\\
I_2 &= \sup_{t\in[0,T]}\sup_{x\in[0,1]}\left|v^{n}\big([mtT^{-1}]\,Tm^{-1},x\big)-v^n(t,x)\right|\\
I_3 &= \sup_{t\in[0,T]}\sup_{x\in[0,1]}\left|v^{n}\big([(mtT^{-1}+1)]\,Tm^{-1},x\big)-v^n(t,x)\right|.
\end{aligned}$$

The inequalities (3.27) and (3.28) in [7] imply that $I_2+I_3\le C\,m^{-\frac12}$. Furthermore, using an estimate of [7], we deduce that

$$I_1 \le C\sup_{n\ge1}\sup_{t\in[0,T]}\sup_{x\in[0,1]}\sum_{j=1}^{n-1} j^{-2}\exp\Big(\lambda_j^n\,[\frac{mt}{T}]\,\frac{T}{m}\Big)\left|1-\exp\Big[[\frac{mt}{T}]\Big(\lambda_j^n\frac{T}{m}+\ln\Big(1-\lambda_j^n\frac{T}{m}\Big)\Big)\Big]\right|.$$

For $t\le T\,l^{-1}$, $[\frac{mt}{T}]=1$ and the right hand-side of the previous inequality is null. If $t\ge T\,m^{-1}$, then there exists a constant $c>0$ such that $\frac{T}{m}\,[\frac{mt}{T}]\ge c\,t$ and using (A.15) we deduce that

$$\begin{aligned}
I_1 &\le C\sup_{n\ge1}\sup_{t\in[\frac{T}{m},T]}\sum_{j=1}^{n-1} j^{-2}\,e^{-ctj^2}\left|1-\exp(-j^4\,t\,m^{-1})\right|\\
&\le \sup_{n\ge1}\sup_{t\in[\frac{T}{m},T]} m^{-1}\sum_{j=1}^{n-1} j^2\,t\,e^{-ctj^2} \le C\sup_{n\ge1}\sup_{t\in[\frac{T}{m},T]} m^{-1}\sum_{j=1}^{n-1} e^{-ctj^2} \le C\,m^{-\frac12}.
\end{aligned}$$

Hence for $d=1$,

$$\sup_{n\geq 1}\ \sup_{t\in[0,T]}\ \sup_{x\in Q} |v^n(t,x)-v^{n,m}(t,x)| \leq C\, m^{-\frac12}\,, \tag{3.28}$$

and an easy argument shows that this inequality can be extended to any $d\geq 1$. Furthermore, for any $m\geq 1$ and $t\in[0,T]$, $\sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}(|w^n(t,x)-w^{n,m}(t,x)|^{2p}) \leq C\,\sum_{i=1}^6 \tilde{B}_i(t)$, where

$$\begin{aligned}
\tilde{B}_1(t) &= \sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q (G_d)^n(t-s,x,y)\,[\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))\\
&\qquad\qquad -\sigma(\Lambda_m(s),\kappa_n(y),u^n(s,\kappa_n(y)))]\,F(ds,dy)\Big|^{2p}\Big)\,,\\
\tilde{B}_2(t) &= \sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q (G_d)^n(t-s,x,y)\,[\sigma(\Lambda_m(s),\kappa_n(y),u^n(\Lambda_m(s),\kappa_n(y)))\\
&\qquad\qquad -\sigma(\Lambda_m(s),\kappa_n(y),u^{n,m}(\Lambda_m(s),\kappa_n(y)))]\,F(ds,dy)\Big|^{2p}\Big)\,,\\
\tilde{B}_3(t) &= \sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q [(G_d)^n(t-s,x,y)-(G_d)^{n,m}(t-s,x,y)]\\
&\qquad\qquad \times\sigma(\Lambda_m(s),\kappa_n(y),u^{n,m}(\Lambda_m(s),\kappa_n(y)))]\,F(ds,dy)\Big|^{2p}\Big)\,,\\
\tilde{B}_4(t) &= \sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q (G_d)^n(t-s,x,y)\,[b(s,\kappa_n(y),u^n(s,\kappa_n(y)))\\
&\qquad\qquad -b(\Lambda_m(s),\kappa_n(y),u(\Lambda_m(s),\kappa_n(y)))]\,dy\,ds\Big|^{2p}\Big)\,,\\
\tilde{B}_5(t) &= \sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q (G_d)^n(t-s,x,y)\,[b(\Lambda_m(s),\kappa_n(y),u^n(\Lambda_m(s),\kappa_n(y)))\\
&\qquad\qquad -b(\Lambda_m(s),\kappa_n(y),u^{n,m}(\Lambda_m(s),\kappa_n(y)))]\,dy\,ds\Big|^{2p}\Big)\,,\\
\tilde{B}_6(t) &= \sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}\Big(\Big|\int_0^t\int_Q [(G_d)^n(t-s,x,y)-(G_d)^{n,m}(t-s,x,y)]\\
&\qquad\qquad \times b(\Lambda_m(s),\kappa_n(y),u^{n,m}(\Lambda_m(s),\kappa_n(y)))]\,dy\,ds\,\Big|^{2p}\Big)\,.
\end{aligned}$$

The argument is similar to that used in the proof of Theorem 3.3; the inequalities (2.7), (A.25), (3.1) and (3.4) provide an upper estimate of $\tilde{B}_1$, (4.60) and (3.1) give an upper estimate of $\tilde{B}_3$ so that $\tilde{B}_1(t)+\tilde{B}_3(t)\leq C\,m^{-(1-\frac{\alpha}{2})p}$. On the other hand, (A.26) and (2.7) show that for some $\lambda\in]0,1[$,

$$\tilde{B}_2(t)\leq \int_0^t (t-s)^{-\lambda}\,\sup_{n\geq 1}\ \sup_{y\in Q}\ \mathbb{E}\big(|u^n(\Lambda_m(s),\kappa_n(y))-u^{n,m}(\Lambda_m(s),\kappa_n(y))|^{2p}\big)\,ds\,.$$

A similar argument based on (A.25), (4.59), (3.1) (3.4) provide an upper estimate of $\tilde{B}_4(t)+\tilde{B}_6(t)\leq C\,m^{-\mu}$ for any $\mu\in]0,1[$ and show that for some $\lambda\in]0,1[$,

$$\tilde{B}_5(t)\leq \int_0^t (t-s)^{-\lambda}\,\sup_{n\geq 1}\ \sup_{y\in Q}\ \mathbb{E}\big(|u^n(\Lambda_m(s),\kappa_n(y))-u^{n,m}(\Lambda_m(s),\kappa_n(y))|^{2p}\big)\,ds\,.$$

Thus, Gronwall's lemma concludes the proof of (3.27). The rest of the proof of the theorem, which is similar to that of Theorem 3.3 is omitted. □

4 Refined estimates of differences of Green functions

This section is devoted to prove some crucial evaluations for the norms of the difference between G_d and its space discretizations $(G_d)^n$, $(G_d)^{n,m}$ or $(G_d)^n_m$; indeed, as shown in the previous section, they provide the speed of convergence of the scheme. We suppose again that these kernels are defined in terms of the homogeneous Dirichlet boundary conditions. Simple modifications of the proof yield the same estimates for the homogeneous Neumann ones.

Lemma 4.1 *There exists some constant $C > 0$ such that for $t > 0$ and $n \geq 2$,*

$$\int_0^{+\infty} \sup_{x\in Q} \int_Q |G_d(t,x,y) - (G_d)^n(t,x,y)|\, dydt \;\leq\; Cn^{-1}\,, \tag{4.1}$$

$$\int_0^{+\infty} \sup_{x\in Q} \|\, |G_d(t,x,\cdot) - (G_d)^n(t,x,\cdot)|\, \|^2_{(\alpha)}\, dt \;\leq\; Cn^{-(2-\alpha)}\,. \tag{4.2}$$

Proof : Let $\gamma > 0$ to be fixed later on; the inequalities (A.11), (A.19), (A.1) and (A.21) imply that for $0 < \lambda < 1$,

$$\int_0^{\gamma n^{-2}} \sup_{x\in Q} \|G_d(t,x,.) - (G_d)^n(t,x,.)\|_1\, dt \;\leq\; C\,[n^{-2} + n^{-2+\lambda}] \leq C\, n^{-2+\lambda}\,, \tag{4.3}$$

$$\int_0^{\gamma n^{-2}} \sup_{x\in Q} \|\, |G_d(t,x,.) - (G_d)^n(t,x,.)|\, \|^2_{(\alpha)}\, dt \;\leq\; C\, n^{-2+\alpha}\,. \tag{4.4}$$

To estimate $\int_{\gamma n^{-2}}^{+\infty} \sup_x \|\, |G_d(t,x,.) - (G_d)^n(t,x,.)|\, \|\, dt$, where $\|\ \|$ denotes either the $\|\ \|_1$ or $\|\ \|_{(\alpha)}$ norm, we first deal with the case $d = 1$ and $\alpha < 1$. As in Gyöngy, we write

$$|G(t,x,y) - G^n(t,x,y)| \leq \sum_{i=1}^{4} T_i(t,x,y)$$

where

$$\begin{aligned}
T_1(t,x,y) &= \left| \sum_{j=n}^{+\infty} e^{-j^2\pi^2 t}\; \varphi_j(x)\; \varphi_j(y) \right|\,, \\
T_2(t,x,y) &= \left| \sum_{j=1}^{n-1} \left[e^{\lambda_j^n t} - e^{-j^2\pi^2 t} \right]\; \varphi_j(x)\; \varphi_j(y) \right|\,, \\
T_3(t,x,y) &= \left| \sum_{j=1}^{n-1} e^{\lambda_j^n t}\; [\varphi_j(x) - \varphi_j^n(x)]\; \varphi_j(y) \right|\,, \\
T_4(t,x,y) &= \left| \sum_{j=1}^{n-1} e^{\lambda_j^n t}\; \varphi_j^n(x)\; [\varphi_j(y) - \varphi_j(k_n(y))] \right|\,;
\end{aligned} \tag{4.5}$$

for $t > 0$

$$\|\, |G(t,x,\cdot) - G^n(t,x,\cdot)|\, \|^2_{(\alpha)} \leq C \sum_{i=1}^{4} \|T_i(t,x\cdot)\|^2_{(\alpha)}\,.$$

Using (A.15) with $\beta = 0$ and $J_0 = n$, we have

$$\sup_{x,y\in[0,1]} |T_1(t,x,y)| \leq C \sum_{j\geq n} e^{-ctj^2} \leq C\, e^{-ctn^2}\, [1 + t^{-\frac{1}{2}}]. \tag{4.6}$$

Furthermore, since $j \to e^{-j^2\pi^2 t}$ decreases, Abel's transform implies that

$$T_1(t,x,y) \le C\, e^{-cn^2 t} \left[\frac{1}{|\sin(\pi \frac{x-y}{2})|} + \frac{1}{|\sin(\pi \frac{x+y}{2})|} \right]. \tag{4.7}$$

Thus, for $\lambda \in [0,1[$ and $t>0$ we have:

$$\sup_{x\in[0,1]} \int_0^1 T_1(t,x,y)\, dy \le C\, e^{-cn^2 t}\, [1+t^{-\frac{\lambda}{2}}] \int_0^2 u^{-1+\lambda}\, du \le C\, e^{-cn^2 t}\, [1+t^{-\frac{\lambda}{2}}]. \tag{4.8}$$

For $x\in[0,1]$ and $i\in\{1,2,3\}$, let

$$A_n^i(x) = \{ y\in[0,1] : |y-x| \le i\, n^{-1} \text{ or } \; y+x \le i\, n^{-1} \text{ or } \; 2-x-y \le i\, n^{-1} \}. \tag{4.9}$$

Then $dy(A_n^i(x)) \le Cn^{-1}$ and for $x\in[0,1]$, $y,z\in A_n^i(x)$, $|y-z| \le 2i\, n^{-1}$; furthermore, for $j\in\{1,\cdots,4\}$,

$$\|T_j(t,x,.)\|^2_{(\alpha)} \le 2\, \left[\|T_j(t,x,.)\, 1_{A_n^2(x)}(.)\|^2_{(\alpha)} + \|T_j(t,x,.)\, 1_{A_n^2(x)^c}(.)\|^2_{(\alpha)} \right].$$

Thus, for $\lambda \in]\alpha,1[$, $\mu\in]0,1-\lambda[$ and $t \ge \gamma n^{-2}$, Set $\mathcal{A}_n^{(1)}(x) = \{(y,z)\in Q^2 : |y-x| \vee |z-x| \le 2\, n^{-1}\}$, $\mathcal{A}_n^{(2)}(x) = \{(y,z)\in Q^2 : |y-x| \vee (x+z) \le 2\, n^{-1}\}$ and $\mathcal{A}_n^{(3)}(x) = \{(y,z)\in Q^2 : |y-x| \vee (2-x-z) \le 2\, n^{-1}\}$. For $\lambda\in]\alpha,1[$, $\mu\in]0,1[$ and $t\ge \gamma n^{-2}$, exchanging y and z if necessary in $\mathcal{A}_n^{(1)}(x)$ and using the two above estimates of $T_1(t,x,.)$, using (4.6) and (4.7), we deduce that:

$$\begin{aligned}
\sup_{x\in[0,1]} \int_{\mathcal{A}_n^{(1)}(x)} T_1(t,x,y)\, |y-z|^{-\alpha}\, T_1(t,x,z)\, dy\, dz &\le C\, e^{-cn^2 t} \int_{|x-y|\le|x-z|\le 2\, n^{-1}} \Big(1+t^{-\frac{\lambda}{2}}\Big) \\
&\quad \times |y-x|^{-(1-\lambda)}\, |y-z|^{-\alpha}\, \Big(1+t^{-\frac{\mu}{2}}\Big)\, |x-z|^{-(1-\mu)}\, dy\, dz \\
\le \;& \Big(1+t^{-\frac{\lambda+\mu}{2}}\Big) \Big(\int_0^{4n^{-1}} u^{-(1-\lambda+\alpha)}\, du\Big) \Big(\int_0^{2n^{-1}} v^{-(1-\mu)}\, dv\Big) \\
\le \;& C\, e^{-cn^2 t}\, n^{\alpha} \Big(1+t^{-\frac{\lambda+\mu}{2}}\Big)\, n^{-(\lambda+\mu)} \le C\, e^{-cn^2 t}\, n^{\alpha}.
\end{aligned}$$

Similar computations for integrals over the sets $\mathcal{A}_n^{(i)}(x)$, $i=2,3$, yield

$$\sup_{x\in[0,1]} \|T_1(t,x,.)\, 1_{A_n^2(x)}(.)\|_{(\alpha)} \le C\, e^{-cn^2 t}\, n^{\alpha}. \tag{4.10}$$

Let $\mathcal{B}_n^{(1)}(x) = \{(y,z)\in Q^2 \,:\, 2\, n^{-1} \le |y-x| \wedge |z-x|\,,\, |y-z| \le 2\, n^{-1}\}$ and $\mathcal{B}_n^{(2)}(x) = \{(y,z)\in Q^2 \,:\, 2\, n^{-1} \le |y-z| \wedge |y-x| \wedge |z-x|\}$. Then $\sup_{x\in[0,1]} \|T_1(t,x,.)\, 1_{A_n^2(x)^c}(.)\|^2_{(\alpha)} \le 2\sum_{i=1}^2 T_{1,i}(t,x)$, where

$$T_1^{(i)}(t,x) = \int_{\mathcal{B}_n^{(i)}(x)} T_1(t,x,y)\, |y-z|^{-\alpha}\, T_1(t,x,z)\, dydz\,.$$

Then, for $T>0$ (4.7) implies:

$$\sup_{x\in[0,1]} T_1^{(1)}(t,x) \le C\, e^{-ctn^2} \int_{\mathcal{B}_n^{(1)}(x)} (|y-x| \wedge |z-x|)^{-2}\, |y-z|^{-\alpha}\, dy\, dz \le C\, e^{-ctn^2}\, n^{\alpha}. \tag{4.11}$$

Similarly, for $(y,z) \in \mathcal{B}^{(2)}$, let $I(y,z) \le M(y,z) \le S(y,z)$ denote the ordered values of $|x-y|, |x-z|, |y-z|$; then (4.7) implies that

$$\sup_{x\in[0,1]} T_1^{(2)}(t,x) \le C\, e^{-ctn^2} \int_{\mathcal{B}_n^2(x)} u^{-(1+\frac{\alpha}{2})}\, M(y,z)^{-(1+\frac{\alpha}{2})}\, dz\, dy \le C\, e^{-ctn^2}\, n^{\alpha}\,. \tag{4.12}$$

The inequalities (4.11) and (4.12) yield:

$$\sup_{x\in[0,1]} \|T_1(t,x,.)\, 1_{A_n(x)^c}(.)\|^2_{(\alpha)} \le C\, n^{\alpha}\, e^{-cn^2 t}\,. \tag{4.13}$$

The inequalities (4.10) and (4.13) imply that for $t \ge \gamma\, n^{-2}$, we have

$$\sup_{x\in[0,1]} \|T_1(t,x,.)\|^2_{(\alpha)} \le C\, n^{\alpha}\, e^{-cn^2 t}\,. \tag{4.14}$$

To study T_2, set $\Delta_j^n(t) := e^{\lambda_j^n t} - e^{-j^2\pi^2 t}$; then for any $A \in [0,2]$ we have

$$0 \le \Delta_j^n(t) \le C\,(j/n)^2\, j^2\, t\, e^{-cj^2 t} \le C\, n^{-A}\, j^A\, e^{-cj^2 t}\,, \tag{4.15}$$

so that (A.16) with $K = A$ yields

$$\sup_{x\in[0,1]} T_2(t,x,y) \le C\, n^{-A}\, t^{-\frac{A+1}{2}}\, e^{-ct}\,. \tag{4.16}$$

Furthermore, since $T_2(t,x,y) \le |\sum_{j=1}^{n-1} e^{\lambda_j^n t}\, \varphi_j(x)\, \varphi_j(y)| + |\sum_{j=1}^{n-1} e^{-j^2\pi^2 t}\, \varphi_j(x)\, \varphi_j(y)|$, Abel's transform yields that for every $1 \le N_1(n) < N_2(n) \le n-1$,

$$\Big| \sum_{j=N_1(n)}^{N_2(n)} \Delta_j^n(t)\, \varphi_j(x)\, \varphi_j(y) \Big| \le e^{-c\, N_1(n)^2\, t} \Big[\frac{1}{|\sin(\frac{\pi\,(x-y)}{2})|} + \frac{1}{|\sin(\frac{\pi\,(x+y)}{2})|} \Big]\,. \tag{4.17}$$

Hence for $A \in]0,2]$ and $\lambda \in]0, \frac{2}{A+1} \wedge 1[$, we have for any $t > 0$:

$$\sup_{x\in[0,1]} \int_0^1 T_2(t,x,y)\, dy \le C\, e^{-ct}\, n^{-A\lambda}\, t^{-\frac{A+1}{2}\lambda} \int_0^2 u^{-1+\lambda}\, du \le C\, e^{-ct}\, n^{-A\lambda}\, t^{-\frac{A+1}{2}\lambda}\,. \tag{4.18}$$

In order to bound the $\|\ \|_{(\alpha)}$ norm of $T_2(t,x,.)$ for $t \ge \gamma\, n^{-2}$, let $T_2(t,x,y) \le \sum_{i=1}^3 T_{2,i}(t,x,y)$, with

$$T_{2,1}(t,x,y) = \sum_{j=1}^{[\sqrt{n}]} |\Delta_j^n(t)|, \quad T_{2,i}(t,x,y) = \Big| \sum_{j=N_{i-1}(n)+1}^{N_i(n)} \Delta_j^n(t)\, \varphi_j(x)\, \varphi_j(y) \Big| \quad, i = 2,3$$

with $N_1(n) = [\sqrt{n}]$, $N_2(n) = [n/2]$ and $N_3(n) = n-1$. The inequalities (4.15) with $A = 2$ and (A.15) with $\beta = 0$ yield

$$\sup_{x,y\in[0,1]} T_{2,1}(t,x,y) \le C \sum_{j=1}^{[\sqrt{n}]} n^{-1} e^{-cj^2 t} \le C\, n^{-1}\, [1 + t^{-\frac{1}{2}}]\, e^{-ct}\,.$$

Furthermore, $\sup\{T_{2,1}(t,x,y)\,;\,(t,x,y)\in]0,+\infty[\times[0,1]^2\}\le C\sqrt{n}$. Hence, raising the first upper estimates of $T_{2,1}(t,x,\xi)$ to the power $1-\frac{\alpha}{3}\in]0,1[$ and the second one to the power $\frac{\alpha}{3}$ separately for $\xi=y$ and $\xi=z$, we deduce that

$$\begin{aligned}\sup_{x\in[0,1]}\|T_{2,1}(t,x,.)\|^2_{(\alpha)} &\le C\,e^{-ct}\int_{[0,1]^2} n^{\frac{1}{3}}\,(1+t^{-(1-\frac{\alpha}{3})})\,n^{-2+\frac{2\alpha}{3}}\,|y-z|^{-\alpha}\,dy\,dz\\ &\le C\,e^{-ct}\,(1+t^{-1+\frac{\alpha}{3}})\,n^{-2+\alpha}\end{aligned}\tag{4.19}$$

Thus, for $t\ge\gamma\,n^{-2}$, if $\mathcal{A}_n^{(i)}(x)$ and $\mathcal{B}_n^{(j)}(x)$, $1\le i\le 3$ and $1\le j\le 2$ denote the sets introduced to estimate $\|T_1(t,x,.)\|^2_{(\alpha)}$. Let $\lambda\in]\alpha,1[$ and $\mu\in]0,1-\lambda[$, using (4.17) and (4.16) with $A=0$, we have for $\lambda\in]\alpha,1[$ and $\mu\in]0,1[$,

$$\begin{aligned}\sup_{x\in[0,1]}\int_{\mathcal{A}_n^{(1)}(x)} &T_{2,3}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,3}(t,x,z)\,dy\,dz\le C\,e^{-cn^2t}\,(1+t^{-\frac{\lambda+\mu}{2}})\\ &\times(|x-y|\wedge|x-z|)^{-1+\lambda}\,|y-z|^{-\alpha}\,(|x-y|\vee|x-z|)^{-1+\mu}\,dy\,dz\\ \le Ce^{-cn^2t}(1+t^{-\frac{\lambda+\mu}{2}})&\Big(\int_0^{4n^{-1}}u^{-1+\lambda-\alpha}\,du\Big)\Big(\int_0^{4n^{-1}}v^{-1+\mu}\,dv\Big)\le Cn^\alpha e^{-ctn^2}[1+(nt^{\frac{1}{2}})^{-(\lambda+\mu)}]\,.\end{aligned}$$

Similar computations for integrals over the sets $\mathcal{A}_n^{(i)}(x)$ for $i=2,3$ yield

$$\sup_{x\in[0,1]}\|T_{2,3}(t,x,.)\,1_{A_n^2(x)}(.)\|^2_{(\alpha)}\le C\,n^\alpha\,e^{-ctn^2}\,.\tag{4.20}$$

Furthermore, (4.17) implies that

$$\begin{aligned}\int_{\mathcal{B}_n^{(1)}(x)}T_{2,3}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,3}(t,x,z)\,dy\,dz &\le C\,e^{-cn^2t}\Big(\int_0^{2\,n^{-1}}u^{-2}\,du\Big)\Big(\int_0^{2\,n^{-1}}v^{-\alpha}\,dv\Big)\\ &\le C\,e^{-cn^2t}\,n^\alpha\,.\end{aligned}\tag{4.21}$$

For $(y,z)\in\mathcal{B}_n^{(2)}(x)$, let $I(y,z)\le M(y,x)\le S(y,z)$ denote the ordered values of $|x-y|$, $|y-z|$ and $|x-z|$; then we have:

$$\begin{aligned}&\int_{\mathcal{B}_n^{(2)}(x)}T_{2,3}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,3}(t,x,z)\;dy\,dz\\ &\le C\,e^{-cn^2t}\int_{2\,n^{-1}\le I(y,z)\le M(y,z)\le S(y,z)\le 2}I(y,z)^{-1-\frac{\alpha}{2}}\,M(y,z)^{-1-\frac{\alpha}{2}}\,dy\,dz\\ &\le C\,e^{-cn^2t}n^\alpha\,.\end{aligned}\tag{4.22}$$

The inequalities (4.20) - (4.22) yield that for $t\ge\gamma n^{-2}$:

$$\sup_{x\in[0,1]}\|T_{2,3}(t,x,.)\|^2_{(\alpha)}\le C\,n^\alpha\,e^{-ctn^2}\,.\tag{4.23}$$

Let $C_0>0$ be a "large" constant to be chosen later on and suppose that $t\ge C_0\,n^{-2}$; in order to use Abel's transform for $T_{2,2}$ and $t\ge C_0 n^{-2}$, we set $\tilde S_j(\xi):=\sum_{i=1}^{j-1}\cos(i\pi\xi)$; then

$$\begin{aligned}T_{2,2}(t,x,y) \le\ &\Big|\big[\tilde S_{[\sqrt{n}]+1}(y-x)-\tilde S_{[\sqrt{n}]+1}(y+x)\big]\ \Delta^n_{[\sqrt{n}]+1}(t)\Big|\\ &+\Big|\sum_{j=[\sqrt{n}]+2}^{[n/2]}\big[\tilde S_j(y-x)-\tilde S_j(y+x)\big]\ [\Delta^n_{j-1}(t)-\Delta^n_j(t)]\Big|\\ &+\Big|\big[\tilde S_{[n/2]+1}(y-x)-\tilde S_{[n/2]+1}(y+x)\big]\ \Delta^n_{[n/2]}(t)\Big|\,.\end{aligned}\tag{4.24}$$

We study the monotonicity of $j \in \big[[\sqrt{n}]+1, [n/2]\big] \longmapsto \Delta_j^n(t)$. For fixed n and t, let $\phi(j) := \Delta_j^n(t)$; then $\phi(j) = \exp\left[-4n^2 t \sin^2\left(\frac{j\pi}{2n}\right)\right] - \exp[-j^2\pi^2 t]$ and

$$\phi'(j) = 2j\pi^2 t \exp[-j^2\pi^2 t]\, \left\{1 - \frac{\sin\left(\frac{j\pi}{n}\right)}{\frac{j\pi}{n}}\, \exp\left[n^2 t\left(\frac{j^2\pi^2}{n^2} - 4\sin^2\left(\frac{j\pi}{2n}\right)\right)\right]\right\}.$$

Let $u := \dfrac{j\pi}{2n}$ and, for $[\sqrt{n}]+1 \le j \le [n/2]$, which implies $\dfrac{\pi}{2\sqrt{n}} \le u \le \dfrac{\pi}{4}$, set

$$\psi(u) := \frac{\sin(2u)}{2u}\, \exp\left[\,4n^2 t\,(u^2 - \sin^2 u)\,\right]\,;$$

in order to study the sign of $\phi'(j)$, we must compare $\psi(u)$ to 1.

For $0 \le u \le \frac{\pi}{4}$ we have

$$u^2 - \sin^2 u \ge u^2 - \left(u - \frac{u^3}{3!} + \frac{u^5}{5!}\right)^2 \ge \frac{u^4}{3} - \frac{2}{45}\,u^6 + \frac{u^8}{5!\,3} - \frac{u^{10}}{(5!)^2} \ge \frac{u^4}{3}\,\left(1 - \frac{2}{15}\,u^2\right)\,;$$

hence we deduce

$$4\,n^2\,t\,(u^2 - \sin^2 u) \ge 4\,n^2\,t\,\frac{u^4}{3}\,\frac{11}{12} \ge \frac{11}{9}\,n^2\,t\,u^4\,.$$

Furthermore, the inequalities $e^x \ge 1+x$ and $\frac{\sin(2u)}{2u} \ge 1 - \frac{2}{3}\,u^2 > 0$ for $0 \le u \le \frac{\pi}{4}$ imply that for $0 \le u \le \frac{\pi}{4}$

$$\begin{aligned}\psi(u) \;&\ge\; \left(1 - \frac{2}{3}\,u^2\right)\,\left(1 + \frac{11}{9}\,n^2\,t\,u^4\right)\\ &\ge\; 1 - \frac{u^2}{3}\,\left[2 - \frac{11}{3}\,n^2\,t\,u^2 + \frac{22}{9}\,n^2\,t\,u^4\right].\end{aligned}$$

Set $R(X) = \frac{22}{9}\,n^2\,t\,X^2 - \frac{11}{3}\,n^2\,t\,X + 2$ and suppose that $C_0 \ge \frac{16}{11}$; then R has two zeros

$$X_1 = \frac{3}{4}\left[1 - \sqrt{1 - \frac{16}{11\,n^2\,t}}\,\right] \qquad \text{and} \qquad X_2 = \frac{3}{4}\left[1 + \sqrt{1 - \frac{16}{11\,n^2\,t}}\,\right].$$

Clearly, $X_2 > (\frac{\pi}{4})^2$; fix $\varepsilon > 0$ and suppose that $\frac{4}{11\,C_0} \le \frac{\varepsilon}{(1+\varepsilon)^2}$, which holds for C_0 large enough. Then $X_1 = \frac{C_1^2}{n^2\,t}$ with $C_1^2 \le \frac{6\,(1+\varepsilon)}{11}$. For $u \in]\frac{C_1}{n\sqrt{t}}, \frac{\pi}{4}]$, $R(u^2) < 0$ and hence $\psi(u) \ge 1$. Let $\tilde{C}_1 = (\frac{2\,C_1}{\pi})^2$; note that for $t \ge \frac{\tilde{C}_1}{n}$, we have $\frac{C_1}{n\sqrt{t}} \le \frac{\pi}{2\sqrt{n}}$, which implies that $\phi(j)$ decreases for $[\sqrt{n}] \le j \le [n/2]$.

On the other hand, for $0 \le u \le \frac{\pi}{4}$, $\sin(u) \ge u - \frac{u^3}{3!}$ so that

$$4\,n^2\,t(u^2 - \sin^2 u) \le \frac{4\,n^2\,t}{3}\,u^4 - \frac{n^2\,t}{9}\,u^6\,,$$

while

$$\frac{\sin(2u)}{2u} \le 1 - \frac{2}{3}\,u^2 + \frac{2}{15}\,u^4\,;$$

hence

$$\psi(u) \le \left(1 - \frac{2}{3}\,u^2 + \frac{2}{15}\,u^4\right)\,\exp\left(\frac{4\,n^2\,t}{3}\,u^4\left[1 - \frac{u^2}{12}\right]\right).$$

For $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$ and $u \in [\frac{\pi}{2\sqrt{n}}, \frac{C_1}{n\sqrt{t}}]$, we have

$$\frac{4\,n^2\,t}{3}\,u^4\,\Big[1-\frac{u^2}{12}\Big] \le \frac{4\,C_1^4}{3\,C_0}\,.$$

Furthermore, if $0 \le r \le A$ and $e^A = 1 + D\,A$ (i.e., $D = [e^A - 1]\,A^{-1} > 1$), then we have $e^r \le 1 + D\,r$. Thus, if C_0 is large enough, for $A = \frac{4\,C_1^4}{3\,C_0} < \frac{1}{2}$ and D defined above, $D \le 2\,\big(e^{\frac{1}{2}} - 1\big)$ we have:

$$\begin{aligned} \psi(u) &\le \Big(1 - \frac{2}{3}\,u^2 + \frac{2}{15}\,u^4\Big)\,\Big[1 + D\,\frac{4\,n^2\,t}{3}\,u^4\,\big[1 - \frac{u^2}{12}\big]\Big] \\ &\le 1 - \frac{2}{3}\,u^2\,\Big[1 - \big(\frac{1}{5} + 2\,D\,n^2\,t\big)\,u^2 + \frac{4\,D\,n^2\,t}{3}\,u^4\,\big(1 - \frac{u^2}{5}\big)\,\big(1 - \frac{u^2}{12}\big)\Big]\,. \end{aligned}$$

If $\frac{C_1^2}{C_0} < \frac{1}{2}$ (which holds if C_0 is large enough), for $u \le \frac{C_1}{n\sqrt{t}}$ and $t \ge \frac{C_0}{n^2}$, one has $\big(1 - \frac{u^2}{5}\big)\,\big(1 - \frac{u^2}{12}\big) \ge (1 - \frac{C_1^2}{5\,C_0})(1 - \frac{C_1^2}{12\,C_0}) > 0$. Let C_2 be a positive constant such that $C_2^2 \le \frac{1}{5(\sqrt{e}-1)}$; then $C_2 < C_1$ and for $t \ge \frac{C_0}{n^2}$ and $u \in [\frac{\pi}{2n}, \frac{C_2}{n\sqrt{t}}]$,

$$1 - \big(\frac{2\,u^2}{5} + 2\,D\,n^2\,t\,u^2\big) \ge 1 - \frac{2\,C_1^2}{5\,C_0} - 2\,D\,C_2^2 \ge 0\,,$$

which implies that $\psi(u) \le 1$. Hence, for $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$, the function $\phi(j)$ increases for $[\sqrt{n}] \le j \le [\frac{2\,C_2}{\pi\,\sqrt{t}}]$ with $0 < C_2 < C_1 < 1$ and decreases for $[\frac{2\,C_1}{\pi\,\sqrt{t}}] + 1 \le j \le [\frac{n}{2}]$, while for $t \ge \frac{2\,C_1}{\pi\,n}$, $\phi(j)$ decreases for $[\sqrt{n}] \le j \le [n/2]$. For $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$ and $i = 1, 2$, set $B_i = \frac{2C_i}{\pi}$ and let

$$\begin{aligned} T_{2,2,1}(t,x,y) &= \Big|\sum_{j=[\sqrt{n}]}^{[B_2\,t^{-\frac{1}{2}}]} \Delta_j^n(t)\,\varphi_j(x)\,\varphi_j(y)\Big| + \Big|\sum_{j=[B_1\,t^{-\frac{1}{2}}]}^{[n/2]} \Delta_j^n(t)\,\varphi_j(x)\,\varphi_j(y)\Big|\,, \\ T_{2,2,2}(t,x,y) &= \Big|\sum_{j=[B_2\,t^{-\frac{1}{2}}]}^{[B_1\,t^{-\frac{1}{2}}]} \Delta_j^n(t)\,\varphi_j(x)\,\varphi_j(y)\Big|\,. \end{aligned}$$

Trivially, for such t, $T_{2,2}(t,x,y) \le \sum_{i=1}^2 T_{2,2,i}$ and there exists a constant C such that

$$\sup_{(x,y)\in[0,1]^2} T_{2,2,1}(t,x,y) \le C\,n\,e^{-ct} \tag{4.25}$$

for every $t \ge \frac{\gamma}{n^2}$. For $t \ge \frac{\tilde{C}_1}{n}$, let $T_{2,2,1}(t,x,y) = T_{2,2}(t,x,y)$. Using (4.18) and (4.24), we deduce that for $t \ge \frac{C_0}{n^2}$ and $\beta \in [0,1]$

$$\begin{aligned} T_{2,2,1}(t,x,y) &\le C\,\Big[\frac{1}{|\sin(\frac{\pi\,(x-y)}{2})|} + \frac{1}{|\sin(\frac{\pi\,(x+y)}{2})|}\Big]\,\Big[\Delta^n_{[\sqrt{n}]+1}(t) + \sum_{i=1}^2 \Delta^n_{[\frac{B_i}{\sqrt{t}}]}(t)\,1_{\{C_0 n^{-2} \le t \le \tilde{C}_1\,n^{-1}\}}\Big] \\ &\le C\,\Big[\frac{1}{|x-y|} + \frac{1}{x+y} + \frac{1}{2-x-y}\Big]\,\Big[n^{-1}\,e^{-ctn} + n^{-2\beta}\,t^{-\beta}\,1_{\{\frac{C_0}{n^2} \le t \le \frac{\tilde{C}_1}{n}\}}\Big]\,. \end{aligned} \tag{4.26}$$

For $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$, it remains to deal with the values of u in the interval $[\frac{C_2}{n\sqrt{t}}, \frac{C_1}{n\sqrt{t}}]$, i.e., to bound directly the sum $T_{2,2,2}(t,x,y)$. The inequality (4.15) implies that for $B_2\,t^{-\frac{1}{2}} \le j \le B_1\,t^{-\frac{1}{2}}$,

$\Delta_j^n(t) \le C n^{-2A}\, t^{-A}\, e^{-c\,t\,j^2}$ for any $A \in [0,1]$. Therefore, the inequality (A.15) implies that for any $A \in [0,1]$,

$$\sup_{x\in[0,1]} T_{2,2,2}(t,x,y) \le C\, n^{-2A}\, t^{-(A+\frac{1}{2})}\,. \tag{4.27}$$

Furthermore, (4.17) implies that

$$\sup_{x\in[0,1]} T_{2,2,2}(t,x,y) \le C\Big[\frac{1}{|x-y|} + \frac{1}{x+y} + \frac{1}{2-x-y}\Big]\,. \tag{4.28}$$

Finally, for $t \in [\frac{C_0}{n^2}, \frac{\tilde C_1}{n}]$ and $u \in [\frac{D_2}{2\,n\,\sqrt{t}}\,, \frac{D_1}{2\,n\,\sqrt{t}}]$ with $D_i = B_i\,\pi,\ i=1,2$:

$$\psi'(u) = \exp[4\,n^2\,t\,(u^2 - \sin^2 u]\, u^{-2}\,[2u\,\cos(2u) - \sin(2u) + 4\,n^2\,t\,u\,(2u - \sin(2u))];$$

Set $L(x) = 2u\,\cos(x) - \sin(x) + 2\,n^2\,t\,x\,(x - \sin(x))$ for $x \in [2\,D_2\,t^{-\frac{1}{2}}, 2\,D_1\,t^{-\frac{1}{2}}]$; then

$$\begin{aligned}
L(x) &\ge -\frac{x^3}{3} + \frac{4\,x^5}{5!} - \frac{x^7}{7!} + 2\,n^2\,t\,\Big[\frac{x^3}{3!} - \frac{x^5}{5!}\Big]\\
&\ge \frac{D_2^3}{3}\,\frac{1}{n\sqrt{t}} - \Big(\frac{1}{3} + \frac{2\,D_2^5}{5!}\Big)\,\frac{1}{(n\sqrt{t})^3} + \frac{4\,D_2}{5!}\,\frac{1}{(n\,\sqrt{t})^5} - \frac{D_1^7}{7!}\,\frac{1}{(n\sqrt{t})^7}\\
&\ge \frac{D_2^3}{3}\,\frac{1}{n\sqrt{t}}\Big[1 - \frac{1}{D_2^3}\,\Big(\frac{1}{3} + \frac{2\,D_2^5}{5!}\Big)\,\frac{1}{C_0}\Big) - \frac{D_1^7}{6!\,D_2^3}\,\frac{1}{C_0^3}\Big] > 0
\end{aligned}$$

if C_0 is large enough. Thus ψ in increasing on the interval $[\frac{D_2}{2\,n\,\sqrt{t}}\,, \frac{D_1}{2\,n\,\sqrt{t}}]$ and one of the following holds: the function ψ remains larger than 1 on this interval, or it remains less than 1 or there exists a unique $D_0 \in [\frac{B_2}{2}, \frac{B_1}{2}]$ such that $\psi(u) \le 1$ for $u \in [\frac{D_2}{2\,n\,\sqrt{t}}\,, \frac{D_0}{2\,n\,\sqrt{t}}]$ and $\psi(u) \ge 1$ for $u \in [\frac{D_0}{2\,n\,\sqrt{t}}\,, \frac{D_1}{2\,n\,\sqrt{t}}]$. Hence the function ϕ is either decreasing, or increasing, or first increasing and then decreasing on the interval $[\frac{B_2}{\sqrt{t}}, \frac{B_1}{\sqrt{t}}]$. Therefore, since $\sup_{B\in[B_2,B_1]} \phi(\frac{B}{\sqrt{t}}) \le C\,n^{-2}\,t^{-1}$, Abel's transform implies that for $t \in [\frac{C_0}{n^2}, \frac{\tilde C_1}{n}]$,

$$\sup_{x\in[0,1]} T_{2,2,2}(t,x,y) \le C\,n^{-2}\,t^{-1}\Big[\frac{1}{|\sin(\pi\,\frac{x-y}{2})|} + \frac{1}{|\sin(\pi\,\frac{(x+y}{2})|}\Big]\,. \tag{4.29}$$

The inequalities (4.26) applied with $\beta = \frac{1}{2}$ and $\beta = 1$ respectively and (4.25) imply that for $\lambda \in]0,\alpha[$ and $\mu \in]0, 1-\lambda[$ there exists a constant $C>0$ such that for every $t \ge \gamma n^{-2}$:

$$\begin{aligned}
&\sup_{x\in[0,1]} \int_{\mathcal{A}_n^{(1)}(x)} T_{2,2,1}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,2,1}(t,x,z)\,dy\,dz \le\\
&\qquad C\,n^{\lambda+\mu}\,n^{-2+\lambda+\mu}\,e^{-ctn}\,\Big(\int_0^{4\,n^{-1}} u^{-1+\lambda-\alpha}\,du\Big)\,\Big(\int_0^{2\,n^{-1}} v^{-1+\mu_1}\,dv\Big)\\
&\qquad + C\,n^{\lambda+\mu}\,n^{-4+2(\lambda+\mu)}\,t^{-(\lambda+\mu)}\,\Big(\int_0^{4\,n^{-1}} u^{-1+\lambda-\alpha}\,du\Big)\,\Big(\int_0^{2\,n^{-1}} v^{-1+\mu}\,dv\Big)\,1_{\{C_0\,n^{-2}\le t\le \tilde C_1\,n^{-1}\}}\\
&\le\ C\,\Big[n^{-2+\alpha+(\lambda+\mu)}\,e^{-ctn} + 1_{\{C_0\,n^{-2}\le t\le \tilde C_1\,n^{-1}\}}n^{-4+\alpha+2(\lambda+\mu)}\,t^{-(\lambda+\mu)}\Big]\,.
\end{aligned} \tag{4.30}$$

Similar computations for the integrals over the sets $\mathcal{A}_n^{(i)}(x)$, $i=2,3$ imply that for $\nu = \lambda+\mu \in]0,1[$ and $t \ge \gamma n^{-2}$:

$$\sup_{x\in[0,1]} \|T_{2,2,1}(t,x,.)\,1_{A_n^{(2)}(x)}(.)\|^2_{(\alpha)} \le C\,\Big[n^{-1+\alpha}\,e^{-ctn} + 1_{\{C_0\,n^{-2}\le t\le \tilde C_1\,n^{-1}\}}n^{-4+\alpha+2\nu}\,t^{-\nu}\Big]\,. \tag{4.31}$$

Furthermore, (4.26) with $\beta \in]\frac{1}{2}, 1[$ yields:

$$\sup_{x\in[0,1]} \int_{\mathcal{B}_n^{(1)}(x)} T_{2,2,1}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,2,1}(t,x,z)\,dy\,dz$$
$$\le C\Big[n^{-2}\,e^{-ctn} + n^{-4\beta}\,t^{-2\beta}\,1_{\{C_0\,n^{-2}\le t\le \tilde{C}_1\,n^{-1}\}}\Big] \int_{\mathcal{B}_n^{(1)}(x)} (|x-y|\wedge|x-z|)^{-2}\,|y-z|^{-\alpha}\,dy\,dz$$
$$\le C\Big(n^{-2+\alpha}\,e^{-ctn} + n^{-4\beta+\alpha}\,t^{-2\beta}\,1_{\{C_0\,n^{-2}\le t\le \tilde{C}_1\,n^{-1}\}}\Big)\,. \tag{4.32}$$

Finally, as in the proof of (4.22), let $I(y,z) \le M(y,z) \le S(y,z)$ denote the ordered values of $|x-y|$, $|x-z|$ and $|y-z|$; using (4.26) with $\beta \in]\frac{1}{2}, 1[$, we deduce:

$$\sup_{x\in[0,1]} \int_{\mathcal{B}_n^{(2)}(x)} T_{2,2,1}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,2,1}(t,x,z)\,dy\,dz$$
$$\le C\Big[n^{-2}\,e^{-ctn} + n^{-4\beta}\,t^{-2\beta}\,1_{\{C_0\,n^{-2}\le t\le \tilde{C}_1\,n^{-1}\}}\Big]\Big(\int_{2\,n^{-1}}^{2} u^{-1-\frac{\alpha}{2}}\,du\Big)^2$$
$$\le C\Big(n^{-2+\alpha}\,e^{-ctn} + n^{-4\beta+\alpha}\,t^{-2\beta}\,1_{\{C_0\,n^{-2}\le t\le \tilde{C}_1\,n^{-1}\}}\Big)\,. \tag{4.33}$$

The inequalities (4.30)-(4.33) yield that for $\nu \in]\alpha, 1[$ and $\beta \in]\frac{1}{2}, 1[$,

$$\sup_{x\in[0,1} \|T_{2,2,1}(t,x,.)\|_{(\alpha)}^2 \le C\Big[n^{-1+\alpha}\,e^{-ctn} + 1_{\{\frac{C_0}{n^2}\le t\le \frac{\tilde{C}_1}{n}\}}\big(n^{-4+\alpha+2\nu}\,t^{-\nu} + n^{-4\beta+\alpha}\,t^{-2\beta}\big)\Big]\,. \tag{4.34}$$

For $t \in [C_0\,n^{-2}, \tilde{C}_1\,n^{-1}]$ we give an upper estimate of $\|T_{2,2,2}(t,x,.)\|_{(\alpha)}^2$; the inequalities (4.27) and (4.28) yield that for $A \in [0,1]$, $\lambda \in]0, \alpha[$ and $\mu \in]0, 1[$:

$$\sup_{x\in[0,1]} \int_{\mathcal{A}_n^{(1)}(x)} T_{2,2,2}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,2,2}(t,x,z)\,dy\,dz \le$$
$$C\,n^{-2A(\lambda+\mu)}\,t^{-(A+\frac{1}{2})(\lambda+\mu)}\Big(\int_0^{4\,n^{-1}} u^{-1+\lambda-\alpha}\,du\Big)\Big(\int_0^{2\,n^{-1}} v^{-1+\mu}\,dv\Big)$$
$$\le \quad C\,n^{-(2A+1)(\lambda+\mu)+\alpha}\,t^{-(A+\frac{1}{2})(\lambda+\mu)}\,. \tag{4.35}$$

A similar computation for the integrals over $\mathcal{A}_n^{(i)}(x)$, $i = 2, 3$ yields for the same choice of A, λ, μ and $\nu = \lambda + \mu \in]0, 1+\alpha[$:

$$\sup_{x\in[0,1]} \|T_{2,2,2}(t,x,y)\,1_{A_n^{(2)}(x)}(.)\|_{(\alpha)}^2 \le C\,n^{-(2A+1)\nu+\alpha}\,t^{-(A+\frac{1}{2})\nu}\,. \tag{4.36}$$

The inequalities (4.27) and (4.28) yield that for $A \in]\frac{1}{2}, 1]$ and $\lambda \in]\frac{1}{2A+1}, \frac{1}{2}[$:

$$\sup_{x\in[0,1]} \int_{\mathcal{B}_n^{(1)}(x)} T_{2,2,2}(t,x,y)\,|y-z|^{-\alpha}\,T_{2,2,2}(t,x,z)\,dy\,dz \le$$
$$C\,n^{-4A\lambda}\,t^{-2\lambda(A+\frac{1}{2})} \int_{\{|x-y|\vee|x-z|\ge 2n^{-1},\,|y-z|\le 2n^{-1}\}} (|x-y|\wedge|x-z|)^{-2(1-\lambda)}\,|y-z|^{-\alpha}\,dy\,dz$$
$$\le \quad C\,n^{-2\lambda(2A+1)+\alpha}\,t^{-\lambda(2A+1)}\,. \tag{4.37}$$

As in the proof of (4.22), let $I(y,z) \le M(y,z) \le S(y,z)$ denote the ordered values of $|x-y|$, $|x-z|$ and $|y-z|$; then for $\alpha \in]\frac{2}{3}, 1[$, $\frac{1}{3} < \lambda < \frac{\alpha}{2}$, the inequalities (4.29) and (4.27) with $A = 1$

imply that for $\lambda \in]0, \frac{\alpha}{2}[$:

$$\sup_{x\in[0,1]} \int_{\mathcal{B}_n^{(2)}(x)} T_{2,2,2}(t,x,y)\, |y-z|^{-\alpha}\, T_{2,2,2}(t,x,z)\, dy\, dz \leq$$
$$C\, n^{-4\lambda}\, t^{-3\lambda}\, n^{-4(1-\lambda)}\, t^{-2(1-\lambda)} \Big(\int_{2\,n^{-1}}^{2} u^{-1+\lambda-\frac{\alpha}{2}}\, du \Big)^2 \leq C\, n^{-4+\alpha}\, t^{-2}\,. \qquad (4.38)$$

The inequalities (4.36)-(4.38) used with $A=1$ imply that for $\mu \in]\frac{1}{3}, \frac{1}{2}[$ there exists a constant $C>0$ such that for every $t \in [C_0\, n^{-2}, \tilde{C}_1\, n^{-1}]$:

$$\sup_{x\in[0,1]} \|T_{2,2,2}(t,x,.)\|_{(\alpha)}^2 \leq C\, \Big[n^{-4+\alpha}\, t^{-2} + n^{-6\mu+\alpha}\, t^{-3\,\mu} \Big]\,. \qquad (4.39)$$

The inequalities (4.34) and (4.39) imply that for $\gamma \geq C_0$ with C_0 large enough, $\nu \in]\alpha, 1[$, $\lambda = 3\mu = 2\beta \in]1, \frac{3}{2}[$, there exits a constant $C>0$ such that for all $t \in [\gamma\, n^{-2}, +\infty[$,

$$\sup_{x\in[0,1]} \|T_{2,2}\|_{(\alpha)}^2 \leq C\, n^{\alpha}\, \Big[n^{-1}\, e^{-ctn} + 1_{\{C_0\, n^{-2} \leq t \leq \tilde{C}_1\, n^{-1}\}} \Big(n^{-4+2\nu}\, t^{-\nu} + n^{-4}\, t^{-2} + n^{-2\lambda}\, t^{-\lambda} \Big) \Big]\,. \qquad (4.40)$$

The inequalities (4.19), (4.23) and (4.40) yield for $\lambda \in]1, \frac{3}{2}[$, $\nu \in]\alpha, 1[$ and $t \geq \gamma n^{-2} \geq C_0\, n^{-2}$:

$$\begin{aligned} \sup_{x\in[0,1]} \|T_2(t,x,.)\|_{(\alpha)}^2 \;\leq\; & C\, n^{\alpha}\, \Big[e^{-ctn^2} + n^{-2}\, e^{-ct}\, (1 + t^{-1+\frac{\alpha}{3}}) + n^{-1}\, e^{-ctn} \\ & + 1_{\{\frac{C_0}{n^2} \leq t \leq \frac{\tilde{C}_1}{n}\}} \Big(n^{-4+2\nu}\, t^{-\nu} + n^{-4}\, t^{-2} + n^{-2\lambda}\, t^{-\lambda} \Big) \Big]\,. \end{aligned} \qquad (4.41)$$

We now turn to the term T_3. For $l \in \{0, \cdots, n-1\}$ and $\frac{l}{n} \leq x < \frac{l+1}{n}$, one has

$$\varphi_j(x) - \varphi_j^n(x) = \sqrt{2}\, n\, \Big[\Big(\frac{l+1}{n} - x \Big) \int_{\frac{l}{n}}^{x} j\pi \cos(j\pi u)\, du - \Big(x - \frac{l}{n} \Big) \int_{x}^{\frac{l+1}{n}} j\pi \cos(j\pi u)\, du \Big]\,.$$

Hence using (A.16) we deduce that

$$T_3(t,x,y) \leq C \int_{\frac{l}{n}}^{\frac{l+1}{n}} \Big| \sum_{j=1}^{n-1} j e^{t\lambda_j^n} \cos(j\pi u) \sin(j\pi y) \Big|\, du\,, \qquad (4.42)$$

where

$$\Big| \sum_{j=1}^{n-1} j\, e^{t\lambda_j^n}\, \cos(j\pi u)\; \sin(j\pi y) \Big| \leq C\, \Big[1 + t^{-1} \Big]\, e^{-ct}\,. \qquad (4.43)$$

Let $S_N(x) := \sum_{i=1}^{N-1} \sin(i\pi x)$ for $N \geq 2$ and $S_1(x) = 0$. Then for $x \in]0,2[$, $\sup_{N\geq 2} |S_N(x)| \leq \frac{C}{|\sin(\frac{\pi x}{2})|}$; thus Abel's transform yields

$$\begin{aligned} T_3(t,x,y) \;\leq\; & \int_{\frac{l}{n}}^{\frac{l+1}{n}} (n-1)\, e^{t\lambda_{n-1}^n}\, |S_n(y+u) + S_n(y-u)|\, du \\ & + \int_{\frac{l}{n}}^{\frac{l+1}{n}} \Big| \sum_{j=2}^{n-1} \big[(j-1)\, e^{t\lambda_{j-1}^n} - j\, e^{t\lambda_j^n} \big]\, \big[S_j(y+u) + S_j(y-u) \big] \Big|\, du. \end{aligned} \qquad (4.44)$$

For $1 \le z \le n-1$, set $H(z) = z \,\exp\Big[-4n^2 t \sin^2\Big(\frac{z\pi}{2n}\Big)\Big]$; then

$$H'(z) = \Big(1 - 2n^2 t \frac{z\pi}{n} \sin\Big(\frac{z\pi}{n}\Big)\Big) \;\exp\Big[-4n^2 t \sin^2\Big(\frac{z\pi}{2n}\Big)\Big].$$

The map $\theta \longmapsto 1-2n^2 t\theta \sin\theta$ decreases on $\big[\frac{\pi}{n}, \theta_0\big]$ and increases on $\big[\theta_0, \frac{(n-1)\pi}{n}\big]$, where $\theta_0 \in]\frac{\pi}{2}, \frac{2\pi}{3}[$ is the unique solution in the interval $]\frac{\pi}{2}, \frac{2\pi}{3}[$ to the equation $\theta + \tan\theta = 0$. Therefore, if $t \ge \tilde{C}_0\, n^{-2}$ with $\tilde{C}_0 = \frac{1}{2\,\theta_0\,\sin(\theta_0)}$, then $H'(\frac{n\,\theta_0}{\pi}) \le 0$. Furthermore, if $t \ge \frac{1}{2n\pi \sin\left(\frac{\pi}{n}\right)} \sim C_1$ then $H'(1) \le 0$ and if $t \ge \frac{1}{2n^2\left(\pi - \frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)} \sim \frac{C_2}{n}$, then $H'(n-1) \le 0$. Hence, for $\tilde{C}_0\, n^{-2} \le t \le \frac{1}{2n^2\left(\pi-\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}$, there exists two integers $j_0 \in [1\,, \frac{\theta_0\, n}{\pi}]$ and $j_1 \in [\frac{\theta_0\, n}{\pi}, n-1]$ such that $H(j)$ increases for $1 \le j \le j_0$ and for $j_1 + 1 \le j \le n-1$ and decreases for $j_0 + 1 \le j \le j_1$. If $\frac{1}{2n^2\left(\pi-\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)} \le t \le \frac{1}{2\,n\,\pi\,\sin\left(\frac{\pi}{n}\right)}$, then there exists a unique integer $j_0 \in [1, \frac{\theta_0\, n}{\pi}]$ such that $H(j)$ increases for $1 \le j \le j_0$ and decreases for $j_0 + 1 \le j \le n-1$. Finally, if $t \ge \frac{1}{2n\pi\sin\left(\frac{\pi}{n}\right)}$, then the function H decreases on $[1, n-1]$. Suppose that $t \ge \pi^{-1}\, n^{-2}$; then $H'(n/2) \le 0$, which implies that $j_0 \le [n/2]$. The inequalities $\frac{2}{\pi}\, u \le \sin(u) \le u$, which hold for $0 \le u \le \frac{\pi}{2}$ imply that for $z \le n$, $H(z) \le z\,\exp(-4\,t\,z^2) \le C\, t^{-\frac{1}{2}}\, e^{-ct}$ and that $H(n-1) \le C\, n\, e^{-c\,t\,n^2} \le C\, t^{-\frac{1}{2}}\, e^{-ctn^{-2}}$ and finally that $H(1) \le C\, e^{-4\,t}$. Set $A(l) := [0,1] \cap \Big(\big[\frac{l-1}{n}, \frac{l+2}{n}\big] \cup \Big[0, \frac{(2-l)^+}{n}\Big] \cup \big[(2 - \frac{l+2}{n}) \wedge 1, 1\big]\Big)$; then $dx(A(l)) \le C\, n^{-1}$. Let $\tilde{C}_0 > 0$ to be chosen later on; the upper estimates of $H(j)$ for $j = 1, j_0$ and $n-1$ together with (4.43) imply that for $\gamma \ge \pi^{-1} \vee \tilde{C}_0$ and $t \ge \gamma\, n^{-2}$, $0 < \lambda < 1$ and $y \in A(l)$,

$$\begin{aligned}
\sup_{x \in [\frac{l}{n}, \frac{l+1}{n}]} T_3(t,x,y) \;&\le\; C\,(1 + t^{-\lambda})\,(1 + t^{-\frac{1-\lambda}{2}})\, e^{-ct} \\
&\qquad \times \int_{\frac{l}{n}}^{\frac{l+1}{n}} \Big[\, \Big|\sin\big(\frac{\pi\,(u-y)}{2}\big)\Big|^{-1+\lambda} + \Big|\sin\big(\frac{\pi\,(u+y)}{2}\big)\Big|^{-1+\lambda} \Big]\, du \\
&\le\; C\,(1 + t^{-\frac{1+\lambda}{2}})\, n^{-\lambda}\, e^{-ct}\,,
\end{aligned} \tag{4.45}$$

while for $y \notin A(l)$,

$$\sup_{x \in [\frac{l}{n}, \frac{l+1}{n}]} T_3(t,x,y) \le C\, n^{-1}\,(1 + t^{-\frac{1}{2}})\, e^{-ct}\, \Big[\, \Big|y - \frac{2l+1}{2n}\Big|^{-1} + \Big|y + \frac{2l+1}{2n}\Big|^{-1} + \Big|2n - y - \frac{2l+1}{2n}\Big|^{-1} \Big]. \tag{4.46}$$

Then, using the partition $A(l), A(l)^c$ and the inequalities (4.45), (4.45) and (4.43), we deduce that for $\bar{\lambda} \in]0,1[$ and $t \ge \gamma\, n^{-2}$:

$$\begin{aligned}
\sup_{x \in [0,1]} T_3(t,x,y) \;&\le\; C\, e^{-ct}\, \Big[(1 + t^{-\frac{1+\bar{\lambda}}{2}})\, n^{-1-\bar{\lambda}} + n^{-1}\,(1 + t^{-\bar{\lambda}})\,(1 + t^{-\frac{1-\bar{\lambda}}{2}}) \int_{2n-1}^{2} u^{-(1-\bar{\lambda})}\, du \Big] \\
&\le\; C\, e^{-ct}\,(1 + t^{-\frac{1+\bar{\lambda}}{2}})\, n^{-1}\,.
\end{aligned} \tag{4.47}$$

Similarly, for $t \ge \gamma\, n^{-2}$ and $\lambda \in]0,1[$ (4.45) implies that there exists a constant $C > 0$ such that for every $l \in \{0, \cdots, n-1\}$ and $x \in [\frac{l}{n}, \frac{l+1}{n}]$,

$$\begin{aligned}
\|T_3(t,x,.)\, 1_{A(l)}(.)\|^2_{(\alpha)} \;&\le\; C\, e^{-ct}\,(1 + t^{-1-\lambda})\, n^{-2\lambda} \int_0^{4\,n^{-1}} \int_0^{4\,n^{-1}} |u-v|^{-\alpha}\, du\, dv \\
&\le\; C\, e^{-ct}\,(1 + t^{-1-\lambda})\, n^{-2+\alpha-2\lambda}\,.
\end{aligned} \tag{4.48}$$

Furthermore, for $t \geq \gamma\, n^{-2}$ the inequalities (4.43) and (4.46) and separate estimates in the cases $y, z \notin A(l)$ and either $|y-z| \leq n^{-1}$ or $|y-z| \geq n^{-1}$ yield that $\nu \in]0, \frac{\alpha}{2}[$ there exists a constant $C > 0$ such that for every $l \in \{0, \cdots, n-1\}$ and $x \in [\frac{l}{n}], \frac{l+1}{n}]$:

$$
\begin{aligned}
\|T_3(t,x,.)\, 1_{A_n(l)^c}(.)\|^2_{(\alpha)} &\leq C\, n^{-2}\, e^{-ct}\, (1 + t^{-\nu}\, t^{-\frac{1-\nu}{2}}\, t^{-\frac{1}{2}}) \\
&\quad \times \Big[\int_{n^{-1}}^{2} \int_{n^{-1}}^{2} 1_{\{|\xi-\eta| \leq n^{-1}\}}\, \xi^{-1+\nu} \eta^{-1}\, |\xi - \eta|^{-\alpha}\, d\xi\, d\eta \\
&\qquad + \int_{n^{-1}}^{2} \int_{n^{-1}}^{2} 1_{\{|\xi-\eta| \geq n^{-1}\}}\, \xi^{-1+\nu}\, |\xi-\eta||^{\alpha}\, \eta^{-1} |d\xi\, d\eta \Big] \\
&\leq C\, n^{-2}\, e^{-ct}\, (1 + t^{-1-\frac{\nu}{2}}) \Big[\Big(\int_{n^{-1}}^{2} u^{-2+\nu}\, du \Big) \Big(\int_{0}^{4n^{-1}} v^{-\alpha}\, dv \Big) \\
&\qquad + \Big(\int_{n^{-1}}^{2} u^{-1+\nu-\frac{\alpha}{2}}\, du \Big) \Big(\int_{n^{-1}}^{2} v^{-1-\frac{\alpha}{2}}\, dv \Big) \Big] \\
&\leq C\, n^{-2-\nu+\alpha}\, e^{-ct}\, (1 + t^{-1-\frac{\nu}{2}})\,. \qquad (4.49)
\end{aligned}
$$

The inequalities (4.48) and (4.49] imply that for $t \geq \gamma\, n^{-2}$, $\lambda = \frac{\nu}{2} \in]0, \frac{\alpha}{4}[$ and $\nu \in]0, \frac{\alpha}{2}[$,

$$
\sup_{x \in [0,1]} \|T_3(t,x,.)\|^2_{(\alpha)} \leq C\, e^{-ct}\, n^{-2+\alpha} n^{-2\lambda}\, (1 + t^{-1-\lambda})\,. \qquad (4.50)
$$

We finally give upper estimates of T_4. We suppose that $x = \dfrac{l}{n}$, $1 \leq l \leq n-1$; the general case is easily deduced by linear interpolation. For $\dfrac{k}{n} \leq y \leq \dfrac{k+1}{n}$, $0 \leq k \leq n-1$, one has $k_n(y) = \dfrac{k}{n}$ and using (A.16), we deduce

$$
\begin{aligned}
T_4(t,x,y) &= \left| \sum_{j=1}^{n-1} e^{t\lambda_j^n} \sin(j\,\pi\, x) \int_{\frac{k}{n}}^{y} j\pi \cos(j\pi u) du \right| \\
&\leq \frac{1}{2} \int_{\frac{k}{n}}^{y} \left| \sum_{j=1}^{n-1} j e^{t\lambda_j^n} \Big[\sin(j\,\pi\,(x+u)) + \sin(j\,\pi\,(x-u)) \Big] \right| du \\
&\leq C\, n^{-1} \Big[t^{-1} + 1 \Big]\, e^{-ct}\,. \qquad (4.51)
\end{aligned}
$$

Let $B(l) := \{u \in [0,1]; \big|\frac{l}{n} - u\big| \leq \frac{1}{n}$ or $\frac{l}{n} - u \leq \frac{1}{n}$ or $2 - \frac{l}{n} - u \leq \frac{1}{n}\}$; as usual, $dx(B(l)) \leq cn^{-1}$. Let then $C^1(l) := \{y \in [0,1]; \exists u \in B(l) \cap [k_n(y), y]\}$ and for $i = 1,2$ $\tilde{C}^i(l) := \{z \in [0,1]; \exists y \in C^1(l), |y-z| \leq \frac{i}{n}\}$. Then $dx(\tilde{C}^i(l)) \leq C\, n^{-1}$ and for $y \notin \tilde{C}^1(l)$, one has $|y-x| \wedge (y+x) \wedge (2-x-y) \geq n^{-1}$. The computations made to prove (4.45) and (4.46) yield for $\lambda \in]0,1[$ the existence of a constant $C > 0$ such that for every $l \in \{0, \cdots, n\}$,

$$
\begin{aligned}
T_4(t, l/n, y)\, 1_{\tilde{C}^2(l)}(y) &\leq C\, n^{-\lambda} \big(1 + t^{-\frac{1+\lambda}{2}}\big) e^{-ct}\,, \\
T_4(t, l/n, y)\, 1_{\tilde{C}^2(l)^c}(y) &\leq C\, n^{-1}\, (1 + t^{-\frac{1}{2}})\, e^{-ct} \Big[\big|y - \frac{l}{n}\big|^{-1} + \big(y - \frac{l}{n}\big)^{-1} + \big|2 - y - \frac{l}{n}\big|^{-1}\,.
\end{aligned}
$$

Computations similar to those proving (4.47), (4.48) and (4.49) imply that for $\lambda \in]0,1[$ and $\nu \in]0, \frac{\alpha}{4}[$ there exists a constant $C > 0$ such that:

$$
\begin{aligned}
\sup_{x \in [0,1]} T_4(t,x,y) &\leq C\, e^{-ct}\, (1 + t^{-\frac{1+\lambda}{2}})\, n^{-1} \qquad (4.52) \\
\sup_{x \in [0,1]} \|T_4(t,x,\cdot)\|^2_{(\alpha)} &\leq n^{-2+\alpha}\, e^{-ct} (1 + t^{-(1+\nu)})\, n^{-2\nu}\,. \qquad (4.53)
\end{aligned}
$$

The inequalities (4.8), (4.18) with $A = 2$ and $\lambda = \frac{1}{2}$, (4.47) and (4.52) with $\bar\lambda = \frac{1}{2}$ imply that for some $\mu \in]0, \frac{1}{2}[$ there exists a constant $C > 0$ such that for every $t \geq \gamma\, n^{-2}$ with a constant $\gamma > 0$ large enough, one has

$$\sup_{x\in[0,1]} \|G(t,x,.-G^n(t,x,.)|\|_1 \leq C\left[n^{-1}\left(1+t^{-\frac{3}{4}}\right)+e^{-c\,t\,n^2}\left(1+t^{-\mu}\right)\right]. \tag{4.54}$$

On the other hand, the inequalities (4.14), (4.41), (4.50) and (4.53) imply that for $\nu \in]0, \frac{\alpha}{4}[$, $\lambda \in]1, \frac{3}{2}[$, $\mu \in]\alpha, 1[$, there exists a constant $C > 0$ such that for $t \geq \gamma\, n^{-2}$ and $\gamma > 0$ large enough, one has

$$\begin{aligned}\sup_{x\in[0,1]} \|\,|G(t,x,.-G^n(t,x,.)|\,\|^2_{(\alpha)} \leq C\, n^\alpha \Big[& e^{-ctn^2} + n^{-1}\, e^{-ctn} \\ & + n^{-2}\, e^{-ct}\left(1+t^{-(1+\nu)}\, n^{-2\nu}+t^{-1+\frac{\alpha}{3}}\right) \\ & + \left(n^{-4+2\mu}\, t^{-\mu} + n^{-4}\, t^{-2} + n^{-2\lambda} t^{-\lambda}\right) 1_{\{\gamma n^{-2}\leq t\leq \tilde C_1 n^{-1}\}}\Big],\end{aligned} \tag{4.55}$$

which proves the desired estimate for $d = 1$.

To conclude the proof, it remains to extend the inequalities (4.54) and (4.55) to any dimension d and to integrate with respect to t. We use the fact that for any $d \geq 2$, we have

$$\begin{aligned}|G_d(t,x,y)-(G_d)^n(t,x,y)| \;\leq\; & \sum_{i=1}^{d}\Big(\prod_{j=1}^{i-1}|G(t,x_j,y_j)|\Big)|G(t,x_i,y_i)-G^n(t,x_i,y_i)| \\ & \times\Big(\prod_{j=i+1}^{d}|G^n(t,x_j,y_j)|\Big).\end{aligned} \tag{4.56}$$

Hence, the inequalities (A.5), (A.19), (4.54) and (4.56) imply that for any $\lambda \in]0, 1[$,and any $\nu \in]0, 1/4[$, there exists a constant $C > 0$ such that for any $t \geq \gamma\, n^{-2}$,

$$\sup_{x\in Q} \|G_d(t,x,.)-(G_d)^n(t,x,.)\|_1 \leq C\, t^{-\nu}\left[n^{-1}\left(1+t^{-\frac{3}{4}}\right)+e^{-c\,t\,n^2}\left(1+t^{-\nu}\right)\right]; \tag{4.57}$$

Using (4.3) and integrating (4.57) with respect to t on $[\gamma\, n^{-2}, +\infty[$ we obtain (4.1). Finally, for $1 \leq k \leq d-1$ set $\alpha_k = \alpha\, 2^{-k}$ and set $\alpha_d = \alpha_{d-1}$; then using (4.56), (2.2), (A.1), (A.21) and (4.55), we deduce that for $\alpha \in]0, 2[$, $C_1 > 0$, $\lambda \in]\alpha, 1[$, $\mu \in]1, \frac{3}{2}[$, $\nu \in]0, \frac{\alpha_d}{4}[$ and for $t \geq \gamma\, n^{-2}$ for $\gamma > 0$ large enough, one has since $t^{-\frac{1}{2}} \leq C\, n$ on the time interval $[\gamma\, n^{-2}, +\infty[$

$$\begin{aligned}\sup_{x\in Q} \|\,|G_d(t,x,.-(G_d)^n(t,x,.)|\,\|_{(\alpha)} \leq\; & C\sum_{i=1}^{d} t^{-\frac{1}{2}\sum_{j=1}^{i-1}\alpha_j} \\ & \times\|\,|G(t,x_i,.)-G^n(t,x_i,.)|\,\|_{(\alpha_i)}\, n^{\sum_{j=i+1}^{d}\alpha_j} \\ \leq C\, n^\alpha\Big[& e^{-ctn^2}+n^{-1}\, e^{-ctn}+n^{-2}\, e^{-ct}\left(t^{-(1+\nu)}\, n^{-2\nu}+t^{-(1+\lambda)}\, n^{-2\lambda}+1\right) \\ & +\left(n^{-4}\, t^{-2}+n^{-4+2\nu}\, t^{-2+\nu}+n^{-2\mu}\, t^{-\mu}\right)1_{\{\gamma n^{-2}\leq t\leq \tilde C_1 n^{-1}\}}\Big].\end{aligned} \tag{4.58}$$

Integrating (4.58) on the time interval $[\gamma\, n^{-2}, +\infty[$ and using (4.4), we deduce (4.2). □

We now estimate the norm of the difference $(G_d)^n$ and $(G_d)^{n,m}$.

Lemma 4.2 *Given any $T > 0$ and $\nu > 0$ there exists $C > 0$ such that*

$$\sup_{x\in Q} \int_0^T \int_Q \big[\, |(G_d)^n(t,x,y) - (G_d)^{n,m}(t+T\,m^{-1},x,y)| + |(G_d)^n(t,x,y) - (G_d)^n_m(t+T\,m^{-1},x,y)|\,\big]\,dydt \le C\,m^{-1+\nu}\,, \tag{4.59}$$

$$\sup_{x\in Q} \int_0^T \big[\, \|\,|(G_d)^n(t,x,\cdot) - (G_d)^{n,m}(t+T\,m^{-1},x,\cdot)|\,\|^2_{(\alpha)} + |(G_d)^n(t,x,.) - (G_d)^n_m(t,x,.)|^2_{(\alpha)}\,\big]\,dt \le C\,m^{-1+\frac{\alpha}{2}}\,. \tag{4.60}$$

Proof : We only prove these inequalities for $G^n - G^{n,m}$ under the homogeneous Dirichlet boundary conditions and first suppose that $d=1$; let $\bar G^n_m = G^n - \tilde G^n_m$ and $\bar G^{n,m} = G^{n,m} - \tilde G^{n,m}$ where $\tilde G^{n,m}$ is defined by (A.27), i.e.,

$$\tilde G^{n,m}(t,x,y) = \sum_{j=1}^{(n\wedge\sqrt{m})-1} \big(1 - T\,m^{-1}\,\lambda^n_j\big)^{[\frac{mt}{T}]}\,\varphi^n_k(x)\varphi_k(\kappa_n(y))\,,$$

and $\tilde G^n_m$ is defined by

$$\tilde G^n_m(t,x,y) = \sum_{j=1}^{(n\wedge\sqrt{m})-1} e^{\lambda^n_j t}\,\varphi^n_k(x)\varphi_k(\kappa_n(y))\,. \tag{4.61}$$

Then (A.35) and (A.38) provide upper estimates of the norms of $\bar G^{n,m}$. Furthermore, if $\bar G^n_m$ is different from zero, then $\sqrt{m} < n$ and (A.15) with $J_0 = \sqrt{m}$ and $\beta = 0$ implies the existence of positive constants c, C such that for every $t > 0$ and $x, y \in [0,1]$:

$$|\bar G^n_m(t,x,y)| \le C\,e^{-ctm}\,\big(1 + t^{-\frac12}\big)\,. \tag{4.62}$$

Since $j \longrightarrow \lambda^n_j$ in decreasing, Abel's transform implies that for $x = l/n$:

$$|\bar G^n_m(t,x,y)| \le C\,e^{-ctm}\Big[\frac{1}{|x-\kappa_n(y)|} + \frac{1}{x+\kappa_n(y)} + \frac{1}{2-x-\kappa_n(y)}\Big]\,. \tag{4.63}$$

Thus, if $\bar D^3_m(l)$ is defined as in the proof of lemma A.5, for some $\bar c > 0$

$$\begin{aligned}
&\int_{\bar D^3_m(l)^c}\int_{\bar D^3_m(l)^c} |\bar G^n_m(t,x,y)|\,|y-z|^{-\alpha}\,|\bar G^n_m(t,x,z)|\,dy\,dz \\
&\quad\le C\,e^{-ctm}\Big[\int_{m^{-\frac12}}^2 u^{-2}\,du \int_0^{\bar c m^{-\frac12}} v^{-\alpha}\,dv + \Big(\int_{m^{-\frac12}}^2 u^{-1-\frac{\alpha}{2}}\,du\Big)^2\,\Big] \\
&\quad\le C\,e^{-ctm}\,m^{\frac{\alpha}{2}}\,,
\end{aligned}$$

while for $x = l/n$, $\mu \in]\alpha,1[$, $\nu \in]0, 1-\mu[$ and $\beta = \mu+\nu \in]\alpha,1[$, we deduce that

$$\begin{aligned}
&\int_{\bar D^3_m(l)}\int_{\bar D^3_m(l)} |\bar G^n_m(t,x,y)|\,|y-z|^{-\alpha}\,|\bar G^n_m(t,x,z)|\,dydz \\
&\quad\le C\,\big(1+t^{-\frac{\beta}{2}}\big)\,e^{-ctm}\int_0^{\bar c m^{-\frac12}} u^{-1+\mu-\alpha}\,du \int_0^{\bar c m^{-\frac12}} v^{-1+\nu}\,dv \\
&\quad\le C\,\big(1+t^{-\frac{\beta}{2}}\big)\,m^{\frac{\alpha-\beta}{2}}\,e^{-ctm}\,.
\end{aligned}$$

Thus, for $\lambda \in]0, \frac{1}{2}[$ and $\beta \in]\alpha, 1[$,

$$\|\bar{G}^n_m(t,x,.)\|_1 \;\leq\; C\left(1+t^{-\lambda}\right) e^{-ctm}\,, \tag{4.64}$$
$$\|\,|\bar{G}^n_m(t,x,.)|\,\|^2_{(\alpha)} \;\leq\; C\, e^{-ctm}\, t^{-\frac{\beta}{2}}\, m^{\frac{\alpha-\beta}{2}}\,. \tag{4.65}$$

Let $\tilde{c}$ be a positive constant to be fixed later on; for $t \leq \tilde{c}\,T\,m^{-1}$ we estimate separately the norms of $\tilde{G}^{n,m}(t,x,.)$ and $\tilde{G}^n_m(t,x,.)$. Inequalities (A.31) and (A.32) provide us the estimates of $\tilde{G}^{n,m}$; for $\tilde{G}^n_m$, we proceed in a similar way. Indeed, $j \to \exp(\lambda^n_j\, t)$ is decreasing and $\exp(\lambda^n_j\, t) \leq e^{-ctj^2}$ for some positive constant c, and $|\tilde{G}^n_m(t,x,y)| \leq C\,(n \wedge \sqrt{m})$. The arguments used to prove (A.31) and (A.32) immediately yield that these inequalities hold with $\tilde{G}^n_m$ instead of $\tilde{G}^{n,m}$; hence for any $\tilde{c} > 0$ there exists a constant $C > 0$ such that for $t \leq \tilde{c}\,T\,m^{-1}$,

$$\|\tilde{G}^n_m(t,x,.) - \tilde{G}^{n,m}(t,x,.)\|_1 \;\leq\; C\,(1+t^{-\lambda})\,, \tag{4.66}$$
$$\|\,|\tilde{G}^n_m(t,x,.) - \tilde{G}^{n,m}(t,x,.)|\,\|^2_{(\alpha)} \;\leq\; C\,(n \wedge \sqrt{m})^{\alpha}\,. \tag{4.67}$$

Furthermore, if $t \in [\tilde{c}\,T\,m^{-1}, T]$, then :

$$|\tilde{G}^n_m(t,x,y) - \tilde{G}^{m,n}(t,x,y)| \leq \bar{T}(t,x,y) = T_1(t,x,y) + T_2(t,x,y)$$

where

$$T_1(t,x,y) \;=\; \left|\sum_{j=1}^{(n\wedge\sqrt{m})-1}\left[\exp\left(\frac{\left(\left[\frac{mt}{T}\right]+1\right)\,\lambda^n_j\,T}{m}\right) - \left(1-\lambda^n_j\,\frac{T}{m}\right)^{-\left(\left[\frac{mt}{T}\right]+1\right)}\right]\varphi^n_j(x)\varphi_j(\kappa_n(y))\right|,$$
$$T_2(t,x,y) \;=\; \left|\sum_{j=1}^{(n\wedge\sqrt{m})-1}\left[\exp\left(\lambda^n_j t\right) - \exp\left(\frac{\left(\left[\frac{mt}{T}\right]+1\right)\,\lambda^n_j\,T}{m}\right)\right]\varphi^n_j(x)\varphi_j(\kappa_n(y))\right|.$$

We first study $T_1(t,x,y)$; for $x \in]0, n \wedge \sqrt{m}[$, set

$$\Phi_1(x) = \exp\left[-\left(\left[\frac{mt}{T}\right]+1\right)\ln\left(1+\frac{4n^2T\sin^2\left(\frac{x\pi}{2n}\right)}{m}\right)\right] - \exp\left(-\frac{\left(\left[\frac{mt}{T}\right]+1\right)\,4n^2T\sin^2\left(\frac{x\pi}{2n}\right)}{m}\right).$$

Then $\Phi_1(x) \geq 0$ and $\Phi_1'(x) = \frac{T}{m}\,\left(\left[\frac{mt}{T}\right]+1\right)\,2n\pi\sin\left(\frac{x\pi}{n}\right)\cdot\psi_1(x)$, where for $a = \left[\frac{mt}{T}\right]+1$ and $u = 4n^2\frac{T}{m}\sin^2\left(\frac{x\pi}{2n}\right) \geq 0$,

$$\psi_1(x) = \Delta(u) := e^{-au} - (1+u)^{-(a+1)}\,.$$

We remark that for fixed $t > 0$, a increases to $+\infty$ with m and that for $t \leq \tilde{c}Tm^{-1}$, $a \geq [\tilde{c}]+1 > \tilde{c}$. We then write

$$\Delta(u) = e^{-au}\,\delta(u),\;\; \delta(u) = 1 - \exp\left[au - (a+1)\ln(1+u)\right].$$

It is easy to see that δ increases on $[0, \frac{1}{a}]$ and decreases on $[\frac{1}{a}, +\infty[$. Since $\delta(0) = 0$, $\delta \geq 0$ on $[0, u_0]$ and $\delta \leq 0$ on $[u_0, +\infty[$ for some $u_0 > \frac{1}{a}$. We remark that for any $c > 0$

$$\delta\left(\frac{c}{a}\right) = 1 - \exp\left[c - (a+1)\,\ln(\left(1+\frac{c}{a}\right)\right] < 0$$

if

$$(a+1)\,\ln\left(1+\frac{c}{a}\right) \leq (a+1)\,\left[\frac{c}{a} - \frac{a^2}{2\,c^2} + 3\,\frac{a^3}{3\,c^3}\right] \leq c\,.$$

This last inequality holds if and only if for $X=\frac{c}{a}$, $\Pi(X)=\frac{1}{3}X^2-\frac{1}{2}X+\frac{1}{a+1}<0$. If $a+1>\frac{16}{3}$, which holds if $t\geq\frac{4T}{m}$, then the discriminant of $\Pi(X)$ is positive and $\Pi(X)<0$ for $X\in]X_1,X_2[$, where

$$X_1=\frac{3}{2}\left(\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{4}{3(a+1)}}\right)<\frac{4}{a+1}\text{ and }X_2=\frac{3}{2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{4}{3(a+1)}}\right)>\frac{3}{4}.$$

For $a\geq 5$, $\frac{4}{a+1}<\frac{3}{4}$ and $\Pi(X)<0$ on $[\frac{4}{a+1},\frac{3}{4}]$, that is $\Pi(\frac{c}{a})>0$ for $\frac{c}{a}=\frac{4}{a+1}$, i.e., for some $c<4$. Furthermore, for this choice of $u_0=\frac{c}{a}$, if $u_0=4\,n^2\,\frac{T}{m}\,\sin^2(\frac{x_0\,\pi}{2n})$, then

$$0\leq\Phi_1(x_0)\leq e^{-a\,\ln(1+\frac{c}{a})}-e^{-c}\leq C\,\left(e^{\frac{c^2}{2a}}-1\right)\leq\frac{C\,T}{mt}.$$

For $t\geq\frac{4T}{m}$, using Abel's transform, we have for $x=\dfrac{l}{n}$ and $\kappa_n(y)=\frac{k}{n}$,

$$|T_1(t,x,y)|\leq C\,\frac{T}{mt}\Big[\frac{1}{|x-\kappa_n(y)|}+\frac{1}{x+\kappa_n(y)}+\frac{1}{2-x-\kappa_n(y)}\Big] \tag{4.68}$$

We now estimate $T_2(t,x,y)$ by Abel's transform if $t\geq\tilde c\,T\,m^{-1}$ for large enough $\tilde c>0$. For $x\in]0,n\wedge\sqrt{m}[$, set

$$\Phi_2(x)=\exp\Big[-4\,n^2\,t\,\sin^2\Big(\frac{x\pi}{2n}\Big)\Big]-\exp\Big[-4\,n^2\,\big([mtT^{-1}+1]\big)\,T\,m^{-1}\,\sin^2\Big(\frac{x\pi}{2n}\Big)\Big].$$

Then $\Phi_2(x)\geq 0$ and for $b=\big([m\,t\,T^{-1}]+1\big)\,T\,m^{-1}$ and $U(x)=4\,n^2\,\sin^2(\frac{x\pi}{2n})$, one has

$$\Phi_2'(x)=2\,n\,\pi\,\sin\Big(\frac{x\pi}{n}\Big)\,\Big[b\,e^{-bU(x)}-t\,e^{-tU(x)}\Big].$$

Since $\frac{\partial}{\partial z}[z\,e^{-z\,U(x)}]=[1-z\,U(x)]\,e^{-zU(x)}$, we deduce that for $b\,U(x)\leq 1$, i.e., $x\leq C_1t^{-\frac{1}{2}}$, we have $\Phi_2'(x)\geq 0$ while for $t\,U(x)\geq 1$, i.e., $x\geq C_2t^{-\frac{1}{2}}$, we have $\Phi_2'(x)\leq 0$. Suppose finally that $t\,U(x)\leq 1\leq b\,U(x)$ and let $\varepsilon_0>0$ be fixed small enough; then for $t\geq\tilde c\,T\,m^{-1}$ for $\tilde c$ large enough, we have $b\leq t+T\,m^{-1}\leq\frac{\tilde c+1}{\tilde c}\,t$, which implies that $b=(1+\varepsilon)\,t$ for $\varepsilon\in]0,\varepsilon_0]$. Thus, for $t\,U(x)\in[(1+\varepsilon)^{-1},1]$

$$\begin{aligned}\Phi_2'(x)&=2\,n\,\pi\,\sin\Big(\frac{x\pi}{n}\Big)\,t\,e^{-t\,U(x)}\,\Big[(1+\varepsilon)\,e^{-\varepsilon tU(x)}-1\Big]\\&=2\,n\,\pi\,\sin\Big(\frac{x\pi}{n}\Big)\,t\,e^{-t\,U(x)}\,\Big[\varepsilon\,(1-tU(x))-\varepsilon^2\,t\Big(1-\frac{t\,U(x)}{2}\Big)+O(\varepsilon^3)\Big].\end{aligned}$$

Set $Z=t\,U(x)$ and set $\Pi(Z)=\frac{\varepsilon}{2}\,Z^2-(1+\varepsilon)\,Z+1$; then Π has two roots, $Z_1=\varepsilon^{-1}\,(1+\varepsilon-\sqrt{1+\varepsilon^2})=1-\frac{\varepsilon}{2}+0(\varepsilon^2)$ and $Z_2=\varepsilon^{-1}\,(1+\varepsilon+\sqrt{1+\varepsilon^2})=\frac{2}{\varepsilon}+1+0(\varepsilon)$. Thus, for $t\,U(x)\in[(1+\varepsilon)^{-1},1-\frac{\varepsilon}{2}[$, $\Phi_2'(x)>0$ while for $t\,U(x)\in]1-\frac{\varepsilon}{2},1]$ $\Phi_2'(x)<0$. Therefore the function $j\in\{1,\cdots,n\wedge\sqrt{m}\}\to\Phi_2(j)$ increases on $\{1,\cdots,j_0\}$ and decreases on $\{j_0+1,n\wedge\sqrt{m}\}$, for some $j_0=C\,t^{-\frac{1}{2}}$ and

$$\Phi_2(j_0)\leq\exp\Big(-4\,n^2\,b\,\sin^2\Big(\frac{c\pi}{2\,n\,\sqrt{t}}\Big)\Big)\,\Big[\exp\Big(4\,n^2\,T\,m^{-1}\,\sin^2\Big(\frac{c\pi}{2\,n\,\sqrt{t}}\Big)\Big)-1\Big]\leq\frac{CT}{mt}.$$

Therefore, given $\tilde c$ large enough, for $t\geq\tilde cTm^{-1}$, we have using Abel's transform:

$$|T_2(t,x,y)|\leq\frac{CT}{mt}\Big[\frac{1}{|x-\kappa_n(y)|}+\frac{1}{x+\kappa_n(y)}+\frac{1}{2-x-\kappa_n(y)}\Big]. \tag{4.69}$$

Furthermore,

$$|\bar{T}(t,x,y)| \leq |T_1(t,x,y)| + |T_2(t,x,y)| \leq C\,(n \wedge \sqrt{m})\,. \tag{4.70}$$

Hence for $\lambda \in]0,1[$, using (4.70), (4.68) and (4.69) we deduce that there exists a constant $C>0$ such that for $t \geq \frac{4T}{m}$:

$$\|\bar{T}(t,x,.)\|_1 \leq C t^{-1+\lambda}\, m^{-1+\frac{3\lambda}{2}}\,, \tag{4.71}$$

and for $\mu \in]\alpha,1[$, $\nu \in]0,1-\mu[$ and $\beta = \mu+\nu \in]\alpha,1[$, using the sets $\mathcal{A}^{(i)}_{n\wedge\sqrt{m}}(x)$ for $i \leq 3$ and $\mathcal{B}^{(j)}_{n\wedge\sqrt{m}}(x)$ for $j=1,2$ and the fact that $\frac{1}{n} \leq \frac{1}{n\wedge\sqrt{m}}$, using (4.68) and (4.69), we deduce that given $\tilde{c}$ large enough, there exists constants $c, C>0$ such that for every $t \in [\frac{\tilde{c}T}{m}, T]$:

$$\begin{aligned}
\|\,|\bar{T}(t,x,.)|\,\|^2_{(\alpha)} \;\leq\;& C\,(n\wedge\sqrt{m})^{\mu+\nu}\,(T\,m^{-1}\,t^{-1})^{2-\mu-\nu} \int_0^{\frac{c}{n\wedge\sqrt{m}}} u^{-1+\mu-\alpha}\,du \int_0^{\frac{c}{n\wedge\sqrt{m}}} v^{-1+\nu}\,dv \\
&+\frac{C\,T^2}{m^2\,t^2}\Big[\Big(\int_{\frac{c}{n\wedge\sqrt{m}}}^2 u^{-2}\,du\Big)\Big(\int_0^{\frac{c}{n\wedge\sqrt{m}}} v^{-\alpha}\,dv\Big) + \Big(\int_{\frac{c}{n\wedge\sqrt{m}}}^2 u^{-1-\frac{\alpha}{2}}\,du\Big)^2\Big] \\
\leq\;& C\,(n\wedge\sqrt{m})^{\alpha}\,m^{-2}\,t^{-2} \leq C\,m^{-2+\frac{\alpha}{2}}\,t^{-2}\,.
\end{aligned} \tag{4.72}$$

For $d=1$, the inequalities (A.35), (4.64), (4.66 and (4.71) imply the existence of $\lambda \in]0,\frac{1}{2}[$ and positive constants c, C such that for any $t \in]0,T]$:

$$\sup_{x\in Q} \|(G_d)^n(t,x,.) - (G_d)^{n,m}(t,x,.)\|_1 \leq C\left[\left(1+t^{-\lambda}\right)e^{-ctm} + t^{-1+\lambda}\,m^{-1+\frac{3\lambda}{2}}\right], \tag{4.73}$$

while the inequalities (A.38), (4.65), (4.67) and (4.72) yield the existence of $\beta \in]0,\alpha\wedge d[$ and positive constants $\tilde{c}, c$ and C such that for every $t \in]0,T]$:

$$\sup_{x\in Q} \|\,|(G_d)^n(t,x,.) - (G_d)^{n,m}(t,x,.)|\,\|^2_\alpha \leq C\,m^{\frac{\alpha}{2}}\left[e^{-ctm}\left(1+(tm)^{-\frac{\beta}{2}}\right) + m^{-2}\,t^{-2}\,1_{[\tilde{c}Tm^{-1},T]}(t)\right]. \tag{4.74}$$

As in the proof of Lemma 4.1, let $\alpha_k = \alpha\,2^k$ for $1 \leq k \leq d-1$ and $\alpha_d = \alpha_{d-1}$. The inequalities (4.73) for $d=1$, (A.19) and (A.25) yield (4.73) for any d, while the inequalities (4.74) for $d=1$, (A.21) and (A.26) yield (4.74) for any d. Integrating with respect to t we deduce the inequalities (4.59) and (4.60). □

5 Some numerical results

In order to study the influence of the correlation coefficient α of the Gaussian noise on the speed of convergence, we have implemented in C the implicit discretization scheme $u^{n,m}$ in the case of homogeneous boundary conditions in dimension $d=1$ for the equation (2.15).

To check the influence of the time mesh, we have fixed the space mesh n^{-1} with $n=500$ and taken the smallest time mesh m_0^{-1} with $m_0 = 20736$. Using one trajectory of the noise F, we have approximated by the Monte-Carlo method $e(m_i) = \mathbb{E}(|u^{n,m_0}(1,.5) - u^{n,m_i}(1,.5)|^2)$ and $\hat{e}(m_i) = \sup_{x\in[0,1]} \mathbb{E}(|u^{n,m_0}(1,x) - u^{n,m_i}(1,x)|^2)$ for 13 divisors m_i of m_0, ranging from $m_1 = 854$ to $m_{13} = 144$. These simulations have been done for various values of α, including the case of the space-time white noise which corresponds to the limit case $\alpha=1$. Assuming that u^{n,m_0} is close to u, according to (3.15) and (3.27), these errors should behave like $C\,[m_i^{-(1-\frac{\alpha}{2})} + n^{-(2-\alpha)}] \sim m_i^{-(1-\frac{\alpha}{2})}$ for this choice of n and m_i. Thus, we have computed the linear regression coefficients

$c(t)$ and $d(t)$ (resp. $\hat{c}(t)$ and $\hat{d}(t)$) of $\ln(e(m_i))$ (resp. of $\ln(\hat{e}(m_i))$), i.e., of the approximation of $\ln(e(m_i))$ by $c(t)\ln(m_i)+d(t)$ as well as the corresponding standard deviation sd (reps. $\hat{sd}$) for $K=3200$ Monte-Carlo iterations in the case $\sigma(x)=0.2\,x+1$ and $b(x)=x+2$, which are summarized in the following:

α	Theoretical exponent	observed $c(t)$ $x=\frac{1}{2}$	sd $x=\frac{1}{2}$	observed $\hat{c}(t)$ $\sup_x$	$\hat{sd}$ $\sup_x$
White noise	0.5	0.6664727	0.006270	0.6329907	0.0107664
0.9	0.55	0.6954329	0.0120673	0.6852743	0.0129754
0.8	0.6	0.7547656	0.0098305	0.7203216	0.0134158
0.7	0.65	0.7511618	0.0089021	0.7507985	0.0185679
0.6	0.7	0.8158496	0.0143157	0.8006539	0.0089716
0.5	0.75	0.8826439	0.0144247	0.8512296	0.0088718
0.4	0.8	0.8986740	0.0099570	0.9111777	0.0112821
0.3	0.85	0.9592507	0.0116868	0.9134532	0.0116631
0.2	0.9	0.9890983	0.0115842	0.9562799	0.0147068
0.1	0.95	1.1797052	0.0114345	1.0219011	0.0120427

The study of the influence of the space mesh is done in a similar way; we fix the time mesh m^{-1} with $m=32000$ and let the smallest space mesh $n_0=432$. Again for various divisors of n_0, using one trajectory of the noise F we have approximated $\varepsilon(n_i)=\mathbb{E}(|u^{n_0,m}(1,.5)-u^{n_i,m}(1,.5)|^2)$ and $\tilde{\varepsilon}(n_i)=\mathbb{E}(|u^{n_0,m}(1,.5)-u^{n_i,m}(1,.5)|^2)$ for the 7 divisors n_i of n_0 ranging from 72 to 12. Assuming that $u^{n_0,m}$ is close to u, according to (3.15) and (3.27), these errors should behave like $C\,[m^{-(1-\frac{\alpha}{2})}+n_i^{-(2-\alpha)}]\sim n_i^{-(2-\alpha)}$ for this choice of n_i and m. Thus, we have computed the linear regression coefficients $\gamma(x)$ and $\delta(x)$ (resp. $\hat{\gamma}(t)$ and $\hat{\delta}(t)$) of $\ln(\varepsilon(n_i))$ (resp. of $\ln(\hat{\varepsilon}(n_i))$), i.e., of the approximation of $\ln(\varepsilon(n_i))$ by $\gamma(x)\ln(n_i)+\delta(x)$ as well as the corresponding standard deviation SD (reps. $\hat{SD}$) for $K=3200$ iterations in the case $\sigma(x)=1$ and $b(x)=2x+3$; these results are summarized in the following:

α	Theoretical exponent	observed $\gamma(x)$ $x=\frac{1}{2}$	SD $x=\frac{1}{2}$	observed $\hat{\gamma}(x)$ $\sup_x$	$\hat{SD}$ $\sup_x$
White noise	1.0	1.2512638	0.0345607	1.2503774	0.0267767
0.9	1.1	1.3466928	0.0340313	1.3360574	0.0200828
0.8	1.2	1.43471	0.0335898	1.4250593	0.0210765
0.7	1.3	1.5460363	0.0305230	1.5050464	0.0298206
0.6	1.4	1.5869267	0.0209759	1.5859167	0.0273728
0.5	1.5	1.6713847	0.0279707	1.667131	0.0271759
0.4	1.6	1.7704079	0.0282726	1.7259275	0.0258797
0.3	1.7	1.8380935	0.0280384	1.7910783	0.0231881
0.2	1.8	1.8978477	0.0274440	1.8503569	0.0208426
0.1	1.9	1.9236069	0.0208369	1.9054399	0.0229467

Finally, since our method applies in the case of non-linear coefficients, we have performed similar computations for $e(m_i)$, $\hat{e}(m_i)$, $\varepsilon(n_j)$ and $\hat{\varepsilon}(n_j)$ for $1 \leq i \leq 13$ and $1 \leq j \leq 7$ with $K = 3000$ iterations in the case $\sigma(x) = b(x) = 1 + 0.2\,\cos(x)$; the corresponding results are summarized in the following:

α	Theoretical exponent	observed $c(t)$ $x=\frac{1}{2}$	sd $x=\frac{1}{2}$	observed $\hat{c}(t)$ $\sup_x$	$\hat{sd}$ $\sup_x$
White noise	0.5	0.4914526	0.0602499	0.5200141	0.0430660
0.8	0.6	0.5550415	0.0449213	0.6069792	0.0495536
0.5	0.75	0.7243992	0.0176232	0.7946869	0.0430660
0.2	0.9	0.8606597	0.0225398	0.8571082	0.0429083
α	Theoretical exponent	observed $\gamma(x)$ $x=\frac{1}{2}$	SD $x=\frac{1}{2}$	observed $\hat{\gamma}(x)$ $\sup_x$	$\hat{SD}$ $\sup_x$
White noise	1.0	1.0277942	0.0789985	0.8263350	0.1056492
0.8	1.2	1.3628039	0.0830278	1.127607	0.0684362
0.5	1.5	1.5626127	0.0710379	1.5506996	0.0685654
0.2	1.8	1.7350629	0.0707697	1.4874854	0.0768065

In this semi-linear (non-linear) case, the speed of convergence is worse and the precision is less than in the previous cases.

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A Appendix.

We start this section by stating and proving some results concerning the Green kernel G_d in arbitrary dimension $d \geq 1$. As in the previous sections, we will suppose that G_d and its discretized versions are defined with the homogeneous Dirichlet conditions on δQ; all the results stated remain true if one replaces the Dirichlet conditions by the Neumann ones.

Lemma A.1 *Let $d \geq 1$ and $\alpha \in]0, 2 \wedge d[$. There exists some constant $C > 0$ depending only on α, such that for all x, x' in $Q = [0,1]^d$ and $0 < t \leq t' \leq T$:*

$$\sup_{y \in Q} \| \, |G_d(t,y,\cdot)| \, \|^2_{(\alpha)} \;\leq\; C\, t^{-\frac{\alpha}{2}}\,, \tag{A.1}$$

$$\int_0^{+\infty} \| \, |G_d(t,x,\cdot) - G_d(t,x',\cdot)| \, \|^2_{(\alpha)} \, dt \;\leq\; C\, |x - x'|^{2-\alpha}\,, \tag{A.2}$$

$$\sup_{x\in Q}\int_0^t \|\,|G_d(t'-s,x,\cdot)-G_d(t-s,x,\cdot)|\,\|^2_{(\alpha)}\,ds \;\leq\; C\,|t'-t|^{1-\frac{\alpha}{2}}\,, \tag{A.3}$$

$$\sup_{x\in Q}\int_t^{t'} \|\,|G_d(t'-s,x,\cdot)|\,\|^2_{(\alpha)}\,ds \;\leq\; C\,|t'-t|^{1-\frac{\alpha}{2}}\,. \tag{A.4}$$

Proof: To prove (A.1), recall first the usual upper estimate of $|G_d|$:

$$|G_d(t,x,y)|\leq C\,t^{-\frac{d}{2}}\,\exp\left(-c\frac{|x-y|^2}{t}\right). \tag{A.5}$$

We remark that, if $|x-y|\geq|y-z|$, one has $\exp\left(-c\,|x-y|^2\,t^{-1}\right)\leq\exp\left(-c\,|y-z|^2\,t^{-1}\right)$, while if $|x-y|\leq|y-z|$, one has $|y-z|^{-\alpha}\geq|x-y|^{-\alpha}$. Hence

$$\begin{aligned}\sup_{x\in Q}\|G_d(t,x,\cdot)\|^2_{(\alpha)} \;&\leq\; C\,t^{-d}\left(\int_0^{+\infty}e^{-c\frac{u^2}{t}}\,u^{-\alpha+d-1}\,du\right)\cdot\left(\int_0^{+\infty}e^{-c\frac{v^2}{t}}\,v^{d-1}\,dv\right)\\ &\leq\; Ct^{-\frac{\alpha}{2}}.\end{aligned}$$

We now prove (A.2) and set $x'=x+v$. Then, for $0<t\leq|v|^2$, we have

$$\|\,|G_d(t,x,\cdot)-G_d(t,x',\cdot)|\,\|^2_{(\alpha)}\leq 2\,[\,\|\,|G_d(t,x,\cdot)\,\|^2_\alpha+\|\,|G_d(t,x',\cdot)\,\|^2_\alpha\,]\,.$$

The change of variables defined by $x-y=|v|\,\eta$, $x-z=|v|\,\xi$ and $t=|v|^2\,s$ in the first integral (and a similar one with x' instead of x in the second one), combined with (A.1), yields

$$\begin{aligned}\int_0^{|v|^2}\|\,|G_d(t,x,\cdot)-G_d(t,x',\cdot)|\,\|^2_{(\alpha)}\;dt &\leq C\int_0^1 s^{-d}\,|v|^{-2d+2}\,ds\\ &\quad\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-c\frac{|\eta|^2}{s}}\,|v|^{-\alpha}\,|\eta-\xi|^{-\alpha}\,e^{-c\frac{|\xi|^2}{s}}\,|v|^{2d}\,d\xi\,d\eta\\ \leq\;& C|v|^{2-\alpha}\int_0^1 s^{-d}\,ds\Big\{\iint_{\{|\xi-\eta|\leq|\eta|\}}e^{-c\frac{|\xi-\eta|^2}{s}}|\xi-\eta|^{-\alpha}e^{-c\frac{|\xi|^2}{s}}d\xi d\eta\\ &\qquad+\iint_{\{|\xi-\eta|\geq|\eta|\}}e^{-c\frac{|\eta|^2}{s}}|\eta|^{-\alpha}\,e^{-c\frac{|\xi|^2}{s}}\,d\xi d\eta\Big\}\\ \leq\;& C\,|v|^{2-\alpha}\int_0^1 s^{-\frac{\alpha}{2}}ds=C\,|v|^{2-\alpha}.\end{aligned} \tag{A.6}$$

On the other hand, if $t\geq|v|^2$, for every $j\in\{1,\cdots,d\}$, we use the following well-known estimate

$$\left|\frac{\partial}{\partial x_j}G_d(t,x,y)\right|\leq Ct^{-\frac{d+1}{2}}\,\exp\left(-c\frac{|x-y|^2}{t}\right) \tag{A.7}$$

and the fact that, if $G=G_1$, we have

$$\begin{aligned}|G_d(t,x,y)-G_d(t,x',y)| \;&\leq\; \sum_{i=1}^d\left(\prod_{j=1}^{i-1}|G(t,x_j,y_j)|\right)\,|G(t,x_i,y_i)-G(t,x'_i,y_i)|\\ &\qquad\times\left(\prod_{j=i+1}^d|G(t,x'_j,y_j)|\right)\,.\end{aligned}$$

Thus, Taylor's formula implies that

$$\begin{aligned}
&\int_{|v|^2}^{+\infty} \| \,|G_d(t,x,\cdot)-G_d(t,x',\cdot)|\, \|_{(\alpha)}^2 \, dt \\
&\quad\leq\; C\sum_{i=1}^d v_i^2 \int_{v_i^2}^{+\infty} t^{-(d+1)}\, dt \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left(\prod_{j=1}^{i-1} e^{-c\frac{|x_j-y_j|^2}{t}}\, |y_j-z_j|^{-\alpha_j}\, e^{-c\frac{|x_j-z_j|^2}{t}} \right) \int_{[0,1]^2} d\lambda_1 d\lambda_2 \\
&\qquad\qquad \times e^{-c\frac{|x_i+\lambda_1 v_i-y_i|^2}{t}}\, |y_i-z_i|^{-\alpha}\, e^{-c\frac{|x_i+\lambda_2 v_i-z_i|^2}{t}} \left(\prod_{j=i+1}^{d} e^{-c\frac{|x'_j-y_j|^2}{t}}\, |y_j-z_j|^{-\alpha_j}\, e^{-c\frac{|x'_j-z_j|^2}{t}} \right).
\end{aligned}$$

Then, for every $i\in\{1\cdots,d\}$ such that $v_i\neq 0$, the changes of variables $x_j-y_j=v_i\,\eta_j$, $x_j-z_j=v_i\,\xi_j$ for $j\leq i$, $x'_j-y_j=v_i\,\eta_j$, $x'_j-z_j=v_i\,\xi_j$ for $j\geq i+1$, and $t=v_i^2\, s$ for every j yield

$$\begin{aligned}
&\int_{|v|^2}^{+\infty} \| \,|G_d(t,x,\cdot)-G_d(t,x',\cdot)|\, \|_{(\alpha)}^2 \, dt \\
&\quad\leq\; C\sum_{i=1}^d 1_{\{v_i\neq 0\}} v_i^2 \int_1^{+\infty} |v_i|^{-2(d+1)}\, v_i^2\, s^{-(d+1)}\, ds \int_{\mathbb{R}^d} d\eta \int_{\mathbb{R}^d} d\xi\, |v_i|^{2d} \\
&\qquad \times \left(\prod_{j\neq i} e^{-c\frac{|\eta_j|^2}{s}}\, |v_i|^{-\alpha_j}\, |\eta_j-\xi_j|^{-\alpha_j}\, e^{-c\frac{|\xi_j|^2}{s}} \right) \int_{[0,1]^2} d\lambda_1\, d\lambda_2\, e^{-c\frac{|\eta_i+\lambda_1|^2}{s}}\, |v_i|^{-\alpha_i}\, e^{-c\frac{|\eta_i+\lambda_2|^2}{s}} \\
&\quad\leq\; C\sum_{i=1}^d 1_{\{v_i\neq 0\}}\, |v_i|^{2-\alpha} \int_1^{+\infty} s^{-(d+1)}\, ds \int_{-1}^1 d\lambda \left(\prod_{j\neq i} \int_{\mathbb{R}^2} e^{-c\frac{|\eta_j|^2}{s}}\, |\eta_j-\xi_j|^{-\alpha_j}\, e^{-c\frac{|\xi_j|^2}{s}}\, d\xi_j\, d\eta_j \right) \\
&\qquad \times \int_{\mathbb{R}^2} e^{-c\frac{|\eta_i+\lambda|^2}{s}}\, |\eta_i-\xi_i|^{-\alpha_i}\, e^{-c\frac{|\xi_i+\lambda|^2}{s}}\, d\xi_i\, d\eta_j\, .
\end{aligned}$$

Splitting again the integrals between $\{|\xi_j-\eta_j|\leq|\eta_j|\}$ and $\{|\xi_j-\eta_j|\geq|\eta_j|\}$ for $j\neq i$ and $\{|\xi_i-\eta_i|\leq|\eta_i+\lambda|\}$ and $\{|\xi_i-\eta_i|\geq|\eta_i+\lambda|\}$ yields

$$\int_{|v|^2}^{+\infty} \| \,|G_d(t,x,\cdot)-G_d(t,x',\cdot)|\, \|_{(\alpha)}^2 \leq C\, |v|^{2-\alpha} \int_1^{+\infty} s^{-(1+\alpha)} ds \leq C\, |v|^{2-\alpha}. \tag{A.8}$$

The inequalities (A.6) and (A.8) give (A.2). We now prove (A.3) and set $h=t'-t$; an argument similar to that used to prove (A.6), based on the change of variables defined by $t-s=h\,r$, $y-x=\sqrt{h}\,\eta$ and $z-x=\sqrt{h}\,\xi$ yields

$$\begin{aligned}
&\int_0^{h\wedge t} \| \,|G_d(t'-s,x,\cdot)-G_d(t-s,x,\cdot)|\, \|_{(\alpha)}^2 \, ds \\
&\quad\leq\; C\int_0^1 h^{-d+1}\, dr \Big[(r+1)^{-d} \int_{\mathbb{R}^d} d\eta \int_{\mathbb{R}^d} d\xi\, h^d\, h\, e^{-c\frac{|\eta|^2}{r+1}}\, h^{-\frac{\alpha}{2}}\, |\xi-\eta|^{-\alpha}\, e^{-c\frac{|\xi|^2}{r+1}} \\
&\qquad\qquad + r^{-d} \int_{\mathbb{R}^d} d\eta \int_{\mathbb{R}^d} d\xi\, h^d\, e^{-c\frac{|\eta|^2}{r}}\, h^{-\frac{\alpha}{2}}\, |\xi-\eta|^{-\alpha}\, e^{-c\frac{|\xi|^2}{r}} \Big] \\
&\quad\leq\; C\, h^{1-\frac{\alpha}{2}} \int_0^1 \left\{ (r+1)^{-\frac{\alpha}{2}} + r^{-\frac{\alpha}{2}} \right\} ds \leq C\, h^{1-\frac{\alpha}{2}}\, .
\end{aligned} \tag{A.9}$$

On the other hand, for $s\geq h$, we use Taylor's formula and the estimate

$$\left| \frac{\partial}{\partial t} G_d(t,x,y) \right| \leq C t^{-\frac{d+2}{2}}\, e^{-c\frac{|x-y|^2}{t}}, \tag{A.10}$$

combined with the previous change of variables used to prove (A.9); thus we obtain

$$\begin{aligned}
&\int_{h\wedge t}^{t} \| \, |G_d(t'-s,x,\cdot) - G_d(t-s,x,\cdot)| \, \|_{(\alpha)}^2 \, ds \\
&\leq \; C\, h^2 \int_{h\wedge t}^{t} ds \int_{[0,1]^2} d\lambda_1 \, d\lambda_2 \int_{Q^2} (s+\lambda_1 h)^{-\frac{d+2}{2}} (s+\lambda_2 h)^{-\frac{d+2}{2}} \, e^{-c\frac{|x-y|^2}{s+\lambda_1 h}} \, |y-z|^{-\alpha} \, e^{-c\frac{|x-z|^2}{s+\lambda_2 h}} \, dy \, dz \\
&\leq \; C\, h^2 \int_1^{+\infty} h^{-(d+2)} \, h \, dr \int_{[0,1]^2} d\lambda_1 \, 1 d\lambda_2 \; (r+\lambda_1)^{-\frac{d+1}{2}} \, (r+\lambda_2)^{-\frac{d+1}{2}} \\
&\qquad\qquad \times \int_{Q^2} h^d \, e^{-c\frac{|\eta|^2}{r+\lambda_1}} \, h^{-\frac{\alpha}{2}} \, |\xi-\eta|^{-\alpha} \, e^{-c\frac{|\xi|^2}{r+\lambda_2}} \, d\xi d\eta \\
&\leq \; C\, h^{1-\frac{\alpha}{2}} \int_1^{+\infty} dr \int_{[0,1]^2} d\lambda_1 \, d\lambda_2 \, (r+\lambda_1)^{-\frac{d+1}{2}} \, (r+\lambda_2)^{-\frac{d+1}{2}} \int_{\mathbb{R}^{2d}} e^{-c\frac{|u|^2}{r+\lambda_1}} |u|^{-\alpha} \, e^{-c\frac{|v|^2}{r+\lambda_2}} |v|^{-\alpha} \, du \, dv \\
&\leq \; C\, h^{1-\frac{\alpha}{2}} \int_1^{+\infty} dr \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 \, (r+\lambda_1)^{-\frac{1+\alpha}{2}} \, (r+\lambda_2)^{-\frac{1+\alpha}{2}} \\
&\leq \; C\, h^{1-\frac{\alpha}{2}} \int_1^{+\infty} dr \, r^{-(1+\alpha)} \leq C\, h^{1-\frac{\alpha}{2}} \, . \qquad \text{(A.11)}
\end{aligned}$$

The inequalities (A.9) and (A.11) yield (A.3). Finally, (A.1) implies that for $h = t'-t > 0$,

$$\int_t^{t'} \| \, |G_d(t'-s,x,\cdot)| \, \|_{(\alpha)}^2 \, ds \leq C \int_0^h r^{-\frac{\alpha}{2}} dr = C\, h^{1-\frac{\alpha}{2}} \, ,$$

which completes the proof of (A.4). □

We recall the following well-known set of estimates, and briefly sketch their proof for the sake of completeness:

Lemma A.2 *For $x, x' \in Q$, $0 \leq t < t' \leq T$ and $\mu \in]0,1[$,*

$$\int_0^{+\infty} \int_Q |G_d(t,x,y) - G_d(t,x',y)| \, dy \, dt \;\leq\; C\, |x-x'| \, , \qquad \text{(A.12)}$$

$$\int_0^t \int_Q |G_d(t-s,x,y) - G_d(t'-s,x,y)| \, dy \, ds \;\leq\; C\, |t'-t|^{\mu} \, , \qquad \text{(A.13)}$$

$$\int_t^{t'} |G_d(t-s,x,y)| \, dy ds \leq C\, |t'-t| \, . \qquad \text{(A.14)}$$

Proof : The inequality (A.14) is a straightforward consequence of (A.5); on the other hand, (A.12) is deduced from Taylor's formula and (A.7). Finally, to prove (A.13), one writes for $\mu \in]0,1[$,

$$\begin{aligned}
&|G_d(s+h,x,y) - G_d(s,x,y)| \\
&\quad \leq \; C \, \left\{ |G_d(s+h,x,y)|^{1-\mu} + |G_d(s,x,y)|^{1-\mu} \right\} \cdot h^{\mu} \int_0^1 \left| \frac{\partial}{\partial s} G_d(s+\lambda h,x,y) \right|^{\mu} d\lambda \, .
\end{aligned}$$

(A.10) yields $\int_0^t \int_Q |G_d(s+h,x,y) - G_d(s,x,y)| \, dy \, ds \leq h^{\mu} \int_0^t s^{-\mu} \, ds$, which implies (A.13). □

The following technical result are needed to obtain refined estimates for the discretized kernels G^n and $G^{n,m}$.

Lemma A.3 *For any $c \geq 0$ there exists a constant $C > 0$ such that, for $K \geq 0$, $\beta \in [0,1[$, $t > 0$, $a > 1$ and $J_0 \geq 1$,*

$$\sum_{j=J_0}^{\infty} j^{-\beta} e^{-ctj^2} \leq C\, e^{-ctJ_0^2} \left[1 + t^{-\frac{1-\beta}{2}}\right], \tag{A.15}$$

$$\sum_{j=1}^{\infty} j^{K} e^{-ctj^2} \leq C \left[1 + t^{-\frac{K+1}{2}}\right] e^{-ct}, \tag{A.16}$$

$$\sum_{j=J_0}^{\infty} \Big(1 + \frac{cTj^2}{m}\Big)^{-1} \leq C\, m^{\frac{1}{2}}\, T^{-\frac{1}{2}}, \tag{A.17}$$

$$\sum_{j=J_0}^{\infty} \Big(1 + \frac{cTj^2}{m}\Big)^{-a} \leq C \frac{m}{J_0\, T\, (a-1)} \left(1 + \frac{c\,T\, J_0^2}{m}\right)^{-a+1}. \tag{A.18}$$

Proof : Since the functions $x \longmapsto x^{-\beta} e^{-ctx^2}$ are decreasing for $0 \leq \beta \leq 1$, we have

$$\begin{aligned}\sum_{j=J_0}^{\infty} j^{-\beta} e^{-ctj^2} &\leq J_0^{-\beta} e^{-ctJ_0^2} + \int_{J_0}^{+\infty} x^{-\beta} e^{-ctx^2} dx \\ &\leq J_0^{-\beta} e^{-ctJ_0^2} + t^{-\frac{1-\beta}{2}} \left[e^{-ctJ_0^2} \int_{J_0\sqrt{t}}^{1} x^{-\beta} dx\, 1_{\{J_0\sqrt{t} \leq 1\}} + \int_{(J_0\sqrt{t})\vee 1}^{+\infty} e^{-cx^2} dx \right].\end{aligned}$$

For $0 \leq \beta < 1$, the last inequality yields (A.15) and also (A.16) for $K = 0$. Finally, given $K \geq 1$, the function $x \longmapsto x^K e^{-ctx^2}$ increases on $\left[0, \sqrt{\frac{K}{2ct}}\right]$ and decreases on $\left[\sqrt{\frac{K}{2ct}}, +\infty\right[$. Therefore, if $t \geq K/(2c)$, a change of variables and integration by parts yield

$$\sum_{j=1}^{\infty} j^K e^{-ctj^2} \leq e^{-ct} + \int_1^{+\infty} x^K e^{-ctx^2} dx \leq e^{-ct} + t^{-\frac{K+1}{2}} \int_{\sqrt{t}}^{+\infty} y^K e^{-cy^2} dy \leq C\, e^{-ct}$$

and on the other hand, if $t \leq K/(2c)$, if $J_1 := \left[\sqrt{K/(2ct)}\right]$, then

$$\begin{aligned}\sum_{j=1}^{\infty} j^K e^{-ctj^2} &\leq \int_1^{J_1} x^K e^{-ctx^2} dx + J_1^K e^{-ctJ_1^2} + \int_{J_1}^{+\infty} x^K e^{-ctx^2} dx \\ &\leq t^{-\frac{K+1}{2}} \int_{\sqrt{t}}^{+\infty} x^K e^{-cx^2} dx + Ct^{-\frac{K}{2}} \leq Ct^{-\frac{K+1}{2}}.\end{aligned}$$

To prove (A.17) and (A.18), we use the fact that $j \to \left(1 + \frac{cTj^2}{m}\right)^{-a}$ decreases for any $a \geq 1$, we deduce that

$$\sum_{j=J_0}^{\infty} \Big(1 + \frac{cTj^2}{m}\Big)^{-a} \leq \int_{J_0}^{\infty} \Big(1 + \frac{cTx^2}{m}\Big)^{-a} dx\,;$$

direct integration for $a = 1$ and the obvious estimate $1 \leq \frac{x}{J_0}$ for $a > 1$ yield the inequalities (A.17) and (A.18). □

The following lemma provides bounds for the $\|\ \|_1$ and $\|\ \|_\alpha$ norms of $|(G_d)^n(t, x, .)|$.

Lemma A.4 *There exists a constant $C > 0$ such that for every $t > 0$, $d \geq 1$, $\lambda > 0$ and $0 < \alpha < \beta < d \wedge 2$,*

$$\sup_n \sup_{x\in Q} \|(G_d)^n(t,x,.)\|_1 \leq C\, t^{-\lambda}\, e^{-ct}\,, \tag{A.19}$$

$$\sup_n \sup_{x\in Q} \|(G_d)^n(t,x,.)\|^2_{(\alpha)} \leq C\, t^{-\frac{\beta}{2}}\, e^{-ct}\,, \tag{A.20}$$

$$\sup_{x\in Q} \|(G_d)^n(t,x,.)\|^2_{(\alpha)} \leq C\, n^{\alpha}\, e^{-ct}\,. \tag{A.21}$$

Proof: It suffices to check these inequalities for $x = l/n$, $0 \leq l \leq n$; we at first prove them for $d = 1$. Let $\tilde{S}_j(x) := \sum_{i=1}^{j-1} \cos(i\pi x)$; using Abel's transform and well-known estimates for $\tilde{S}_j(x)$ we have, since $j \longmapsto \lambda_j^n = -4n^2 \sin^2\left(\frac{j\pi}{2n}\right)$ is decreasing, for $\kappa = \kappa_n(y)$,

$$\begin{aligned}
|G^n(t,x,y)| &= C\left|\sum_{j=1}^{n-1} e^{\lambda_j^n t}\big[\tilde{S}_{j+1}(\pi\frac{l-\kappa}{n}) - \tilde{S}_{j+1}(\pi\frac{l+\kappa}{n}) - \tilde{S}_j(\pi\frac{l-k}{n}) + \tilde{S}_j(\pi\frac{l+k}{n}\big]\right| \\
&\leq C\left|e^{\lambda^n_{n-1}t}\big[\tilde{S}_n(\pi\frac{l-\kappa}{n}) - \tilde{S}_n(\pi\frac{l+\kappa}{n})\big]\right| + C\left|\sum_{j=2}^{n-1}\big[\tilde{S}_j(\pi\frac{l-\kappa}{n}) - \tilde{S}_j(\pi\frac{l+\kappa}{n})\big]\,\big[e^{\lambda^n_{j-1}t} - e^{\lambda^n_j t}\big]\right| \\
&\leq C\, e^{\lambda_1^n t}\left\{\left|\sin\left(\pi\frac{x-\kappa_n(y)}{2}\right)\right|^{-1} + \left|\sin\left(\pi\frac{x+\kappa_n(y)}{2}\right)\right|^{-1}\right\}.
\end{aligned} \tag{A.22}$$

Fix $0 < \lambda < \frac{1}{2}$ and $t > 0$; using (A.15) with $\beta = 0$, $J_0 = 1$, and (A.22) we deduce that

$$\begin{aligned}
\sup_{x\in[0,1]} \|\, G^n(t,x,.)\,\|_1 &\leq C\, e^{-ct}\,(1+t^{-\lambda}) \sup_{x\in[0,1]}\int_0^1 \Big[\frac{1}{|x-\kappa_n(y)|^{1-2\lambda}} + \frac{1}{|x+\kappa_n(y)|^{1-2\lambda}} \\
&\qquad + \frac{1}{|2-x-\kappa_n(y)|^{1-2\lambda}}\Big]\, dy \\
&\leq C\, e^{-ct}(1+t^{-\lambda})\,.
\end{aligned} \tag{A.23}$$

Let $A_n^i(x)$ be the sets defined by (4.9); for $0 \leq l \leq n$, set $D_n^{(i)}(l) = A_n^i(l/n)$. Then $dx\left(D_n^{(i)}(l)\right) \leq \frac{C}{n}$ and, if $y \notin D_n^{(3)}(l)$, one has $|x-k_n(y)| \geq \frac{2}{3}|x-y|$, $|x+k_n(y)| \geq \frac{2}{3}|x+y|$ and similarly, if $y \notin D_n^{(2)}(l)$, $|x-k_n(y)| \geq \frac{1}{2}|x-y|$, $|x+k_n(y)| \geq \frac{1}{2}|x+y|$. Thus, for every $n \geq 1$ and $0 \leq l \leq n$, $\|\,|G^n(t,l/n,.)|\,\|^2_{(\alpha)} \leq C(T_1+T_2+T_3)$, where T_i is the integral of $|G^n(t,l/n,y)|\,|y-z|^{-\alpha}\,|G(t,l/n,z)|$ respectively on the sets $A_1 = \{(y,z)\,:\, y \in D_n^{(2)}(l), z \in D_n^{(2)}(l)\}$, $A_2 = \{(y,z)\,:\, y \in D_n^{(2)}(l)^c,\, z \in D_n^{(2)}(l)^c,\, |y-z| \leq n^{-1}\}$ and $A_3 = \{(y,z)\,:\, y \in D_n^{(2)}(l)^c,\, z \in D_n^{(2)}(l)^c,\, |y-z| \geq n^{-1}\}$.

Thus, using (A.15) and (A.22), we deduce that for $0 < \alpha < \lambda < 1$, $0 < \mu < 1$,

$$T_1 \leq C\, e^{-ct}\,(1+t^{-\frac{\lambda+\mu}{2}})\Big(\int_0^{3n^{-1}} u^{-1-\alpha+\lambda}\, du\Big)\Big(\int_0^{3n^{-1}} v^{-1+\mu}\, dv\Big)\,;$$

a similar computation implies that

$$T_3 \leq C\, e^{-ct}\,(1+t^{-\frac{\lambda+\mu}{2}})\Big(\int_{n^{-1}}^2 u^{-1-\alpha+\lambda}\, du\Big)\Big(\int_{n^{-1}}^2 v^{-1+\mu}\, dv\Big)\,.$$

Finally, for $0 < \mu < 1$,

$$T_2 \le C\, e^{-ct}\, (1 + t^{-\frac{\alpha+\mu}{2}})\, n^{1-\alpha} \Big(\int_0^{n^{-1}} u^{-\alpha}\, du \Big) \Big(\int_{n^{-1}}^{2} v^{-1+\mu} \Big) .$$

The upper estimates of T_i for $1 \le i \le 3$ clearly imply (A.20) when $d = 1$.

Again, to prove (A.21), it suffices to show that this inequality holds for $x = l/n$, i.e., that $\sup_{0\le l \le n} \| \, |G^n(t, l/n, .)| \, \|^2_{(\alpha)} \le C\, e^{-ct}\, n^\alpha$. Using the sets A_i, $1 \le i \le 3$, the inequality (A.22), the crude estimate

$$G^n(t, x, y) \le C\, n\, e^{-ct} , \tag{A.24}$$

and replacing in products involving two of the terms $|x - y|^{-1}$, $|x - z|^{-1}$ and $|y - z|^{-\frac{\alpha}{2}}$ the largest norm by the smallest one, we obtain for $\alpha < \lambda < 1$ and $0 < \mu < 1$,

$$\begin{aligned} \| \, |G^n(t, x, .)| \, \|^2_{(\alpha)} &\le C\, e^{-ct} \Bigg[n^{\lambda+\mu} \Bigg(\int_0^{3n^{-1}} u^{-1+\lambda-\alpha}\, du \Bigg) \Bigg(\int_0^{3n^{-1}} v^{-1+\mu}\, dv \Bigg) \\ &\qquad \times + \Bigg(\int_0^{n^{-1}} u^{-\alpha}\, du \Bigg) \Bigg(\int_{n^{-1}}^{2} v^2\, dv \Bigg) + \Bigg(\int_{n^{-1}}^{2} u^{-1-\frac{\alpha}{2}}\, du \Bigg)^2 \Bigg] \\ &\le C \Big[n^{\lambda+\mu}\, n^{-\lambda-\mu+\alpha} + n^\alpha \Big] e^{-ct} \le C\, n^\alpha\, e^{-ct} . \end{aligned}$$

Since $(G_d)^n(t, x, y) = \prod_{i=1}^d G^n(t, x_i, y_i)$, (A.23) immediately yields (A.19). For $d \ge 2$, and $1 \le i \le d-1$, set $\alpha_i = \alpha\, 2^{-i}$ and set $\alpha_d = \alpha_{d-1}$; then using (2.2), the inequality (A.20) (resp. (A.21)) for $d = 1$ yield (A.20) (resp. (A.21)) for every d. □

We now prove a similar result for the norms of $(G_d)^{n,m}(t, x, .)$.

Lemma A.5 *For every $\lambda \in]0, \frac{1}{2}[$ and $\beta \in]\alpha, d \wedge 2[$, there exist positive constants c and C such that for every $t \in]0, T]$,*

$$\sup_{x\in Q} \|(G_d)^{n,m}(t, x, .)\|_1 \le C\, e^{-ct} \big(1 + t^{-\lambda} \big) , \tag{A.25}$$

and

$$\sup_{x\in Q} \| \, |(G_d)^{n,m}(t, x, .)| \, \|^2_{(\alpha)} \le C\, e^{-ct} \Big[(n \wedge \sqrt{m})^\alpha \wedge \big(1 + t^{-\frac{\beta}{2}} \big) \Big] + C\, e^{-ctm}\, t^{-\frac{\beta}{2}} . \tag{A.26}$$

Proof : For $m \ge 1$, set $(\bar G_d)^{n,m}(t, x, y) := (G_d)^{n,m}(t, x, y) - (\tilde G_d)^{n,m}(t, x, y)$,

$$(\tilde G_d)^{n,m}(t, x, y) = \sum_{\underline{\mathbf{k}} \in \{1, \cdots, [(n\wedge\sqrt{m})-1)]\}^d} \prod_{i=1}^{d} \big(1 - T\, m^{-1}\, \lambda^n_{k_i} \big)^{-[\frac{mt}{T}]}\, \varphi^n_{\underline{\mathbf{k}}}(x)\, \varphi_{\underline{\mathbf{k}}}(\kappa_n(y)) . \tag{A.27}$$

Let $(\tilde G_1)^{n,m} = \tilde G^{n,m}$ and $(\bar G_1)^{n,m} = \bar G^{n,m}$. Since $j \to \big(1 - \frac{T}{m}\, \lambda^n_j \big)^{-[\frac{mt}{T}]}$ is decreasing,

$$|\tilde G^{n,m}(t + T\, m^{-1}, x, y)| \le C\, (n \wedge \sqrt{m})\, e^{-ct} \tag{A.28}$$

and Abel's transform yields that for $x = \frac{l}{n}$ and $\kappa_n(y) = \frac{k}{n}$:

$$|\tilde G^{m,n}(t + T\, m^{-1}, x, y)| \le C\, e^{-ct} \Big[\frac{1}{|x - \kappa_n(y)|} + \frac{1}{x + \kappa_n(y)} + \frac{1}{2 - x - \kappa_n(y)} \Big] . \tag{A.29}$$

Finally, since for $j \leq \sqrt{m}$, $\ln\Big(1+\frac{T}{m}\,4n^2\,\sin\big(\frac{j\pi}{2n}\big)\Big) \geq C\,j^2\,T\,m^{-1}$,

$$|\tilde{G}^{n,m}(t+T\,m^{-1},x,y)| \leq C \sum_{j=1}^{(n\wedge\sqrt{m})-1} e^{-ctj^2} \leq C\,e^{-ct}\left(1+t^{-\frac{1}{2}}\right). \tag{A.30}$$

Thus, repeating the arguments used to prove (A.19) - (A.21) we deduce that for $\lambda > 0$, $0 < \alpha < \beta < d\wedge 2$,

$$\sup_{x\in Q} \|\tilde{G}^{n,m}(t,x,.)\|_1 \;\leq\; C\,e^{-ct}\left(1+t^{-\lambda}\right), \tag{A.31}$$

$$\sup_{x\in Q} \|\tilde{G}^{n,m}(t,x,.)\|^2_{(\alpha)} \;\leq\; C\,e^{-ct}\left[\left(1+t^{-\frac{\beta}{2}}\right)\wedge (n\wedge\sqrt{m})^{\alpha}\right] \tag{A.32}$$

We finally give an upper estimate of the norms of $|(\bar{G}_d)^{n,m}(t,x,.)|$ and thus we suppose that $\sqrt{m} < n$. Using (A.17) we deduce that there exist positive constants c, C such that for any $t \leq \frac{2T}{m}$,

$$\sup_{x,y\in Q} |\bar{G}^{n,m}(t,x,y)| \leq C\int_{\sqrt{m-1}}^{+\infty}\big(1+\frac{cT}{m}\,x^2\big)^{-([\frac{mt}{T}]+1)}\,dx \leq C\int_{\sqrt{cT2^{-1}}}^{+\infty}(mT)^{-\frac{1}{2}}\,(1+y^2)^{-([\frac{mt}{T}]+1)}\,dy\,.$$

Hence for $t \leq 2Tm^{-1}$, since $[\frac{mt}{T}] = 1$ or 2, for $x,y\in Q$, $|\bar{G}^{n,m}(t,x,y)| \leq C \leq \sqrt{m} \leq t^{-\frac{1}{2}}$ while for $t \geq 2Tm^{-1}$,

$$|\bar{G}^{n,m}(t,x,y)| \leq C\,(mT)^{-\frac{1}{2}}\int_{\sqrt{cT}}^{+\infty} y\,(1+y^2)^{-([\frac{mt}{T}]+1)}\,dy \leq C\,t^{-1}\,m^{-\frac{1}{2}} \leq C\,t^{-\frac{1}{2}}\,.$$

This implies that

$$\sup_{x,y\in Q} |\bar{G}^{n,m}(t,x,y)| \leq C\left(1+t^{-\frac{1}{2}}\right). \tag{A.33}$$

Furthermore, since $j \to (1-T\,m^{-1}\,\lambda^n_j)^{-[\frac{mt}{T}]}$ decreases, Abel's transform implies that for $x = l/n$ and $\kappa_n(y) = k/n$,

$$\begin{aligned} |\bar{G}^{n,m}(t+T\,m^{-1},x,y)| &\leq C\left(1-T\,m^{-1}\lambda^n_{\sqrt{m}}\right)^{-([\frac{mt}{T}]+1)}\Big[\frac{1}{|\sin\big(\frac{\pi(x-\kappa_n(y))}{2}\big)|}+\frac{1}{|\sin\big(\frac{\pi(x+\kappa_n(y))}{2}\big)|}\Big] \\ &\leq C\,e^{-ctm}\Big[\frac{1}{|x-\kappa_n(y)|}+\frac{1}{x+\kappa_n(y)}+\frac{1}{2-x-\kappa_n(y)}\Big]. \end{aligned} \tag{A.34}$$

An argument similar to that used to prove (A.19) implies that for $\lambda \in]0,\frac{1}{2}[$, there exists $C > 0$ such that for $t \in]0,T[$,

$$\sup_{x\in Q} \|\bar{G}^{n,m}(t,x,.)\|_1 \leq C\left(1+t^{-\lambda}\right). \tag{A.35}$$

Finally, for $x = l/n$ let $\bar{D}^i_m(l) = \{z\in[0,1] : |x-z| \leq i\sqrt{m}, \text{or} x+z \leq i\sqrt{m}, \text{or} 2-x-z \leq i\sqrt{m}\}$. Then since $n \geq \sqrt{m}$, for $y \notin \bar{D}^3_m(l)$ we deduce that $|x-\kappa_n(y) \geq \frac{1}{2}|x-y|$, $|x+\kappa_n(y) \geq \frac{1}{2}|x-y|$ and $|2-x-\kappa_n(y) \geq \frac{1}{2}|x-y|$. Hence, the arguments used to prove (A.20) with $d=1$ and (A.21) show that

$$\begin{aligned} &\int_{\bar{D}^3_m(l)^c}\int_{\bar{D}^3_m(l)^c} |\bar{G}^{n,m}(t+T\,m^{-1},x,y)|\,|y-z|^{-\alpha}\,|\bar{G}^{n,m}(t+T\,m^{-1},x,z)|\,dy\,dz \\ &\leq C\,e^{-ctm}\Big[\int_{c\,m^{-\frac{1}{2}}}^{2}\int_0^{c\,m^{-\frac{1}{2}}} u^{-2}\,v^{-\alpha}\,du\,dv + \Big(\int_{c\,m^{-\frac{1}{2}}}^{2} u^{-(1-\frac{1}{2})}\,du\Big)^2\Big] \\ &\leq C\,e^{-ctm}\,m^{\frac{\alpha}{2}}\,. \end{aligned} \tag{A.36}$$

Furthermore, (A.33)-(A.34) imply that for $\mu \in]\alpha,1[$, $\nu \in]0,1-\mu[$ and $\beta=\mu+\nu \in]\alpha,1[$,

$$\int_{\bar D^3_m(l)}\int_{\bar D^3_m(l)} |\bar G^{n,m}(t+T\,m^{-1},x,y)|\,|y-z|^{-\alpha}\,|\bar G^{n,m}(t+T\,m^{-1},x,z)|\,dy\,dz$$

$$\leq C\,t^{-\frac{\mu+\nu}{2}}\,e^{-ctm}\int_0^{c\,m^{-\frac12}} u^{-1+\mu-\alpha}\,du\int_0^{c\,m^{-\frac12}} v^{-1+\nu}\,dv \leq C\,e^{-ctm}\,m^{\frac{\alpha}{2}}\,. \quad (A.37)$$

The inequalities (A.36) and (A.37) imply that for $\beta \in]\alpha, d\wedge 2[$,

$$\sup_{x\in[0,1]} \||\bar G^{n,m}(t,x,.)|\|^2_{(\alpha)} \leq C\,e^{-ctm}\,m^{\frac{\alpha}{2}}\left[1+(tm)^{-\frac{\beta}{2}}\right]. \quad (A.38)$$

Hence (A.38) and (A.32) imply that (A.26) holds for $d=1$. We conclude that this last inequality holds for any $d\geq 1$ as in the proof of Lemma A.4 $\square$

We finally prove upper estimates for the norms of time increments of $(G_d)^n$.

Lemma A.6 *For any $\lambda>0$ and $T>0$, there exists $C>0$ such that for any $h>0$*

$$\sup_{n\geq 1}\sup_{x\in Q}\sup_{t\in[0,T]}\int_0^t \|(G_d)^n(t-s,x,.)-(G_d)^n(t+h-s,x,.)\|_1\,ds \leq C\,h^{\frac12}\,, \quad (A.39)$$

$$\sup_{n\geq 1}\sup_{x\in Q}\sup_{t\in[0,T]}\int_t^{t+h} \|(G_d)^n(t+h-s,x,.)\|_1\,ds \leq C\,h^{1-\lambda}\,, \quad (A.40)$$

$$\sup_{n\geq 1}\sup_{x\in Q}\sup_{t\in[0,T]}\int_0^t \|\,|(G_d)^n(t-s,x,.)-(G_d)^n(t+h-s,x,.)|\,\|^2_{(\alpha)}\,ds \leq C\,h^{1-\frac{\alpha}{2}}\,, \quad (A.41)$$

$$\sup_{n\geq 1}\sup_{x\in Q}\sup_{t\in[0,T]}\int_t^{t+h} \|\,|(G_d)^n(t+h-s,x,.)\,|\|^2_{(\alpha)}\,ds \leq C\,h^{1-\frac{\alpha}{2}}\,. \quad (A.42)$$

Proof: Inequality (A.19) implies that for $x\in Q$, $\lambda>0$ and $n\geq 1$, $\int_0^h\int_Q |(G_d)^n(s,x,y)|\,dy\,ds \leq C\int_0^h s^{-\lambda}\,ds \leq C\,h^{1-\lambda}$, which proves (A.40). Using (A.21) we deduce that for any $\tilde c>0$ and $h\leq n^{-2}$,

$$\int_0^{\tilde c h} \|\,|(G_d)^n(s,x,.)|\,\|^2_{(\alpha)}\,ds \leq \int_0^{\tilde c h} n^\alpha\,ds \leq C\,n^\alpha\,h \leq h^{1-\frac{\alpha}{2}}\,. \quad (A.43)$$

Suppose that $s\geq n^{-2}$ and suppose that $d=1$. Then by (A.15), $|G^n(s,x,y)| \leq C\,(1+s^{-\frac12})$; using (A.22) and proceeding as in the proof of (A.21), replacing the sets $\mathcal{A}^i_n(x)$ defined by (4.9) by the sets $\mathcal{A}^i_h(x)=\{y\in[0,1] : |y-x|\leq i\sqrt{h} \text{ or } y+x\leq i\sqrt{h} \text{ or } 2-x-y\leq i\sqrt{h}\}$, since we have assumed that $n^{-1}\leq\sqrt{s}$, we deduce that $\|\,|G^n(s,x,.)|\,\|^2_{(\alpha)} \leq C\,s^{-\frac{\alpha}{2}}\,e^{-ct}$. Let $\alpha_k=\alpha\,2^{-k}$ for $1\leq k\leq d-1$ and $\alpha_d=\alpha_{d-1}$; using the inequality (2.2), we deduce that for $s\geq n^{-2}$, then

$$\|\,|(G_d)^n(s,x,.)|\,\|^2_{(\alpha)} \leq C\,s^{-\frac{\alpha}{2}}\,e^{-ct}\,. \quad (A.44)$$

Hence the inequalities (A.43) and (A.44) imply

$$\int_0^h \|\,|(G_d)^n(s,x,.)|\,\|^2_{(\alpha)}\,ds \leq C\Big[\int_0^{h\wedge n^{-2}} n^\alpha\,ds + \int_{h\wedge n^{-2}}^h s^{-\frac{\alpha}{2}}\,ds\Big] \leq C\,h^{1-\frac{\alpha}{2}}\,,$$

which proves (A.42). To prove (A.39) and (A.41), set $t'=t+h$; then for $d=1$ and $x=l\,n^{-1}$

$$|G^n(t,x,y)-G^n(t',x,y)| = \Big|\sum_{j=1}^{n-1} e^{-4tn^2\sin^2\left(\frac{j\pi}{2n}\right)}\left[1-e^{-4n^2h\sin^2\left(\frac{j\pi}{2n}\right)}\right]\varphi_j(x)\varphi_j(\kappa_n(y))\Big|.$$

Thus (A.16) implies the existence of a constant C such that for any $n \geq 1$and $x = l\, n^{-1}$,

$$\begin{aligned}\int_0^t \int_Q |G^n(t-s,x,y) - G^n(t'-s,x,y)|\, dy\, ds &\leq C \sum_{j=1}^{n-1} \int_0^t e^{-crj^2}\, dr \left[1 - e^{-4n^2 h \sin^2\left(\frac{j\pi}{2n}\right)}\right] \\ &\leq C \sum_{j=1}^{n-1} j^{-2} \left[(j^2\, h) \wedge 1\right] \leq C\, h^{\frac{1}{2}}\,,\end{aligned}$$

which proves (A.39). The inequality (A.43) proves that given any $\tilde{c} > 0$,

$$\sup_{n\geq 1} \sup_{x\in[0,1]} \int_O^{\tilde{c}h} \| \, |G^n(s,x,.)| + |G^n(s+h,x,.)| \, \|_{(\alpha)}^2 \, ds \leq C\, h^{1-\frac{\alpha}{2}}\,. \tag{A.45}$$

Fix $\tilde{c} > 0$ large enough, let $t \geq \tilde{c}\, h$ and set $\Phi(j) = \exp(-4n^2 t\, \sin^2(\frac{j\pi}{2n})) - \exp(-4n^2 t'\, \sin^2(\frac{j\pi}{2n})) \geq 0$. Then

$$\Phi'(j) = 2n \sin\left(\frac{j\pi}{n}\right) \left[t' \exp\Big(-4n^2 t' \sin^2\Big(\frac{j\pi}{2n}\Big)\Big) - t\, \exp\Big(-4n^2 t' \sin^2\Big(\frac{j\pi}{2n}\Big)\Big) \right].$$

Then the arguments used to estimate $\Phi_2(x)$ and then $|T_2(t,x,y)|$ in the proof of Lemma 4.2 show that there exists $C > 0$ such that for any $s \in [\tilde{c}h, T]$, $x = l\, n^{-1}$ and $y \in [0,1]$,

$$\begin{aligned}\Big| \sum_{j=[h^{-\frac{1}{2}}]}^{n-1} \big(e^{\lambda_j^n s} - e^{\lambda_j^n (s+h)}\big)\, \varphi_j(x)\, \varphi_j(\kappa_n(y)\Big| &\leq C\, \exp\big(\lambda^n_{[h^{-\frac{1}{2}}]}\, s\big) \\ &\times \Big[\frac{1}{|x-\kappa_n(y|} + \frac{1}{x+\kappa_n(y)} + \frac{1}{2-x-\kappa_n(y)} \Big]\,,\end{aligned}$$

while (A.15) implies that

$$\Big| \sum_{j=[h^{-\frac{1}{2}}]}^{n-1} \big(e^{\lambda_j^n t} - e^{\lambda_j^n t'}\big)\, \varphi_j(x)\, \varphi_j(\kappa_n(y)\Big| \leq C\, \exp\big(\lambda^n_{[h^{-\frac{1}{2}}]} t\big)\,.$$

Using again the sets $\mathcal{A}_h^i(x)$, we deduce that for any $s \in [\tilde{c}h, T]$

$$\sup_{n\geq 1} \sup_{x\in[0,1]} \| \, |G^n(s,x,.) - G^n(s+h,x,.)| \, \|_{(\alpha)}^2 \leq C\, h^{1-\frac{\alpha}{2}}\,.$$

Thus,

$$\int_{\tilde{c}h}^T \| \, |G^n(s,x,.) - G^n(s+h,x,.)| \, \|_{(\alpha)}^2 \, ds \leq C\, h^{1-\frac{\alpha}{2}}\,. \tag{A.46}$$

The inequalities (A.45) and (A.46) yield (A.41) for $d = 1$. Using (A.39) for $d = 1$ and (A.19) we deduce that (A.39) holds for every d. Let $\alpha_k = \alpha\, 2^{-k}$ for $1 \leq k \leq d-1$ and $\alpha_d = \alpha_{d-1}$; then using (2.2) and either (A.45) for $t \leq \tilde{c}h$, (A.44) and (A.46) for $t \geq \tilde{c}h$, we deduce that (A.41) holds for any d; this concludes the proof. □

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