Singularity functions for fractional processes: application to the fractional Brownian sheet

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Abstract

In this paper almost sure convergence and asymptotic normality of generalized quadratic variation are studied. The main result in this paper extend classical results from Baxter and Gladyshev so that they can be applied to fractional Gaussian processes. An application to the estimation of the true axes of a fractional Brownian sheet is also obtained.

Résumé

On étudie dans ce papier les propriétés de convergence et de normalité asymptotique des variations quadratiques généralisées d'un champ brownien fractionnaire. Le résultat principal est une extension des résultats classiques de Baxter et Gladyshev au cas de processus gaussiens fractionnaires. Ceci est enfin appliqué à l'estimation de la direction priprivilégiée de tels processus.

keywords: fractional Brownian sheet; estimation; anisotropy; generalized quadratic variations

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1 Introduction

Since [2] it is classical to show that quadratic variations of a Gaussian process converge to an integral of the singularity function if it exists. This result is extended in [8] to some fractional processes including the well known fractional Brownian motion. In [8] the mesh of the increments that define the quadratic

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variation is $1/2^n$ and an almost sure convergence result is obtained when $n \to \infty$. Actually it is known (see [12]) that the almost convergence is true when the mesh is $o(1/\log(n))$. Nevertheless quadratic variations are not suitable when one is interested in a central limit theorem for fractional Brownian motion. Actually it is shown in [9] that for $H > 3/4$ the central limit theorem is false for fractional Brownian motion. To overcome this problem the usual increments in the quadratic variations are replaced by second order increments to yield the generalized quadratic variations. In this case generalized quadratic variations have been used in [10] for Gaussian processes with stationary increments and in [3] for filtered white noise that are fractional processes with non-stationary increments.

In the first part of the paper the almost sure convergence of the generalized quadratic variations is obtained under general assumption on the correlation of the Gaussian process when the mesh is $1/n$. When the limit is not vanishing it leads to a generalization of the singularity function in the spirit of Gladyshev. Then the asymptotic normality of the generalized quadratic variations is established in this general setting. In the second part an example of application of this theorem to spatial estimation for a Gaussian field is given. Actually we consider a fractional Brownian sheet which is a centered Gaussian field $(W_{H_1,H_2}(x,y))_{(x,y) \in \mathbb{R}^2}$ such that the correlation function is :

$$E[W(x_1,y_1).W(x_2,y_2)] = \frac{1}{4}(|x_1|^{2H_1} + |x_2|^{2H_1} - |x_1 - x_2|^{2H_1})$$
$$(|y_1|^{2H_2} + |y_2|^{2H_2} - |y_1 - y_2|^{2H_2}) \quad (1)$$

where $0 < H_1, H_2 < 1$.

With no loss of generality one can assume that the Ox axis is the one corresponding to the minimum of H_1, H_2; hence, we denote by H_1 this minimum from now on. This field has been introduced in [11] and studied in [7, 1]. The estimation of the parameter H_1, H_2 is done in [13] under the assumption that the axes of the fractional Brownian sheet are known. We refer to [13] for a discussion of the application of this estimation to the detection of osteoporosis with medical X-rays.

In our paper the parameters H_1, H_2 are identified with the help of generalized quadratic variations of the fractional Brownian sheet restricted to some segments and we do not assume that the axes of the fractional Brownian sheet are known as in [13]. Please note that the use of generalized quadratic variations is quite important because one constructs a confidence interval with the help of the central limit theorem. Moreover we address the problem of estimating the rotation between the axes of the fractional Brownian sheet and the axes of the observations. Please note that a similar problem is solved in [5] for standard Brownian sheet i.e. $H_1 = H_2 = 1/2$, (see [14] for a general reference).

Strongly consistent estimators of H_1, H_2 and θ the angle between the axes of observations and the true axes of the fractional Brownian sheet are given. Moreover a confidence interval is obtained for H_1.

In the second section almost sure convergence of the generalized quadratic variations is established, a Central Limit Theorem is also obtained. Section 3 is devoted to the application to the fractional Brownian sheet.

2   Singularity functions for fractional processes

In the following theorem the almost sure convergence of the generalized quadratic variations is proved. Let us stress the fact that this theorem is classical for quadratic variations (cf. [8]). In [8] a bound for the second derivative of the correlation function $r(s,t)$ is needed when $s \neq t$. In the following theorem this bound is replaced by (3) which is a bound on the fourth derivative of the covariance function $r(s,t)$. Moreover the singularity function (See remark 2 for a precise definition.) is now obtained in (6) as a limit of a fourth order difference operator applied to r. Those are the key points for describing the limit behavior of the generalized quadratic variations.

Theorem 1 *Let $(\xi_t, t \in [0,1])$ be a real process with Gaussian increments such that*

1. *$m_t = \mathbb{E}(\xi_t)$ exists and has a bounded derivative on the segment $[0,1]$.*

2. *There exists a correlation function*

$$r(s,t) = \mathbb{E}\left((\xi(s) - \mathbb{E}\xi(s))(\xi(t) - \mathbb{E}\xi(t))\right) \tag{2}$$

   having the following properties :

   (a) r is continuous on $[0,1]^2$,

   (b) There exist two function $c_1(s)$, $c_2(t)$ such that $\dfrac{\partial^4(r - c_1 - c_2)}{\partial t^2 \partial s^2}$ exists and is a continuous function on $[0,1]^2 \setminus Diag$, where $Diag = \{(u,v) : \text{such that } u = v\}$ and there exists a constant C_0 and a real number $\gamma \in (0,2)$ such that

$$\left| \frac{\partial^4(r - c_1 - c_2)(s,t)}{\partial t^2 \partial s^2} \right| \leq \frac{C_0}{|s-t|^{\gamma+2}}. \tag{3}$$

   (c) Let us define two order increments :

$$\delta_1^h f(s,t) = f(s+h,t) + f(s-h,t) - 2f(s,t) \tag{4}$$
$$\delta_2^h f(s,t) = f(s,t+h) + f(s,t-h) - 2f(s,t) \tag{5}$$

   and let us suppose that there exists a bounded function g defined on $(0,1)$ such that :

$$\lim_{h \to 0^+} \sup_{t \in [h,1-h]} \left| \frac{(\delta_1^h \circ \delta_2^h r)(t,t)}{h^{2-\gamma}} - g(t) \right| = 0. \tag{6}$$

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Then

$$\lim_{N \to \infty} N^{1-\gamma} \sum_{k=1}^{N-1} [\xi_{(k+1)/N} + \xi_{(k-1)/N} - 2\xi_{k/N}]^2 = \int_0^1 g(t)dt. \qquad (7)$$

Remark 1 *Please remark that g is continuous on $(0,1)$ because of (6).*

Remark 2 *If assumption (2b) and assumption (2c) are satisfied for $\gamma_0 \in (0,2)$, they are also satisfied for $\gamma > \gamma_0$ but the corresponding function g_γ is vanishing. When γ_0 is the infimum of the real number such that assumptions (2b) and (2c) are satisfied g_{γ_0} can be viewed as a generalization of the singularity function introduced in [2].*

Remark 3 *In assumption (2b) the functions $c_1(s)$, $c_2(t)$ are introduced so that theorem 1 can be applied to the fractional Brownian motion with covariance*

$$r(s,t) = \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\} \qquad (8)$$

where $0 < H < 1$. Actually the partial derivatives of the covariance r do not exist when $s = 0$ but assumption (2b) still holds. Assumption (2a) is clearly fulfilled for the fractional Brownian motion. Since

$$\delta_1^h \circ \delta_2^h r(t,t) = 4r(t,t) + 2r(t-h,t+h) - 4r(t+h,t) - 4r(t-h,t)$$
$$+ r(t+h,t+h) + r(t-h,t-h), \quad (9)$$

we get $\delta_1^h \circ \delta_2^h r(t,t) = (4 - 2^{2H})h^{2H}$. If one makes the choice $\gamma = 2 - 2H$ then $\forall t \in [h, 1-h] \quad g(t) = (4 - 2^{2H})$, assumption (2c) is fulfilled.

Proof of theorem 1.

One can suppose that $m(t) = 0$ for $t \in [0,1]$ by considering the process $\tilde{\xi}(t) = \xi(t) - \mathbb{E}\xi(t)$. Because of assumption 1, as $N \to \infty$

$$N^{1-\gamma} \sum_{k=1}^{N-1} [m_{(k+1)/N} + m_{(k-1)/N} - 2m_{k/N}]^2 = O(N^{-\gamma}), \qquad (10)$$

the existence of the limit (7) yields the result for a non centered $\xi(t)$. We suppose that $m(t) = 0$ for $t \in [0,1]$ until the end of the proof.

Let us define $\Delta\xi_k = \xi_{(k+1)/N} + \xi_{(k-1)/N} - 2\xi_{k/N}$, and

$$\eta_N = N^{1-\gamma} \sum_{k=1}^{N-1} (\Delta\xi_k)^2.$$

The following notations are introduced :

$$d_{j,k} = N^{1-\gamma} \mathbb{E}[\Delta\xi_j \Delta\xi_k] \qquad (11)$$

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and

$$d_N = 2 \sum_{k=1}^{N-1} d_{k,k}^2 + 4 \sum_{1 \le k < j \le N-1} d_{j,k}^2. \tag{12}$$

Let us first prove $d_N = \text{var}(\eta_N)$. Actually

$$\eta_N^2 = N^{2-2\gamma} \left[\sum_{k=1}^{N-1} (\Delta \xi_k)^4 + 2 \sum_{1 \le k < j \le N-1} (\Delta \xi_k)^2 (\Delta \xi_j)^2 \right].$$

Since $(\Delta \xi_k, \ k = 1, \cdots, N-1)$ is a Gaussian vector

$$\mathbb{E}[(\Delta \xi_k)^4] = 3(\mathbb{E}[(\Delta \xi_k)^2])^2$$

and

$$\mathbb{E}[(\Delta \xi_k)^2 (\Delta \xi_j)^2] = \mathbb{E}[(\Delta \xi_k)^2]\mathbb{E}[(\Delta \xi_j)^2] + 2(\mathbb{E}[\Delta \xi_k \Delta \xi_j])^2.$$

Then

$$\mathbb{E}[\eta_N^2] = 3 \sum_{k=1}^{N-1} d_{k,k}^2 + 2 \sum_{1 \le k < j \le N-1} [d_{k,k} d_{j,j} + 2 d_{j,k}^2]$$

that yields

$$\text{var}(\eta_N) = 2 \sum_{k=1}^{N-1} d_{k,k}^2 + 4 \sum_{1 \le k < j \le N-1} d_{j,k}^2 = d_N.$$

Moreover we will show that there exists a generic constant C such that

$$\mathbb{E}[(\eta_N - \mathbb{E}(\eta_N))^4] \le C d_N^2. \tag{13}$$

Let us remark that η_N is the square of the Euclidean norm of the Gaussian vector $N^{\frac{1-\gamma}{2}}(\Delta \xi_k, \ k = 1, \cdots, N-1)$. The classical Cochran theorem yields $k_N \le N-1$ non-negative real numbers $(\lambda_{1,N}, \cdots, \lambda_{k_N,N})$ and a k_N-dimensional Gaussian vector ζ such that its components are independent reduced Gaussian variables and

$$\eta_N = \sum_{j=1}^{k_N} \lambda_{j,N} \zeta_{j,N}^2.$$

The components of ζ are obtained by a simple orthogonalization procedure. Then

$$
\begin{aligned}
\mathbb{E}[(\eta_N - \mathbb{E}(\eta_N))^4] &= E[(\sum_{j=1}^{k_N} \lambda_{j,N} (\zeta_{j,N}^2 - 1))^4] \\
&= \mathbb{E}[(\zeta_{1,N}^2 - 1)^4] \sum_{j=1}^{k_N} \lambda_{j,N}^4 \\
&+ 6(\mathbb{E}[(\zeta_{1,N}^2 - 1)^2])^2 \sum_{1 \ge i < j \ge k_N} \lambda_{i,N}^2 \lambda_{j,N}^2 \\
&\le C[\sum_{j=1}^{k_N} \lambda_{j,N}^2]^2.
\end{aligned}
$$

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Moreover $d_N = \text{var}\,(\eta_N) = \mathbb{E}[(\zeta_{1,N}^2 - 1)^2] \sum_{j=1}^{k_N} \lambda_{j,N}^2$ yields the inequality (13).

Let us remark that

$$d_{j,k} = N^{1-\gamma}(\delta_1^{1/N} \circ \delta_2^{1/N} r)(\frac{j}{N}, \frac{k}{N}) \tag{14}$$

and that

$$\mathbb{E}(\eta_N) = \sum_{k=1}^{N-1} d_{k,k}. \tag{15}$$

Moreover the limit (6) yields

$$\sup_{1 \leq k \leq N-1} |N d_{k,k} - g(k/N)| \to 0 \tag{16}$$

as $N \to \infty$. Hence, for $M > \sup_{t \in (0,1)} |g(t)|$

$$|d_{k,k}| < \frac{M}{N} \tag{17}$$

for N large enough.

Please remark that g is continuous (remark 1) so that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g(k/N) = \int_0^1 g(t)dt.$$

So we get

$$\lim_{N \to \infty} \mathbb{E}(\eta_N) = \int_0^1 g(t)dt. \tag{18}$$

The next step is to apply Borel-Cantelli lemma. To prove the almost sure convergence of η_N, an estimate of the asymptotic of d_N when $N \to \infty$ is needed. Let us split the study of the asymptotic of (12) in three parts.

- We know that $d_{k,k} < \frac{M}{N}$ (cf. (17)), then

$$\sum_{k=1}^{N-1} d_{k,k}^2 < \frac{M^2}{N} \tag{19}$$

for $M > \sup_{t \in (0,1)} |g(t)|$ and N large enough.

- For $j = k+1,\ k+2$, as $|d_{j,k}| \leq \sqrt{d_{k,k} d_{j,j}}$, we have

$$|d_{j,k}| \leq \frac{M}{N} \tag{20}$$

for $M > \sup_{t \in (0,1)} |g(t)|$ and N large enough.

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- A sharper estimate is needed for $|j - k| > 2$.

Lemma 1 *If* $[s-h, s+h] \times [t-h, t+h] \cap Diag = \emptyset$, *where* $Diag = \{(u,v) : such that u = v\}$, *and* $[s - h, s + h] \times [t - h, t + h] \subset [0, 1]^2$, *then*

$$\left| (\delta_1^h \circ \delta_2^h r)(s, t) \right| \leq \frac{C_0 h^4}{|s - t - 2h|^{\gamma+2}} \qquad (21)$$

for the constant C_0 of (3).

Proof of lemma 1.

One can rewrite

$$\delta_1^h \circ \delta_2^h (r - c_1 - c_2)(s, t) = \int_s^{s+h} du \int_{u-h}^u dx \int_t^{t+h} dv \int_{v-h}^v dy \frac{\partial^4 (r - c_1 - c_2)}{\partial s^2 \partial t^2}(x, y).$$
$$(22)$$

Because of (3)

$$\left| \frac{\partial^4 (r - c_1 - c_2)}{\partial s^2 \partial t^2}(x, y) \right| < \frac{C_0}{|x - y|^{\gamma+2}}$$

on the set $\{s - h \leq u - h \leq x \leq u \leq s + h \; ; \; t - h \leq v - h \leq y \leq v \leq t + h\}$ where $|x - y| > |s - t - 2h|$. Last $\delta_1^h \circ \delta_2^h (r - c_1 - c_2)(s, t) = (\delta_1^h \circ \delta_2^h r)(s, t)$.

$\bullet$

One can deduce from the preceding lemma that for $|j - k| > 2$

$$|d_{j,k}| \leq \frac{C_0 N^{-1}}{|j - k - 2|^{\gamma+2}}. \qquad (23)$$

Hence for $N \geq 5$

$$\sum_{1 \leq k \leq j-3 \leq N-4} d_{j,k}^2 \leq C_0^2 N^{-2} \sum_{1 \leq k \leq j-3 \leq N-4} |j - k - 2|^{-2\gamma-4} \qquad (24)$$

$$= C_0^2 N^{-2} \sum_{t=1}^{N-4} \frac{N - 3 - t}{t^{2\gamma+4}} \qquad (25)$$

$$\leq C_0^2 N^{-1} \left(\int_1^N \frac{dt}{t^{2\gamma+4}} + 1 \right) \qquad (26)$$

$$\leq \frac{4C_0^2}{3N}. \qquad (27)$$

Since the number of terms that satisfy (20) is bounded by $2N$, because of (12), of (19) and of (27) we get

$$N d_N < 10M^2 + \frac{16}{3} C_0^2 \qquad (28)$$

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for $M > \sup_{t \in (0,1)} |g(t)|$ and N large enough.

Because of (13), for $\delta > 0$

$$\mathbb{P}\{|\eta_N - \mathbb{E}(\eta_N)| > N^{\frac{\delta-1}{4}}\} \leq N^{1-\delta} C d_N^2 \qquad (29)$$

by Borel-Cantelli lemma for $\delta \in (0, 1)$, one can deduce

$$\lim_{N \to \infty} (\eta_N - \mathbb{E}(\eta_N)) = 0 \quad \text{a.s.}$$

and the convergence (7) comes from (18). $\qquad\qquad\qquad\qquad\bullet$

Let us now state some central limit theorems.

Theorem 2 *Under the assumptions of theorem 1, if $m(t) = 0 \quad \forall t \in [0,1]$, and if $g(t)$ is a non-vanishing function*

$$\frac{N^{1-\gamma} \left[\sum_{k=1}^{N-1} \Delta \xi_k^2 - \mathbb{E} \sum_{k=1}^{N-1} \Delta \xi_k^2 \right]}{\sqrt{d_N}}$$

converges to a standard centered Gaussian variable, as $N \to +\infty$.

Proof of theorem 2.

Let us recall

$$\eta_N = \sum_{j=1}^{k_N} \lambda_{j,N} \zeta_{j,N}^2,$$

the central limit theorem with Lindeberg condition is applied to

$$S_N = \sum_{j=1}^{k_N} \lambda_{j,N} (\zeta_{j,N}^2 - 1).$$

Hence we have to check that $\max_{1 \leq j \leq k_N} \lambda_{j,N} = o(\sqrt{\operatorname{var}(S_N)})$. This last condition is a consequence of an elementary result in linear algebra:

$$\max_{1 \leq j \leq k_N} \lambda_{j,N} \leq \max_{1 \leq j \leq N-1} \sum_{i=1}^{N-1} N^{1-\gamma} |\mathbb{E}(\Delta \xi_i \Delta \xi_j)|.$$

We already know estimates on the upper bound :

$$
\begin{aligned}
N^{1-\gamma} \max_{1 \leq j \leq N-1} \sum_{i=1}^{N-1} |\mathbb{E}(\Delta \xi_i \Delta \xi_j)| &= \max_{1 \leq j \leq N-1} \sum_{i=1}^{N-1} |d_{i,j}| \\
&\leq [O(N^{-1}) + \\
&\quad \max_{1 \leq j \leq N-1} \sum_{|i-j|>2} C_0 N^{-1} (i-j-2)^{-\gamma-2}] \\
&= O(N^{-1}).
\end{aligned}
$$

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A lower bound is needed for $\mathrm{var}\,(S_N) = \mathrm{var}\,(\eta_N) = d_N$. Because of (16) and the assumption that g is non-vanishing function

$$
\begin{aligned}
d_N &\geq 2 \sum_{k=1}^{N-1} d_{k,k}^2 \\
&\geq 1/N^2 \sum_{k=1}^{N-1} g^2(k/N) + o(1/N) \\
&\geq 1/N \int_0^1 g^2(t)dt + o(1/N).
\end{aligned}
$$

Hence $\mathrm{var} S_N = d_N > C/N$ for a generic positive constant C. The convergence in distribution to a standard Gaussian variable of $\dfrac{N^{1-\gamma}\left[\sum_{k=1}^{N-1}\Delta\xi_k^2 - \mathbb{E}\sum_{k=1}^{N-1}\Delta\xi_k^2\right]}{\sqrt{d_N}}$ is proved. $\qquad\bullet$

Remark 4 *If $\gamma > 1/2$ theorem 2 is still true even if the process ξ is not centered. It is a consequence of (10) and of $d_N > C/N$.*

Let us now see when the bias term $\mathbb{E}[\eta_N] - \int_0^1 g(t)dt$ is negligible. Assumptions *(2b')* and (32) are introduced below to ensure the convergence of Nd_N. One can check that there are fulfilled for the fractional Brownian motion, fractional Brownian sheet is another instance where the following theorem can be applied.

Theorem 3 *Let ξ be a centered Gaussian process such that assumptions (2a) is fulfilled.*

Assumption (2b) is replaced by

(2b') There exist two functions $c_1(s)$, $c_2(t)$ such that $\frac{\partial^4(r-c_1-c_2)}{\partial s^2\partial t^2}$ exists on $[0,1]^2 \setminus \{(u,u) : 0 \leq u \leq 1\}$. Let $T^+ = \{(s,t) : 0 \leq s \leq t \leq 1\}$ be the upper triangle in $[0,1]^2$ (respectively $T^- = \{(s,t) : 0 \leq t \leq s \leq 1\}$ be the lower triangle) and we assume the existence of a continuous function C_+ on T^+ (resp C_- on T^-) such that

$$
|s-t|^{\gamma+2}\frac{\partial^4(r-c_1-c_2)}{\partial s^2\partial t^2}(s,t) = C_+(s,t) \quad s,\ t \in \{(s,t) : 0 \leq s < t \leq 1\} \quad (30)
$$

(Resp. C_- on $\{(s,t) : 0 \leq t < s \leq 1\}$.)

The assumption (2c) is replaced by:

let us suppose that there exists a function g such that

$$
\sup_{t\in[h,1-h]}\left|\frac{\delta_1^h \circ \delta_2^h r(t,t)}{|h|^{2-\gamma}} - g(t)\right| < Ch^{\epsilon+1/2} \quad (31)
$$

where $\epsilon > 0$, $h \to 0$, C is a generic constant. Let us assume that g is a $1/2+\epsilon$ Hölder continuous non-vanishing function.

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Moreover we assume that there exists a function $\tilde{g}$

$$\lim_{h \to 0^+} \sup_{t \in [h, 1-2h]} \left| \frac{\delta_1^h \circ \delta_2^h r(t+h, t)}{|h|^{2-\gamma}} - \tilde{g}(t) \right| = 0. \tag{32}$$

Then

$$\sqrt{N} \left(N^{1-\gamma} \sum_{k=1}^{N-1} \Delta \xi_k^2 - \int_0^1 g(t) dt \right)$$

converges to a centered Gaussian random variable.

Proof of theorem 3

The bound (31) and (15) yield for $h = \frac{1}{N}$:

$$\left| \mathbb{E}[\eta_N] - \frac{1}{N} \sum_{k=1}^{N-1} g(k/N) \right| < C/N^{\epsilon+1/2}.$$

Since g is Hölder continuous

$$\left| \frac{1}{N} \sum_{k=1}^{N-1} g(k/N) - \int_0^1 g(t) dt \right| < C/N^{\epsilon+1/2}.$$

Let us show that $\lim_{N \to \infty} N d_N$ exists. Recall (12)

$$d_N = 2 \sum_{k=1}^{N-1} d_{k,k}^2 + 4 \sum_{1 \le k < j \le N-1} d_{j,k}^2.$$

Because of (16)

$$\lim_{N \to \infty} N \sum_{k=1}^{N-1} d_{k,k}^2 = \int_0^1 g^2(t) dt.$$

Let us now consider the case when $j - k \ge 3$. In this case because of (14) and (22)

$$N d_{j,k} = N^{2-\gamma} \int_{j/N}^{j+1/N} du \int_{u-1/N}^{u} dx \int_{k/N}^{k+1/N} dv \int_{v-1/N}^{v} dy \frac{C_+(x,y)}{(x-y)^{\gamma+2}}. \tag{33}$$

Then

$$\left| N d_{j,k} - N^{2-\gamma} \int_{j/N}^{j+1/N} du \int_{u-1/N}^{u} dx \int_{k/N}^{k+1/N} dv \int_{v-1/N}^{v} dy \frac{C_+(j/N, k/N)}{(x-y)^{\gamma+2}} \right| \le \frac{\epsilon_N}{(j-k-2)^{\gamma+2}}$$

with $\epsilon_N \to 0$ when $N \to \infty$. Actually we use $|C_+(x,y) - C_+(j/N, k/N)| \to 0$ uniformly with respect to j, k when $N \to \infty$. Moreover

$$N^{2-\gamma} \int_{j/N}^{j+1/N} du \int_{u-1/N}^{u} dx \int_{k/N}^{k+1/N} dv \int_{v-1/N}^{v} dy \frac{1}{(x-y)^{\gamma+2}}$$

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is a function of $j - k$ that will be denoted by $f(j - k)$ such that

$$|f(j - k)| \leq \frac{M}{(j - k - 2)^{\gamma + 2}}$$

for M a generic constant. Then

$$N \sum_{1 \leq k < j - 2 \leq N - 4} d_{j,k}^2 = 1/N \sum_{1 \leq k < j - 2 \leq N - 4} C_+^2(j/N, k/N) f^2(j - k)$$

$$= \sum_{l=3}^{N-3} f^2(l) \left(1/N \sum_{k=1}^{N-1-l} C_+^2(l + k/N, k/N) \right).$$

When l is fixed one can show that

$$\lim_{N \to \infty} 1/N \sum_{k=1}^{N-1-l} C_+^2(l + k/N, k/N) = \int_0^1 C_+^2(x, x) dx$$

and since $\sum_{l=3}^{N-3} f^2(l) < +\infty$, the limit of $N \sum_{1 \leq k < j - 2 \leq N - 4} d_{j,k}^2$ exists. At this point we are reduced to prove the convergence $N \sum_{1 \leq k \leq N - 2} d_{k+2,k}^2$ and $N \sum_{1 \leq k \leq N - 1} d_{k+1,k}^2$. For the first limit one uses a perturbation argument that can be found in [8], and (32) yields the existence of the second limit. Similar arguments hold when $k > j$ and we skip the technical details.

●

3 Application to fractional Brownian sheet

3.1 Application of theorem 1

In this part, theorems of the sections 2 are applied to the identification of the axes of the fractional Brownian sheet. They are identified with the help of generalized quadratic variations of the fractional Brownian sheet restricted to some segments. The geometry of the problem is described in figure 1.

In this paper radial segments $[A, B]$ with length L are considered. The distance of the segment to the origin O is $L\varepsilon > 0$ (see section 3.2 when $\varepsilon = 0$). The angle of $[A, B]$ with respect to the axes of observations Oxy is β a parameter under our control. The oriented angle θ between Oxy and $Ox'y'$ the true axes of the fractional Brownian sheet, is the parameter we want to estimate. Let us recall the correlation function of the fractional Brownian sheet $(W_{H_1, H_2}(x', y'))_{(x', y') \in \mathbb{R}^2}$:

$$E[W(x_1', y_1') . W(x_2', y_2')] = \frac{1}{4}(|x_1'|^{2H_1} + |x_2'|^{2H_1} - |x_1' - x_2'|^{2H_1})$$

$$(|y_1'|^{2H_2} + |y_2'|^{2H_2} - |y_1' - y_2'|^{2H_2}) \quad (34)$$

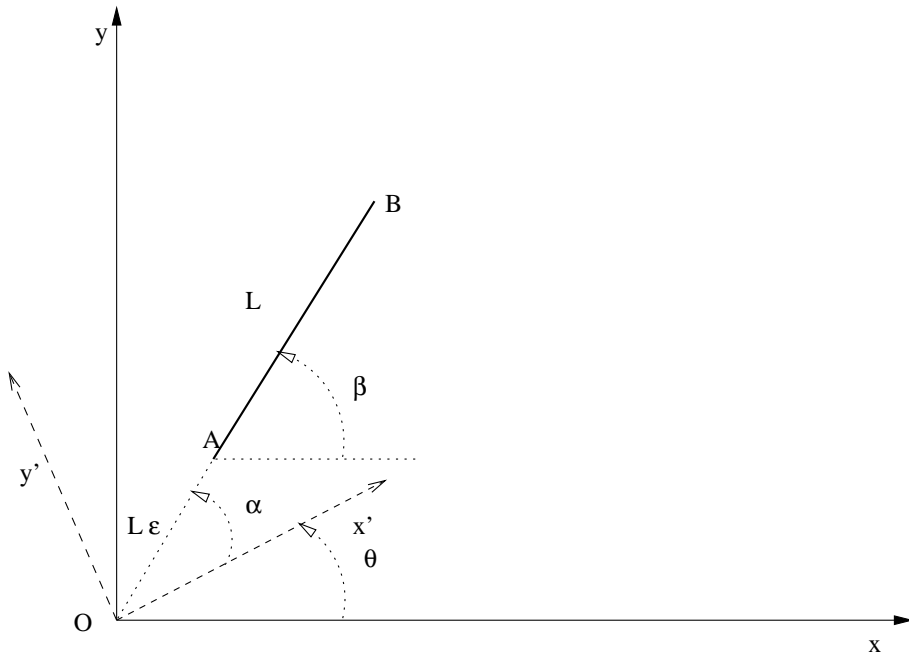where $0 < H_1, \ H_2 < 1$.

Figure 1: Geometry of the observations.

Let us discuss why we consider θ as an oriented angle and what is the range of θ. It is obvious that the distribution of the fractional Brownian sheet is invariant with respect to the transformation $y' \mapsto -y'$. Hence, one can always assume that the orthonormal frames Oxy and $Ox'y'$, have the same orientation. Moreover, since the distribution of the fractional Brownian sheet is invariant with respect to the transformation $x' \mapsto -x'$ we are actually interested in the angle between the two lines Ox and Ox' with no privileged sense $x' > 0$. Since we are working with oriented frames, it means that one can always assume that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let us denote the oriented angles β and θ as shown in figure 1. Then, $\alpha = \theta - \beta$ is the angle of $[A, B]$ with respect to the true axes of $Ox'y'$ of the fractional Brownian sheet.

Let us call $(Z_t, t \in [0,1])$ the restriction of the fractional Brownian sheet to segment $[A, B]$, which can be parameterized as follows :

$$
\begin{aligned}
x'_t &= L(t + \varepsilon) \cos \alpha, \\
y'_t &= L(t + \varepsilon) \sin \alpha,
\end{aligned}
$$

$Z_t = W(x'_t, y'_t)$ for $t \in [0,1]$. Then the covariance of Z, $r(s,t) = E(Z_s Z_t)$, is

$$
r(s,t) = \frac{1}{4} L^{2(H_1 + H_2)} |\cos \alpha|^{2H_1} |\sin \alpha|^{2H_2} (|t + \varepsilon|^{2H_1} + |s + \varepsilon|^{2H_1} - |t - s|^{2H_1})
$$
$$
(|t + \varepsilon|^{2H_2} + |s + \varepsilon|^{2H_2} - |t - s|^{2H_2}). \quad (35)
$$

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By applying theorem 1 we will prove the following result.

Proposition 1

$$\lim_{N\to\infty} N^{2H_1-1} \sum_{k=1}^{N-1} (Z_{(k+1)/N} - 2Z_{k/N} + Z_{(k-1)/N})^2 = \int_0^1 g(t)dt$$

almost surely where :

$$g(t) = (4 - 2^{2H_1})L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2}(t+\varepsilon)^{2H_2}$$

when $H_1 < H_2$ and

$$g(t) = 2(4 - 2^{2H_1})L^{4H_1}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_1}(t+\varepsilon)^{2H_1}$$

when $H_1 = H_2$.

Proof of the proposition 1

We have to check assumptions of theorem 1. First Z is a centered Gaussian process and clearly the covariance function is continuous on $[0,1]^2$.
Concerning assumption (2b), the partial derivatives of r exist in $[0,1]^2 \setminus \{(u,u) : 0 \le u \le 1\}$, let us now check (3) for $\gamma = 2(1 - H_1)$ and $c_1 = c_2 = 0$. Since r is up to a multiplicative constant the product of $f_i(t,s) = (|t+\varepsilon|^{2H_i} + |s+\varepsilon|^{2H_i} - |t-s|^{2H_i})$, for $i = 1, 2$, by Leibnitz rule it is enough to have bounds on partial derivatives of the f_i's. Elementary computations postponed to the appendix 4 yields the bound for C_0:

$$C_0(\varepsilon) \le \frac{1}{4}L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2}[4(1+\varepsilon)^{2H_2}+$$

$$8\frac{2}{3\sqrt{3}}(1 + \max(\varepsilon^{2H_1-1}, (1+\varepsilon)^{2H_1-1}, \varepsilon^{2H_2-1}, (1+\varepsilon)^{2H_2-1})$$

$$+ 8\frac{2}{3\sqrt{3}}\max(1, (1+\varepsilon)^{2H_1-1}, (1+\varepsilon)^{2H_2-1})$$

$$+ 8\max(1, \varepsilon^{2H_2-2}, \varepsilon^{2H_1-2}) + 16]. \quad (36)$$

Let us denote

$$C(\alpha, H_1, H_2, L) = \frac{1}{4}L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2}.$$

Thus we have in case $\varepsilon = 1$:

$$C_0(1) \le C(\alpha, H_1, H_2, L)(40 + \frac{80}{3\sqrt{3}}) \quad (37)$$

where $C_0(1)$ is a constant for inequality (3).

Let us now check assumption (2c) for $t + \varepsilon \ge h \ge 0$

$$(t + \varepsilon + \delta_j h)^{2H_i} = (t + \varepsilon)^{2H_i} + 2H_i(t+\varepsilon)^{2H_i-1}\delta_j h + O((t+\varepsilon)^{2H_i-2}h^2) \quad (38)$$

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for $i = 1, 2$; $j = 1, 2, 3$, $\delta_j = +1, -1, 0$. Moreover

$$f_i(t + \delta_j h, t + \delta_k h) = 2(t + \varepsilon)^{2H_i} + 2H_i(\delta_j + \delta_k)(t + \varepsilon)^{2H_i - 1}h - (|\delta_j - \delta_k|h)^{2H_i}$$
$$+ H_i(2H_i - 1)(t + \varepsilon)^{2H_i - 2}(\delta_j^2 + \delta_k^2)h^2 + O(h^3)(t + \varepsilon)^{2H_i - 3}. \quad (39)$$

This yields an asymptotic expansion for $r(t + \delta_j h, t + \delta_k h)$ and because of (9) one gets for $h > 0$

$$\delta_2^h \circ \delta_1^h r(t, t) = 4C(\alpha, H_1, H_2, L)h^{2H_1}(t + \varepsilon)^{2H_2}(4 - 2^{2H_1})$$
$$[1 + O(h^{2(H_2 - H_1)}) + O(h^{2(1 - H_1)}) + O(h)] \quad (40)$$

if $H_1 < H_2$. Then, if we set

$$g(t) = (4 - 2^{2H_1})L^{2H}|\cos \alpha|^{2H_1}|\sin \alpha|^{2H_2}(t + \varepsilon)^{2H_2},$$

$$\left| \frac{\delta_1^h \circ \delta_2^h r(t, t)}{h^{2 - \gamma}} - g(t) \right| = O(h^{2(H_2 - H_1)}(t + \varepsilon)^{2H_1}), \quad (41)$$

which establishes (6).
Whereas when $H_1 = H_2$,

$$\delta_2^h \circ \delta_1^h r(t, t) = 8C(\alpha, H_1, H_1, L)h^{2H_1}(t + \varepsilon)^{2H_1}(4 - 2^{2H_1})[1$$
$$+ O(h^{2(1 - H_1)}) + O(h)] \quad (42)$$

and

$$g(t) = 2(4 - 2^{2H_1})L^{2H}|\cos \alpha|^{2H_1}|\sin \alpha|^{2H_1}(t + \varepsilon)^{2H_1}$$

and

$$\left| \frac{\delta_1^h \circ \delta_2^h r(t, t)}{h^{2 - \gamma}} - g(t) \right| = O(h^{2 - 2H_1}(t + \varepsilon)^{4H_1 - 2}). \quad (43)$$

$\bullet$

Let us denote by

$$V_N(\beta, L, \varepsilon) = \sum_{k=1}^{N-1}(Z_{(k+1)/N} - 2Z_{k/N} + Z_{(k-1)/N})^2 \quad (44)$$

and state a central limit theorem for $V_N(\beta, L, \varepsilon)$.

Proposition 2 *If $H_1 + 1/4 < H_2$ then*

$$D(H_1, H_2, \alpha, L, \varepsilon) = (4 - 2^{2H_1})L^{2(H_1 + H_2)}|\cos \alpha|^{2H_1}|\sin \alpha|^{2H_2}$$
$$\frac{(1 + \varepsilon)^{2H_2 + 1} - \varepsilon^{2H_2 + 1}}{2H_2 + 1}$$

then

$$\sqrt{N}\left(N^{2H_1 - 1}V_N(\beta, L, \varepsilon) - D(H_1, H_2, \theta - \beta, L, \varepsilon)\right) \quad (45)$$

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converges to a standard centered Gaussian random variable.

If $H_1 = H_2$, $H_1 < 3/4$ then

$$\tilde{D}(H_1, \alpha, L, \varepsilon) = 2(4 - 2^{2H_1})L^{4H_1}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_1}\frac{(1+\varepsilon)^{2H_1+1} - \varepsilon^{2H_1+1}}{2H_1+1}$$

$$\sqrt{N}\left(N^{2H_1-1}V_N(\beta, L, \varepsilon) - \tilde{D}(H_1, \theta - \beta, L, \varepsilon)\right) \tag{46}$$

converges to a centered Gaussian random variable.

Proof of the proposition 2

When $H_1 + 1/4 < H_2$, (31) is satisfied because $2(H_2 - H_1) > 1/2$ and we have (41). Moreover g is continuously differentiable. Since g is non-vanishing and since

$$\int_0^1 g(t)dt = (4 - 2^{2H_1})L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2}\frac{(1+\varepsilon)^{2H_2+1} - \varepsilon^{2H_2+1}}{2H_2+1},$$

one can check (32) with an asymptotic developpement of $\delta_2^h \circ \delta_1^h r(t+h, t)$ similar to (40). The existence of C_+ and C_- is a consequence of the computation of $\frac{\partial^4 r}{\partial s^2 \partial t^2}(s, t)$ done in the appendix. One can apply theorem 3. The first part of the proposition is proved.

When $H_1 = H_2$, $2 - 2H_1 > 1/2$ and equation (43) allow application of theorem 3 with the same arguments as in the previous case.

●

Remark 5 *Let us define $X_N = N^{2H_1-1}V_N(\beta, L, \varepsilon)$. Note that with the same arguments, we prove the asymptotic normality of a pair (X_N, X_{2N}) of rescaled quadratic variations since the central limit theorem can be proved for every linear combination $\lambda X_N + \mu X_{2N}$ which is still a rescaled quadratic variation.*

3.2 Case where $\varepsilon = 0$

In the previous section, the fractional Brownian sheet is restricted to a segment that does not intersect the origin. Technically it yields $\varepsilon > 0$. One may wonder what happens when $\varepsilon = 0$. Although this choice is in some sense more natural the application of the theorems of the section 2 is more difficult when $\varepsilon = 0$. For instance the covariance of Z is no longer differentiable when $t = 0$.

In this section one extend the result of proposition 1 when $\varepsilon = 0$. Let us now consider the restriction of the fractional Brownian sheet to $[O, B]$ parameterized by

$$\begin{aligned} x'_t &= Lt\cos\alpha, \\ y'_t &= Lt\sin\alpha, \end{aligned}$$

$\tilde{Z}_t = W(x'_t, y'_t)$ for $t \in [0, 1]$,

Proposition 3 *If $H_2 > H_1 > 3/4$, then*

$$\lim_{N\to\infty} N^{2H_1-1} \sum_{k=1}^{N-1} (\tilde{Z}_{(k+1)/N} - 2\tilde{Z}_{k/N} + \tilde{Z}_{(k-1)/N})^2 =$$
$$\frac{(4 - 2^{2H_1})L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2}}{2H_2 + 1} \quad (47)$$

If $H_2 = H_1 > 3/4$, then

$$\lim_{N\to\infty} N^{2H_1-1} \sum_{k=1}^{N-1} (\tilde{Z}_{(k+1)/N} - 2\tilde{Z}_{k/N} + \tilde{Z}_{(k-1)/N})^2 =$$
$$\frac{2(4 - 2^{2H_1})L^{4H_1}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_1}}{2H_1 + 1}. \quad (48)$$

Proof of the proposition 3

The assumption (2b) is not satisfied for $\tilde{Z}$. More precisely, the bound obtained in (36) shows that the covariance r of $\tilde{Z}$ satisfies for $1 > \varepsilon > 0$ and $s, t \in [\varepsilon, 1]^2 \setminus \{(u, u) : \varepsilon \le u \le 1\}$,

$$\left| \frac{\partial^4 r(s,t)}{\partial t^2 \partial s^2} \right| \le \frac{C_2 \varepsilon^{2H_1-2}}{|s-t|^{\gamma+2}}$$

for some positive constant C_2. Let $\varepsilon_N \to +\infty$ be such that $\frac{\varepsilon_N}{N} \to 0$ when $N \to +\infty$. This yields

$$|d_{j,k}| \le \frac{C_2 N^{-1}}{|j-k-2|^{\gamma+2}} \left(\frac{\varepsilon_N}{N} \right)^{2H_1-2}.$$

One can show as in (24) that

$$\sum_{\varepsilon_N \le k \le j-3 \le N-4} d_{j,k}^2 \le C_2^2 N^{-2} \left(\frac{\varepsilon_N}{N} \right)^{4H_1-4} \sum_{\varepsilon_N \le k \le j-3 \le N-4} |j-k-2|^{-2\gamma-4}$$
$$= O\left(\frac{\varepsilon_N^{4H_1-4}}{N^{4H_1-3}} \right),$$

and that

$$\sum_{1 \le k \le j-3 \le N-4; k < \varepsilon_N} d_{j,k}^2 \le C \frac{\varepsilon_N}{N},$$

for a generic constant C, because $d_{j,k} = O(\frac{1}{N})$. Let us choose $\varepsilon_N = N^{\frac{4-4H_1}{5-4H_1}}$, then $d_N = O(N^{\frac{1}{4H_1-5}})$. Because of (29) we get the almost sure convergence if

$$1/2 < \frac{1}{5 - 4H_1}$$

which leads to $H_1 > 3/4$. The same arguments hold for $H_1 = H_2 > 3/4$. $\bullet$

3.3 Estimators of the parameters H_1, H_2 and θ

Let us recall that $H_1 \le H_2$ where H_1 is attached to axe Ox' of the fractional Brownian sheet. In this part, we use previous results to construct consistent estimators of H_1, H_2 and $\theta = (Ox, Ox')$, the angle between the Ox' axis of the fractional Brownian sheet with the observation axis Ox. Estimators of H_1, H_2 are studied in propositions 4 and 5, while proposition 6 gives an estimator of θ.

In the following propositions we assume that the segment is not contained in one of the axes of the fractional Brownian sheet i.e. $\beta \ne \theta + \frac{k\pi}{2}$, $\quad \forall k \in \mathbb{Z}$.

Proposition 4 *If $\beta \ne \theta + \frac{k\pi}{2}$, $\quad \forall k \in \mathbb{Z}$, and if V_N is defined by (44)*

$$\hat{H}_{1N} = \frac{1 + \log(V_N(\beta, L, \varepsilon)/V_{2N}(\beta, L, \varepsilon))/\log 2}{2} \tag{49}$$

and

$$\hat{H}_{2N} = \frac{1}{2\log 2} \log\left[\frac{V_N(\beta, 2L, \varepsilon)}{V_N(\beta, L, \varepsilon)}\right] - \hat{H}_{1N} \tag{50}$$

are respectively strongly consistent estimators of H_1 and H_2.

Proof of the proposition 4
Because of proposition 1,

$$\lim_{N \to +\infty} \frac{V_N(\beta, L, \varepsilon)}{N^{1-2H_1}} = \int_0^1 g(t)dt.$$

When $\beta \ne \theta + \frac{k\pi}{2}$, $\quad \forall k \in \mathbb{Z}$,

$$\int_0^1 g(t)dt \ne 0$$

and (49) is established.
To prove (50) let us remark that

$$\frac{V_N(\beta, 2L, \varepsilon)}{V_N(\beta, L, \varepsilon)} \to 2^{2(H_1 + H_2)}$$

when $N \to +\infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ •
Then we can give a confidence interval for the estimation of H_1.

Proposition 5 *Let $H_1 + 1/4 < H_2$, and let $\alpha \ne \frac{k\pi}{2}$, $\quad \forall k \in \mathbb{Z}$.*
Let $m = D(H_1, H_2, \alpha, L, \varepsilon)$ defined in proposition 2. If Γ denotes $\lim_{N \to \infty} N\mathbb{E}(X_{2N} - X_N)^2$, then $\Gamma > 0$ and

$$\mathbb{P}\{2m\Gamma^{-1/2}\log 2\sqrt{N}|H_{1N} - H_1| \le 1.96\} \to 0.95$$

when $N \to +\infty$.

Proof of the proposition 5

Because of remark 5 (X_N, X_{2N}) is asymptotically normal and the limit $\lim_{N \to \infty} N\mathbb{E}(X_{2N} - X_N)^2$ exists. Then $N\mathbb{E}(X_{2N} - X_N)^2 \leq (\frac{3}{2} - \sqrt{2}) \lim_{N \to +\infty} Nd_N$. Hence, $\Gamma > 0$ and because of Slutsky theorem :

$$\sqrt{N} \log(X_N/X_{2N}) \to \mathcal{N}(0, \sigma^2) \tag{51}$$

when $N \to \infty$, where the convergence is in distribution, and where

$$\sigma^2 = \lim_{N \to \infty} \frac{N\mathbb{E}(X_{2N} - X_N)^2}{m^2}.$$

Since

$$\log(X_N/X_{2N}) = 2 \log 2(\hat{H}_{1N} - H_1),$$

and $\sigma^2 = \Gamma/m^2$, the proposition is proved. $\qquad\bullet$

Please note that the rate of convergence $O(1/\sqrt{N})$ is known to be of the same order as the one given by the Cramér-Rao bound (cf. [6] and [4] when the Hurst exponent of a fractional Brownian Motion is identified). In our case, since the process Z is not with stationary increments, the Cramér-Rao bound seems difficult to estimate.

Let us give a numerical application of the previous proposition. In practice we have to compute $m^{-2}\Gamma$. Actually the computation of Γ seems delicate but $N\mathbb{E}(X_{2N} - X_N)^2 \leq 3N\mathbb{E}(X_N - m)^2 = 3Nd_N$ with d_N defined by (12) with Z in place of ξ. And we have an upper bound for Nd_N. More precisely, because of (28)

$$Nd_N \leq 10M^2 + \frac{16}{3}C_0^2(\varepsilon).$$

Moreover $g(1)$ is the supremum of g and we can take

$$M > 4(4 - 2^{2H_1})C(\alpha, H_1, H_2, L)(1 + \varepsilon)^{2H_2}$$

(for instance $M = 4(4 - 2^{2H_1})C(\alpha, H_1, H_2, L)(1 + \varepsilon)^2$). Let us assume $\varepsilon = 1$ and recall that

$$C_0(1) \leq (40 + \frac{80}{3\sqrt{3}})C(\alpha, H_1, H_2, L)$$

then

$$Nd_N \leq C^2(\alpha, H_1, H_2, L)[160(4 - 2^{2H_1})^2 4 + \frac{16}{3}(40 + \frac{80}{3\sqrt{3}})^2].$$

Moreover $m = \frac{4(4 - 2^{2H_1})(2^{2H_2+1} - 1)C(\alpha, H_1, H_2, L)}{2H_2 + 1}$ for $\varepsilon = 1$ and

$$\frac{\Gamma}{m^2} \leq \frac{3[160(4 - 2^{2H_1})^2 4 + \frac{16}{3}(40 + \frac{80}{3\sqrt{3}})^2](2H_2 + 1)^2}{16(4 - 2^{2H_1})^2(2^{2H_2+1} - 1)^2}$$

$$= 3[40 + \frac{16}{3(4 - 2^{2H_1})^2}(10 + \frac{20}{3\sqrt{3}})^2](\frac{2H_2 + 1}{2^{2H_2+1} - 1})^2.$$

Let us assume $H_1 < \frac{1}{2} < H_2$ as an example to have an order of how big shall be N to have a meaningful interval. Then, $\frac{2H_2+1}{2^{2H_2+1}-1} \leq 1$ and $4 - 2^{2H_1} > 2$.

$$\frac{1.96\sqrt{\Gamma}}{m} < 58.39$$

and the confidence interval length is bounded with $\frac{58.39}{\sqrt{N}}$.

In practical situations, we may assume $H_1 \neq H_2$. In this case we can detect the anisotropy of the fractional Brownian sheet, which means than we can estimate θ.

Proposition 6 *If $H_1 \neq H_2$, $\theta \in (\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, and $\beta \neq \theta + \frac{k\pi}{2}$, $\forall k \in \mathbb{Z}$, set*

$$f_N(\beta) = \arctan\left(\left(\frac{V_N(\beta + \pi/2, L, \varepsilon)}{V_N(\beta, L, \varepsilon)}\right)^{\frac{1}{2(H_{1N} - H_{2N})}}\right)$$

then

$$\hat{\theta}_N = sign(f_N(\frac{-\pi}{4}) - f_N(\frac{\pi}{4}))f_N(0) \qquad (52)$$

is a strongly consistent estimator of θ, where $sign(x) = 1$ if $x \geq 0$ and $sign(x) = -1$ if $x < 0$.

Proof of the proposition 6

Because of proposition 1

$$\frac{V_N(\beta + \pi/2, L, \varepsilon)}{V_N(\beta, L, \varepsilon)} \rightarrow |\tan(\theta - \beta)|^{2(H_1 - H_2)}$$

when $N \rightarrow +\infty$. Hence, with our assumptions,

$$f_N(\beta) \rightarrow \arctan(|\tan(\theta - \beta)|),$$

when $N \rightarrow +\infty$. Since $\theta \in (\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, $f_N(0) \rightarrow |\theta|$ when $N \rightarrow +\infty$. Please note that the last assumption is not a real loss of generality since in general we may choose $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$. Elementary computations show that

$$\arctan(|\tan(\theta + \frac{\pi}{4})|) - \arctan(|\tan(\theta - \frac{\pi}{4})| > 0$$

if and only if $\theta \in (0, \frac{\pi}{2})$. Since

$$f_N(\frac{-\pi}{4}) - f_N(\frac{\pi}{4}) \rightarrow \arctan(|\tan(\theta + \frac{\pi}{4})|) - \arctan(|\tan(\theta - \frac{\pi}{4})|$$

when $N \rightarrow +\infty$, the proof is complete.

$\bullet$

Please remark that we do not use all the information available to construct our estimators. This might be an advantage if the parameters are actually functions of the position, which means that the fractional Brownian sheet is actually a first order simplification. To better use the available information we could consider K generalized quadratic variations for different β_j, $j = 1, \ldots, K$ we get estimators $\hat{H}_{1N}^j$, $\hat{H}_{2N}^j$, $\hat{\theta}_n^j$. Thus, we could propose a more robust version of the estimators as the empirical means of these estimators. Other problems may also be addressed in the future such as estimation of the variance of the estimators proposed above.

4 Appendix

Let us prove the upper bound (36). For $1 \geq s > t \geq 0$,

$$\frac{\partial f_i}{\partial t}(t,s) = 2H_i((t+\varepsilon)^{2H_i-1} + (s-t)^{2H_i-1}) \tag{53}$$

$$\frac{\partial f_i}{\partial s}(t,s) = 2H_i((s+\varepsilon)^{2H_i-1} - (s-t)^{2H_i-1}) \tag{54}$$

$$\frac{\partial^2 f_i}{\partial t^2}(t,s) = 2H_i(2H_i-1)[(t+\varepsilon)^{2H_i-2} - (s-t)^{2H_i-2}] \tag{55}$$

$$\frac{\partial^2 f_i}{\partial s^2}(t,s) = 2H_i(2H_i-1)[(s+\varepsilon)^{2H_i-2} - (s-t)^{2H_i-2}] \tag{56}$$

$$\frac{\partial^2 f_i}{\partial s \partial t}(t,s) = 2H_i(2H_i-1)(s-t)^{2H_i-2} \tag{57}$$

$$\frac{\partial^3 f_i}{\partial s^2 \partial t}(t,s) = 2H_i(2H_i-1)(2H_i-2)(s-t)^{2H_i-3} \tag{58}$$

$$\frac{\partial^4 f_i}{\partial s^2 \partial t^2}(t,s) = -2H_i(2H_i-1)(2H_i-2)(2H_i-3)(s-t)^{2H_i-4} \tag{59}$$

for $i = 1$, 2. Hence :

$$(s-t)^{\gamma+2}\frac{\partial^4 f_1}{\partial s^2 \partial t^2}f_2 = -2H_1(2H_1-1)(2H_1-2)(2H_1-3)f_2 \tag{60}$$

$$(s-t)^{\gamma+2}\frac{\partial^3 f_1}{\partial s^2 \partial t}\frac{\partial f_2}{\partial t} = 2H_1(2H_1-1)(2H_1-2)(s-t) \tag{61}$$
$$2H_2[((t+\varepsilon)^{2H_2-1} + (s-t)^{2H_2-1})]$$

$$(s-t)^{\gamma+2}\frac{\partial^3 f_1}{\partial t^2 \partial s}\frac{\partial f_2}{\partial t} = -2H_1(2H_1-1)(2H_1-2)(s-t) \tag{62}$$
$$2H_2[((s+\varepsilon)^{2H_2-1} - (s-t)^{2H_2-1})]$$

$$(s-t)^{\gamma+2}\frac{\partial^2 f_1}{\partial s^2}\frac{\partial^2 f_2}{\partial t^2} = 4\prod_{i=1,2}H_i(2H_i-1)$$
$$\cdot \ [(s+\varepsilon)^{2H_1-2}(s-t)^{2-H_1}-(s-t)^{H_1}] \qquad (63)$$
$$\cdot \ [(t+\varepsilon)^{2H_2-2}(s-t)^{2-H_1}-(s-t)^{2H_2-H_1}]$$
$$(s-t)^{\gamma+2}\frac{\partial^2 f_1}{\partial s\partial t}\frac{\partial^2 f_2}{\partial s\partial t} = 4\prod_{i=1,2}H_i(2H_i-1)(s-t)^{2H_2} \qquad (64)$$

are bounded on $[0,1]^2$, and consequently assumption (2b) is fulfilled. More precisely, since $H_i \in]0,1[$, for $i=1,2$:

$$|2H_i| \ \leq \ 2,$$
$$|2H_i(2H_i-1)| \ \leq \ 2,$$
$$|2H_i(2H_i-1)(2H_i-2)| \ \leq \ \frac{2}{3\sqrt{3}}$$
$$|2H_i(2H_i-1)(2H_i-2)(2H_i-3)| \ \leq \ 1$$

and

$$\sup_{0\leq t<s\leq 1}|(t+\varepsilon)^{2H_i}+(s+\varepsilon)^{2H_i}-(s-t)^{2H_i}|=2(1+\varepsilon)^{2H_i},$$
$$\sup_{0\leq t<s\leq 1}|((s-t)(t+\varepsilon)^{2H_i-1}+(s-t)^{2H_i})| \leq$$
$$(1+\max(\varepsilon^{2H_i-1},(1+\varepsilon)^{2H_i-1})),$$
$$\sup_{0\leq t<s\leq 1}|((s-t)(t+\varepsilon)^{2H_i-1}-(s-t)^{2H_i})| \leq$$
$$\max(1,(1+\varepsilon)^{2H_i-1})),$$
$$\sup_{0\leq t<s\leq 1}|(s+\varepsilon)^{2H_i-2}(s-t)^{2-H_i}-(s-t)^{H_i}|=\sup_{0\leq t<s\leq 1}\bigg|(s-t)^{H_i}$$
$$\bigg[1-(\frac{s-t}{s+\varepsilon})^{2-2H_i}\bigg]\bigg|$$
$$\leq 1,$$
$$\sup_{0\leq s<t\leq 1}|(s+\varepsilon)^{2H_1-2}(s-t)^{2-H_1}-(s-t)^{H_1}| \leq \max(1,\varepsilon^{2H_1-2}),$$
$$\sup_{0\leq t<s\leq 1}|(t+\varepsilon)^{2H_2-2}(s-t)^{2-H_1}-(s-t)^{2H_2-H_1}|=\sup_{0\leq t<s\leq 1}\bigg|(s-t)^{2H_2-H_1}$$
$$\bigg[1-(\frac{s-t}{t+\varepsilon})^{2-2H_2}\bigg]\bigg|$$
$$\leq \max(1,\varepsilon^{2H_2-2}),$$
$$\sup_{0\leq t<s\leq 1}(s-t)^{2H_i}=1.$$

The fourth derivative of r is the sum of terms of the following form:
$$(s-t)^{\gamma+2}(\frac{\partial^4}{\partial s^2\partial t^2}f_1.f_2+\frac{\partial^4}{\partial s^2\partial t^2}f_2.f_1) \leq 4(1+\varepsilon)^{2H_2},$$

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$(s-t)^{\gamma+2}\frac{\partial^3}{\partial s^2 \partial t}f_1 \cdot \frac{\partial}{\partial t}f_2 \leq \frac{4}{3\sqrt{3}}(1+\max(\varepsilon^{2H_1-1},(1+\varepsilon)^{2H_1-1}))$ two of them are in this form,

similarly $(s-t)^{\gamma+2}\frac{\partial^3}{\partial s^2 \partial t}f_2 \cdot \frac{\partial}{\partial t}f_1 \leq \frac{4}{3\sqrt{3}}(1+\max(\varepsilon^{2H_2-1},(1+\varepsilon)^{2H_2-1}))$

$(s-t)^{\gamma+2}\frac{\partial^3}{\partial t^2 \partial s}f_1 \cdot \frac{\partial}{\partial s}f_2 \leq \frac{4}{3\sqrt{3}}\max(1,(1+\varepsilon)^{2H_2-1})$ two of them are in this form, similar terms exist when we interchange f_1 and f_2,

$(s-t)^{\gamma+2}\frac{\partial^2}{\partial s^2}f_1 \cdot \frac{\partial^2}{\partial t^2}f_2 \leq 4\max(1,\varepsilon^{2H_2-2},\varepsilon^{2H_1-2})$ and two of them are in this form, so they add $8\max(1,\varepsilon^{2H_2-2},\varepsilon^{2H_1-2})$,

$(s-t)^{\gamma+2}\frac{\partial^2}{\partial s \partial t}f_1 \cdot \frac{\partial^2}{\partial s \partial t}f_2 \leq 4$ and four of them are in this form, so they add 16.

This sum of terms has to be multiplied by the coefficient

$$\frac{1}{4}L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2},$$

and

$$C_0(\varepsilon) \leq \frac{1}{4}L^{2(H_1+H_2)}|\cos\alpha|^{2H_1}|\sin\alpha|^{2H_2}[4(1+\varepsilon)^{2H_2}+$$
$$8\frac{2}{3\sqrt{3}}(1+\max(\varepsilon^{2H_1-1},(1+\varepsilon)^{2H_1-1},\varepsilon^{2H_2-1},(1+\varepsilon)^{2H_2-1})$$
$$+8\frac{2}{3\sqrt{3}}\max(1,(1+\varepsilon)^{2H_1-1},(1+\varepsilon)^{2H_2-1})$$
$$+8\max(1,\varepsilon^{2H_2-2},\varepsilon^{2H_1-2})+16].$$

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