Identification of the multiscale fractional Brownian motion with biomechanical applications

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In some applications, for instance biomechanics, turbulence, finance, or internet traffic, it appears relevant to model data with a generalization of a fractional Brownian motion for which the Hurst parameter H is depending on the frequency as a piece-wise constant function. These processes are called multiscale fractional Brownian motions. In this contribution, we give a statistical study of the multiscale fractional Brownian motions. We propose a method based on wavelet analysis to detect the frequency changes, estimate the different parameters and test the goodness-of-fit. Biomechanical data are then studied with these new tools, that leads to interesting conclusions.

Keywords: Biomechanics; Detection of change; Goodness-of-fit test; Fractional Brownian motion; Semi-parametric estimation; Wavelet analysis.

1 Introduction

Fractional Brownian motions (f.B.m.) were popularized by Mandelbrot and Van Ness (1968) who suggest the study of their properties as a typical example of non-Markovian process. F.B.m. are centered Gaussian processes with stationary increments. These processes are self-similar and their increments are short or long memory processes and both these properties are controlled by the same parameter : the Hurst index H. But f.B.m. appears as an ideal mathematical model. In some applications, real data lead to the modelling by locally self-similar processes with a time-varying Hurst index $H(t)$ (see Cohen (2000) and the references therein). As a consequence, when the Hurst index is a time-varying function, the increments of the process are no more stationary.

Here, we consider Gaussian processes having stationary increments with a Hurst index varying with the frequencies. These kinds of processes have been introduced (implicitly) by Collins and de Luca (1993) in a statistical study of the position of the center of pressure in upright position, by Rogers (1997) in a discussion where he rejects the f.B.m. (with a constant Hurst index) as an admissible model for the stock prices, and by Benassi and Deguy (1999) for image analysis and image synthesis (they have proposed a model with two different Hurst indexes at low and high frequencies and one frequency of change ω_c). An application to finance of the model proposed by Rogers (1997) is developed in Cheridito (2000).

The previous examples have leaden us to propose a model of generalized f.B.m. with a finite number K of change points of the Hurst index (following the frequencies). We called it (M_K) multiscale fractional Brownian motion, and we provide the main probabilistic properties of this model in Bardet and Bertrand (2003). In this paper, we deal with the statistical study of the multiscale f.B.m. and we focus on the application to biomechanics.

Our plan will be the following : In the following section, we describe the biomechanical data and the corresponding statistical problem. In Section 3, after a brief reminder of the definition of the multiscale fractional Brownian motion and its main probabilistic properties, we show that the variogram method is not relevant for the estimation of the different parameters of a (M_K)-f.B.m. We develop another statistical study based on wavelet analysis and we state (and proved in appendix) a functional central limit theorem for the empirical wavelet coefficients. It leads, in Section 4, to estimations of the different frequency changes and Hurst parameters and to a goodness of fit test. Finally, in Section 5, the biomechanical data are studied with the tools developed in Section 4. All the proofs are given in appendices.

2 The Biomechanical Problem

One motivation of this work is the modeling of biomechanical data corresponding to the regulation of the upright position of the human being. Using a force platform, the position of the center of pressure (C.O.P.) during quiet postural stance is determined. This position is usually measured at a frequency of 100 Hz for the period of one minute, which yields a data set of 6000 observations. The experimental conditions are conformed to the standards of the Association Française de Posturologie (AFP), for instance the feet position (angle and clearance), the eyes open or closed.

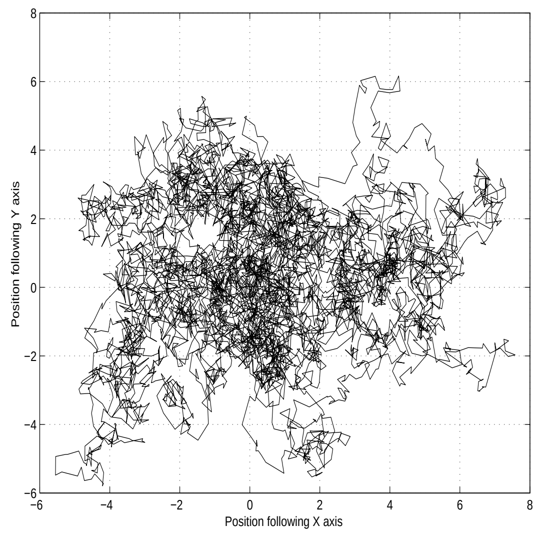

Figure 1 : An example [1] of the trajectory of the C.O.P. during 60s at 100Hz (in mm)

The X axis of the platform corresponds to the fore-aft direction and the Y axis corresponds to the medio-lateral direction. During the 1970's, these data were analyzed as a set of points, *i.e.* without taking into account their order. During the following decade some studies consider them as a process, and Collins and de Luca (1993) introduced the use of f.B.m. to model these data. Actually, they use a generalization of f.B.m. More precisely, the position X_i of the C.O.P. is observed at times $t_i = i\Delta$ for $i = 1, \ldots, N$ ($\Delta = 0.01\ s$). The study of Collins and de Luca is based on the empirical variogram

$$V_N(\delta) = \frac{1}{(N-\delta)} \sum_{i=1}^{N-\delta} \left(X_{(i+\delta)\Delta} - X_{i\Delta}\right)^2 \tag{1}$$

where $\delta \in \mathbb{N}^*$. For a f.B.m., we have $\mathbb{E} V_N(\delta) = \sigma^2\, \Delta^{2H} \times \delta^{2H}$ and after plotting the log-log graph of the variogram as a function of the lag time, i.e. $(\log \delta, \log V_N(\delta))$, a linear regression provides the slope $2H$. Typically, one gets the following type of figure (see Figure 2). It is considered by Collins and de Luca to be a "f.B.m." with two regimes : with slope $2H_0$ *(short term)* and with slope $2H_1$ *(long term)* separated

[1]these experimental data were realized by A. Mouzat and are used in [11].

by a critical time lag δ_c and these parameters are estimated graphically :

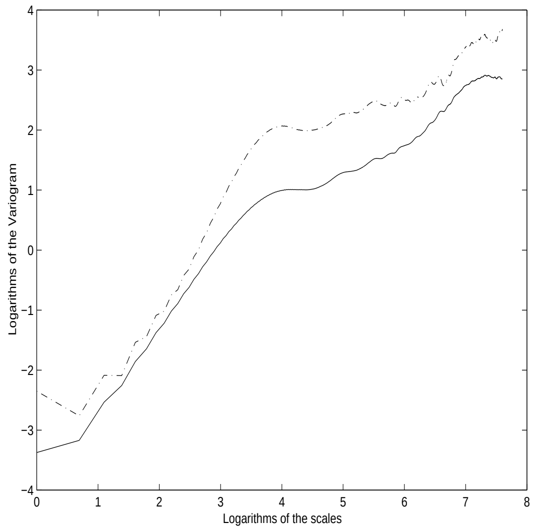

Figure 2 : An example of the log-log graph of the variogram for the previous trajectories X (-.) and Y (-).

They found $H_0 > 0.5$, $H_1 < 0.5$ and a critical time lag $\delta_c \simeq 1\ s$. These results are interpreted as corresponding to two different kinds of regulation of human stance : at *long term* $H_1 < 0.5$ the process is anti-persistent, at *short term* $H_0 > 0.5$ and the process is persistent. This method has been used many times in biomechanics with different experimental conditions (opened eyes versus closed eyes, different feet angles,...), but is missing a mathematical model and its statistical study to obtain confidence intervals on the two slopes $2H_0$, $2H_1$ and the critical time lag δ_c.

3 The multiscale fractional Brownian motion and its statistical study based on wavelet analysis

3.1 Description of the model

A fractional Brownian motion $B_H = \{B_H(t),\, t \in \mathbb{R}_+\}$ of parameters (H, σ) is a real centered Gaussian process with stationary increments and $\mathbb{E}\,|B_H(s) - B_H(t)|^2 = \sigma^2\,|t-s|^{2H}, \quad \forall (s,t) \in \mathbb{R}_+^2$ where $H \in]0,1[$ and $\sigma > 0$. The fractional Brownian motion (f.B.m.) has been proposed by Kolmogorov (1940) who defined it by the harmonizable representation

$$B_H(t) = \int_{\mathbb{R}} \frac{\left(e^{it\xi} - 1\right)}{|\xi|^{H+1/2}}\,\widehat{\overline{W}}(d\xi) \tag{2}$$

where $W(dx)$ is a standard Brownian measure. and $\widehat{W}(d\xi)$ its Fourier transform. We refer to Samorodnitsky and Taqqu, 1994 for the question of equivalence of the different representations of the f.B.m. From the harmonizable representation, a natural generalization is the multiscale fractional Brownian motion with an Hurst index depending on the frequency. More precisely, we define :

Definition 3.1 *For $K \in \mathbb{N}$, a (M_K)-multiscale fractional Brownian motion $X_\rho = \{X_\rho(t), t \in \mathbb{R}_+\}$ (simplify by (M_K)-f.B.m.) is a process such as*

$$X_\rho(t) = 2\sum_{j=0}^{K} \int_{\omega_j}^{\omega_{j+1}} \sigma_j\,\frac{(e^{it\xi} - 1\ )}{|\xi|^{H_j+1/2}}\,\widehat{\overline{W}}(d\xi) \quad \textit{for } t \in \mathbb{R}_+ \tag{3}$$

with $\omega_0 = 0 < \omega_1 < \cdots < \omega_K < \omega_{K+1} = \infty$ *by convention,* $\sigma_i \in I\!R_+^K$ *and* $H_i \in]0,1[^K$.

The (M_K)-f.B.m. was notably introduced in order to relax the self-similarity property of f.B.m. The self-similarity is a form of invariance with respect to changes of time scale [23] and it links the behavior at high frequencies to the behavior at low frequencies.

In Bardet and Bertrand (2003), the main properties of such a process are provided : X_ρ is a Gaussian centered process with stationary increments, its trajectories are a.s. of Hölder regularity α, for every $0 \le \alpha < H_K$ and its increments form a long-memory process (except if the different parameters verify a particular relations, *i.e.*, if its spectral density is a continuous function).

3.2 The question of the choice of the estimator

Let X_ρ be a (M_K)-f.B.m. defined by (3). We observe one path of the process X_ρ on the interval $[0, T_N]$ at the discrete times $t_i = i \times \Delta_N$ for $i = 1, \ldots, N$ and $T_N = N \times \Delta_N$. We consider the asymptotic $N \to \infty$, $\Delta_N \to 0$ and $T_N \to \infty$ and want to estimate the parameters of the (M_K)-f.B.m. that are $(H_0, H_1, \ldots, H_K)$, $(\sigma_0, \sigma_1, \ldots, \sigma_K)$ and $(\omega_1, \ldots, \omega_K)$. *** Changem

Even if the model is defined as a parametric model, in the following we prefer to use a semi-parametric statistics for different reasons. Firstly, the spectral density of X_ρ is not continuous (as a general case) and thus, the classical results of the consistence of the maximum likelihood or Whittle maximum likelihood estimators in such a case of long memory process (see Fox and Taqqu, 1986, Dahlhaus, 1989 or Giraitis and Surgailis, 1990) can not be used. Secondly, the following semi-parametric statistics is more robust than a parametric one if the model is misspecified : for example, if the function $H(\xi)$ is a not exactly a piece-wise constant function, but a constant function on several intervals.

A semi-parametric method of estimation built on the variogram was developed from the seminal paper of Istas and Lang (1997) and gave good results in case of the f.B.m. (see Bardet, 2000) or of the multi-fractionnal f.B.m (see Benassi *et al.*, 1998). But there are difficulties to identify the model (M_K)-f.B.m. with such a method. Indeed, it is obvious to prove that for $\delta > 0$:

$$\mathcal{V}(\delta) = I\!E\left(X_\rho(t+\delta) - X_\rho(t)\right)^2 \quad = \quad 4\sum_{j=0}^{K} \delta^{2H_j}\,\sigma_j^2 \int_{\delta\omega_j}^{\delta\omega_{j+1}} \frac{(1-\cos v)}{v^{2H_j+1}}\,dv. \tag{4}$$

The principle of the variogram's method ensues from the writing of $\log\Big(\mathcal{V}(\delta)\Big)$ as an affine function of $\log\delta$. For a (M_K)-f.B.m. and with $C(H_i) = \log\left(4\sigma_i^2 \int_0^\infty \frac{(1-\cos v)}{v^{2H_i+1}}\,dv\right)$ for $i = 0, 1, \ldots, K$, three situations could provide such a relation :

1. for $\delta \to \infty$, $\log\Big(\mathcal{V}(\delta)\Big) = 2H_0\log\delta + \log\sigma_0^2 + C(H_0) + \mathcal{O}(\delta^{-2H_0})$;

2. for $\delta \to 0$, $\log\Big(\mathcal{V}(\delta)\Big) = 2H_K\log\delta + \log\sigma_K^2 + C(H_K) + \mathcal{O}(\delta^{2-2H_K})$;

3. if $K \ge 2$, and if there exists $j \in \{1, \ldots, K-1\}$ such as $\frac{\omega_{j+1}}{\omega_j} \to \infty$ and more precisely such as

$$\frac{\min_{j\leq i\leq K-1}\left(\omega_{j+1}^{H_i/H_j}\right)}{\max_{1\leq i\leq j}\left(\omega_j^{(1-H_i)/(1-H_j)}\right)} \to \infty, \text{ for } \begin{cases} \delta \times \min_{j\leq i\leq K-1}\left(\omega_{j+1}^{H_i/H_j}\right) \to \infty \\ \delta \times \max_{1\leq i\leq j}\left(\omega_j^{(1-H_i)/(1-H_j)}\right) \to 0 \end{cases}, \text{ then}$$

$$\log\Big(\mathcal{V}(\delta)\Big) = 2H_j\log\delta + \log\sigma_j^2 + C(H_j) + \mathcal{O}\left(\delta^{2-2H_j}\times\max_{1\leq i\leq j}\left(\omega_j^{2-H_i}\right)\right) + \cdots$$
$$\cdots + \mathcal{O}\left(\frac{1}{\delta^{2Hj}\times\min_{j\leq i\leq K-1}\left(\omega_{j+1}^{2H_i}\right)}\right) \quad (5)$$

(the proofs of those three expansions come from $\int_0^\varepsilon \frac{(1-\cos v)}{v^{2H+1}}\,dv = \mathcal{O}(\varepsilon^{2-2H})$ for $\varepsilon \to 0$ and $\int_x^\infty \frac{(1-\cos v)}{v^{2H+1}}\,dv = \mathcal{O}(x^{-2H})$ for $x \to \infty$). In those three situations, if one can show that there is a convergent estimator $V_N(\delta)$ of $\mathcal{V}(\delta)$, then a log-log regression of $\log\Big(V_N(\delta)\Big)$ on $\log\delta$ could allow an estimation of the different parameters. However such a method should have a lot of drawbacks. At first, the estimation of "intermediate" parameters $(H_j)_{1\leq j\leq K-1}$ and $(\sigma_j^2)_{1\leq j\leq K-1}$ requires very specific asymptotic properties between all the frequency changes $(\omega_j)_{1\leq j\leq K-1}$ that translate a lake of generality of the method. Moreover, concretely, the frequency changes are fixed and one obtains rough approximation instead of asymptotic properties (and numerical simulations show that the log-log plot of the variogram does not exhibit any intermediate linear part). Secondly, when the model is misspecified the variogram model could lead to inadequate results. For example the following picture gives the case of a (M_2)-f.B.m. where the variogram method would detect only one frequency change and could not precisely estimate its value. Finally, the variogram's method could peharps be applied in the two first previous situations 1. and 2., *i.e.* for the estimation of (H_0, σ_0^2) or (H_0, σ_0^2). But in such cases, δ will have to be a function of N (number of data) and its choice of function will depend on the unknowns parameters H_0 or H_1 for obtaining central limit theorems for $\log\Big(V_N(\delta)\Big)$... (see the same kind of problem in *et al.*, 2002).

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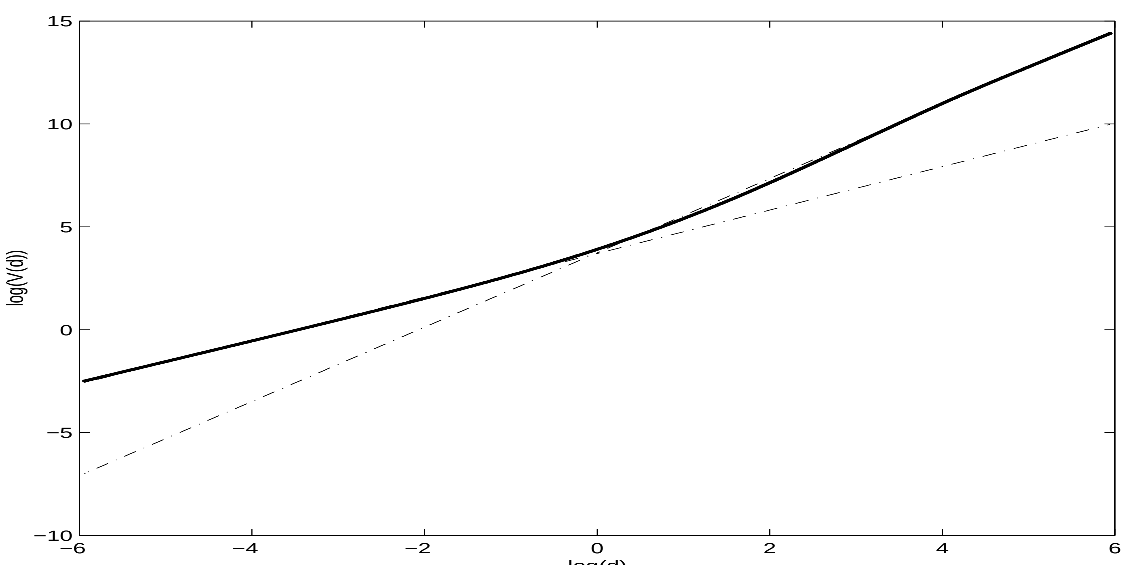


Figure 3: An example of a theoretical variogram for a (M_2)-f.B.m, with $H_0 = 0.9$, $H_1 = 0.2$, $H_2 = 0.5$, and $\sigma_0 = \sigma_1 = \sigma_2 = 5$ and $\omega_1 = 0.05$, $\omega_2 = 0.5$ (in solid, the theoretical variogram, in dot-dashed, its theoretical asymptotes for $\delta \to 0$ and $\delta \to \infty$).

We deduce from the definition of the model that an interesting method to estimate the parameters of a (M_K)-f.B.m. could be a spectral domain method. We chose a method based on a wavelet analysis. This method has been introduced by Flandrin (1992) and was developed by Abry *et al.* (2002) and Bardet *et al.* (2000). We also use here the same results of the wavelet analysis obtained in Bardet and Bertrand (2003).

3.3 A statistical study based on wavelet analysis

We consider a "mother" wavelet ψ such as:

Assumption (A1): $\psi: \mathbb{R} \mapsto \mathbb{R}$ *is a* $\mathcal{C}^\infty$ *function verifying:*

- *for all* $m \in \mathbb{R}$, $\displaystyle\int_{\mathbb{R}} |t^m \psi(t)|\, dt < \infty$,
- *its Fourier transform* $\widehat{\psi}(\xi)$ *is an even function compactly supported on* $[-\beta, -\alpha] \cup [\alpha, \beta]$ *with* $0 < \alpha < \beta$.

These conditions are sufficiently mild and are satisfied by famous wavelets (in particular, Lemarié-Meyer wavelet). As a consequence of the second condition , for all $m \in \mathbb{N}$,

$$\int_{\mathbb{R}} t^m \psi(t) dt = 0. \tag{6}$$

Note that it is not mandatory to choose ψ to be a "mother" wavelet associated to a multiresolution analysis of $\mathbb{L}^2(\mathbb{R})$ and the whole theory can be developed without resorting to this assumption : the choice of ψ is then very large.

Let $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$, define $\lambda = (a, b)$ and the family of functions $(\psi_\lambda)_\lambda$ defined by $\psi_\lambda(t) = \dfrac{1}{\sqrt{a}} \psi\left(\dfrac{t}{a} - b\right)$. Parameters a and b are so-called the scale and the shift of the wavelet transform (here we consider a continue wavelet transform). Let $d_X(a, b)$ be the wavelet coefficient of the process X for the scale a and the shift b, with

$$d_X(a, b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(\frac{t}{a} - b) X(t) dt =< \psi_\lambda, X >_{L^2(\mathbb{R})} .$$

If ψ verifies Assumption (A1) and X_ρ is a (M_K)-f.B.m. the family of wavelet coefficients verifies the following properties (see Bardet and Bertrand, 2003) :

1. for $a > 0$, $(d_{X_\rho}(a, b))_{b \in \mathbb{R}}$ is a stationary centered Gaussian process such as :
$$\mathbb{E}\left(d^2_{X_\rho}(a, .)\right) = \mathcal{I}_1(a) = a \int_{\mathbb{R}} |\widehat{\psi}(au)|^2 \rho^{-2}(u)\, du. \tag{7}$$
2. for all $i = 0, 1, \cdots, K$, if the scale a is such as $[\dfrac{\alpha}{a}, \dfrac{\beta}{a}] \subset [\omega_i, \omega_{i+1}]$, then
$$\mathbb{E}\left(d^2_{X_\rho}(a, .)\right) = a^{2H_i+1} \sigma_i^2 K_{H_i}(\psi), \quad \text{with } K_H(\psi) = \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(u)\right|^2}{|u|^{2H+1}} du. \tag{8}$$

The property (8) is very interesting for the estimation of the parameters of X_ρ. Thus, if we consider a convergent estimator of $\log\left(\mathbb{E}\left(d^2_{X_\rho}(a, .)\right)\right)$, it provides a linear model in $\log a$ and $\log \sigma_i^2$. This natural estimator is $\log I_N(a)$ with

$$I_N(a) = \frac{1}{[N/a] - 1} \sum_{k=1}^{[N/a]-1} d^2_{X_\rho}(a, k\Delta_N). \tag{9}$$

We have a functional central limit theorem for $(\log I_N(a))_{a_{min} \leq a \leq a_{max}}$ (see the proof in Bardet and Bertrand, 2003) :

Proposition 3.1 *Let X_ρ be a (M_K)-f.B.m., $0 < a_{min} < a_{max}$ and ψ verify Assumption (A1). Then :*

$$\sqrt{N\Delta_N}\left(\log I_N(a) - \log \mathcal{I}_1(a)\right)_{a_{min}\le a\le a_{max}} \xrightarrow[N\to\infty]{\mathcal{D}} \left(Z(a)\right)_{a_{min}\le a\le a_{max}} \tag{10}$$

with $(Z(a))$ a centered Gaussian process such as for $(a_1,a_2)\in[a_{min},a_{max}]^2$,

$$cov(Z(a_1),Z(a_2)) = \frac{2a_1\, a_2}{\mathcal{I}_1(a_1)\,\mathcal{I}_1(a_2)} \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)}{|\rho(\xi)|^2} e^{-iu\xi} d\xi\right)^2 du. \tag{11}$$

Then, if we precise the localization of scales, *i.e.* frequencies, we obtain the following consequence :

Corollary 3.1 *Let $i\in\{0,1,\cdots,K\}$ and assume that $\dfrac{\beta}{\alpha}\le\dfrac{\omega_{i+1}}{\omega_i}$. Then,*

$$\sqrt{N\Delta_N}\left(\log I_N(1/f)+(2H_i+1)\log f - \log\sigma_i^2 - \log K_{H_i}(\psi)\right)_{\omega_i/\alpha\le f\le\omega_{i+1}/\beta} \xrightarrow[N\to\infty]{\mathcal{D}} \left(Z(1/f)\right)_{\omega_i/\alpha\le f\le\omega_{i+1}/\beta} \tag{12}$$

with the centered Gaussian process $(Z(.))$ such as for $(f_1,f_2)\in[\dfrac{\omega_i}{\alpha},\dfrac{\omega_{i+1}}{\beta}]^2$,

$$cov(Z(1/f_1),Z(1/f_2)) = \frac{2\,(f_1\,f_2)^{2H_i}}{K^2_{H_i}(\psi)} \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{\overline{\widehat{\psi}}(\xi/f_1)\widehat{\psi}(\xi/f_2)}{|\xi|^{2H_i+1}} e^{-iu\xi} d\xi\right)^2 du. \tag{13}$$

For Δ_N small enough, this result shows that all parameters H_i and σ_i^2 could be estimated by using a linear regression of $\log I_N(1/f_j)$ versus $\log f_j$, when the frequencies ω_i are known. Moreover, this central limit theorem shows that a graph of $(\log f, \log I_N(1/f))$ for $f>0$ exhibits different areas of asymptotic linearity : it suggests the procedure of the following section to estimate and test the frequency changes (see for instance figures 4 or 6).

3.4 The discretization problem

But before, we have to solve the discretization problem. In fact, the definition of wavelet coefficients of X_ρ needs the knowledge of a continuous path of X_ρ. But we suppose here that only a time series $(X_\rho(0), X_\rho(\Delta_N),\cdots,X_\rho(N\Delta_N))$ from X_ρ is provided. Thus, there is a difference between the previous theoretical wavelet coefficients and the empirical wavelet coefficients. A similar problem was considered in Bardet (2002). We follow the procedure and the results of this paper. So, the following empirical wavelet coefficients $e_{X_\rho}(a,b)$ have to be considered, with

$$e_{X_\rho}(a,b) = \frac{\Delta_N}{\sqrt{a}}\sum_{p=0}^{N}\psi\left(\frac{p\Delta_N}{a}-b\right)X_\rho(p\Delta_N).$$

As a consequence, we have to consider $J_N(a)$ instead of $I_N(a)$ with

$$J_N(a) = \frac{1}{[N/a]-1}\sum_{k=1}^{[N/a]-1} e^2_{X_\rho}(a,k\Delta_N). \tag{14}$$

The empirical wavelet coefficients $e_{X_\rho}(a,k\Delta_N)$ converge to theoretical one $d_{X_\rho}(a,k\Delta_N)$ when $\Delta_N\to 0$ (convergence of a Riemann sum) and for k such as $k\Delta_N\to\infty$ and $N-k\Delta_N\to\infty$. More precisely, we have the following property:

Property 3.1 *Let ψ verify Assumption (A1) and X_ρ be a (M_K)-f.B.m. Then, for $a>0$*

1. *for all* $m \in \mathbb{N}$,

$$\mathbb{E}\left(e^2_{X_\rho}(a, k\Delta_N)\right) = \mathbb{E}\left(d^2_{X_\rho}(a, k)\right) + \mathcal{O}\left(\Delta_N + \frac{1}{(k\Delta_N)^m} + \frac{1}{(N - k\Delta_N)^m}\right). \tag{15}$$

2. $\mathbb{E} I_N(a) = \mathbb{E} J_N(a) + \mathcal{O}(\Delta_N)$

Remark 3.1 *When $H > 1/2$, we could approximate $d(a, b)$ by the trapezoid method which would lead in Property 3.1.2 to an error in $\mathcal{O}(\Delta_N^2)$.*

Now, it is possible to provide the functional central limit theorem for $(\log J_N(a))_{a_{min}\le a\le a_{max}}$ computed from $(X_\rho(0), X_\rho(\Delta_N), \cdots, X_\rho(N\Delta_N))$:

Proposition 3.2 *Under assumptions of Proposition 3.1 and with Δ_N such as $N\Delta_N \to \infty$ and $N(\Delta_N)^3 \to 0$ when $N \to \infty$. Then, with the same process Z than in (10),*

$$\sqrt{N\Delta_N}\,(\log J_N(a) - \log \mathcal{I}_1(a))_{a_{min}\le a\le a_{max}} \xrightarrow[N\to\infty]{\mathcal{D}} (Z(a))_{a_{min}\le a\le a_{max}}. \tag{16}$$

As a particular case, for $i \in \{0, 1, \cdots, K\}$ and if $\dfrac{\beta}{\alpha} \le \dfrac{\omega_{i+1}}{\omega_i}$, then

$$\sqrt{N\Delta_N}\left(\log J_N(1/f) + (2H_i+1)\log f - \log \sigma_i^2 - \log K_{H_i}(\psi)\right)_{\omega_i/\alpha\le f\le\omega_{i+1}/\beta} \xrightarrow[N\to\infty]{\mathcal{D}} (Z(1/f))_{\omega_i/\alpha\le f\le\omega_{i+1}/\beta}. \tag{17}$$

A consequence of the discretization problem is that the convergence rate of the central limit theorem (16) that is $\sqrt{N\Delta_N}$ and thus the maximum convergence rate is $o(N^{1/3})$ from the previous conditions on Δ_N.

4 Identification of the parameters

First, let us describe the method at an heuristic level. From Proposition 3.2, Formula (17), we have

$$\log J_N(1/f) = -(2H_i + 1) \times \log(f) + \log\left(\sigma_i^2\right) + \log\left(K_{H_i}(\psi)\right) + \varepsilon_f^{(N)}, \tag{18}$$

for the frequencies f which satisfy the condition

$$\log(\omega_i) - \log(\alpha) \le \log(f) \le \log(\omega_{i+1}) - \log(\beta). \tag{19}$$

Moreover we have $(N\Delta_N)^{1/2}\left(\varepsilon_{f_j}^{(N)}\right)_{1\le j\le m} \xrightarrow[N\to\infty]{\mathcal{D}} (Z(1/f_j))_{1\le j\le m}$. Formula (18) and condition (19) mean that for $\log(f) \in [\log(\omega_i) - \log(\alpha), \log(\omega_{i+1}) - \log(\beta)]$, we have a linear regression of $\log J_N(1/f)$ onto $\log(f)$ with slope $-(2H_i + 1)$ and intercept
$\log \sigma_i^2 + \log K_{H_i}(\psi)$ and for $\log(f) \in [\log(\omega_{i+1}) - \log(\alpha), \log(\omega_{i+2}) - \log(\beta)]$ a linear regression with slope $-(2H_{i+1}+1)$ and intercept $\log \sigma_{i+1}^2 + \log K_{H_{i+1}}(\psi)$. This is a problem of detection of abrupt change on the parameters of a linear regression, but with a transition zone for $\log(f) \in]\log(\omega_{i+1}) - \log(\beta),\ \log(\omega_{i+1}) - \log(\alpha)[$.

Remark 4.1 *Condition (19) implies that $\omega_{i+1} > \dfrac{\beta}{\alpha} \times \omega_i$. Therefore we could only detect the frequency changes sufficiently spaced. For instance, if we choose the Lemarié-Meyer wavelet, we get $\beta/\alpha = 4$ which leads to the condition $\omega_{i+1} > 4 \times \omega_i$.*

In this section, we describe the estimation of the parameters and a goodness of fit test. Both of them are based on the following assumption :

Assumption (B_K) **:** *The process X_ρ is a (M_K)-multiscale fractional Brownian motion. This process is characterized by the parameters Ω^*, H^* and σ^* where $\Omega^* = (\omega_1^*, \cdots, \omega_K^*)$ with $H^* = (H_0^*, H_1^*, \dots, H_K^*)$ and $\sigma^* = (\sigma_0^*, \sigma_1^*, \dots, \sigma_K^*)$. Moreover the following conditions are fulfilled*

- $\omega_{i+1}^* > \dfrac{\beta}{\alpha} \times \omega_i^*$ *for* $i = 1, \cdots, K-1$*;*
- $\min_{0 \le i \le (K-1)} \left\{ \left(H_{i+1}^* - H_i^*\right)^2 + \left(\sigma_{i+1}^* - \sigma_i^*\right)^2 \right\} > 0$ *and*
- *there exists a compact set* $\mathcal{K} \subset]0,1[\times]0,\infty[$ *such as* $(H_i^*, \sigma_i^*) \in \mathcal{K}$ *for all* $i = 0, 1, \cdots, K$.

4.1 Estimation of the parameters

Let X_ρ be a (M_K)-f.B.m. satisfying the assumption (B_K) with K a known integer number. We observe one path of the process at N discrete times, that $(X_\rho(0), X_\rho(\Delta_N), \cdots, X_\rho(N\Delta_N))$. Let $[f_{min},\, f_{max}]$, with $0 < f_{min} < f_{max}$, be the chosen frequency band (see section 5, for an example). We discretize a (slightly modified) frequency band and compute the wavelet coefficients at the frequencies $(f_k)_{0 \le k \le a_N}$ where

$$f_k = \frac{f_{min}}{\beta}\,(q_N)^k \qquad \text{for } k = 0, \cdots, a_N, \quad q_N = \left(\frac{f_{max}}{f_{min}}\,\frac{\beta}{\alpha}\right)^{1/a_N} \qquad \text{and} \quad a_N = N\Delta_N.$$

For notational convenience , we assume here that $N\Delta_N$ is an integer number. By definition, we have $f_0 = f_{min}/\beta$ and $f_{a_N} = f_{max}/\alpha$, then, using the wavelet coefficients at the frequencies $(f_k)_{0 \le k \le a_N}$, we could detect all frequency changes (ω_i^*) included in the band $]f_{min}, f_{max}[$. To simplify the notations, we use the following assumption :

Assumption (C) : $\omega_i^* \in]f_{min}, f_{max}[$ *for all* $i = 1, \dots, K$.

In this framework, the estimation of the different parameters of X_ρ becomes a problem of linear regression with a known number of changes; thus, we follow the same method as in Bai (1994), Bai and Perron (1998), Lavielle (1999) or Lavielle and Moulines (2000) and define the estimated parameters $(\widehat{T}^{(N)}, \widehat{\Lambda}^{(N)})$ as the couple of vectors which minimizes the quadratic criterion :

$$Q^{(N)}(T,\Lambda) = \sum_{j=0}^{K+1} \sum_{i=1+t_j}^{t_{j+1}-\tau_N} \left|Y_i - X_i \lambda_j\right|^2, \text{and thus}$$

$$(\widehat{T}^{(N)}, \widehat{\Lambda}^{(N)}) = \text{Argmin} \left\{ Q^{(N)}(T,\Lambda);\; T \in \mathcal{A}_K^{(N)}, \Lambda \in \mathcal{B}_K \right\}$$

with

- $Y_i = \log\left(J_N(1/f_i)\right)$, $X_i = (\log f_i, 1)$ for $i = 0, \cdots, a_N$;
- $\tau_N = \left[\dfrac{\log(\beta/\alpha)}{\log q_N}\right]$, where $[x]$ is the integer part of x.
- $T = (t_0, t_1, \cdots, t_{K+1}) \in \mathcal{A}_K^{(N)}$ where

$$\mathcal{A}_K^{(N)} = \left\{ (t_0, \cdots, t_{K+1}) \in \mathbb{N}^{K+2};\, t_0 = 0, t_{K+1} = a_N + \tau_N, t_{j+1} - t_j > \tau_N \text{ for } j = 0, \cdots, K \right\};$$

- $\Lambda = (\lambda_0, \cdots, \lambda_K) \in \mathcal{B}_K$ where $\lambda_j = \begin{pmatrix} -(2H_j+1) \\ \log \sigma_j^2 + \log K_{H_j}(\psi) \end{pmatrix}$ and then

$$\mathcal{B}_K = \Big\{ (\lambda_0, \cdots, \lambda_K) \ \text{ with } \ (H_j, \sigma_j^2) \in \mathcal{K} \ \text{ for all } \ j \in \{0, 1, \cdots, K\} \Big\}.$$

The integer τ_N corresponds to the number of frequencies in the transition zones and $\log f_{i+\tau_N} = \log f_i + \log(\beta/\alpha)$. Obviously, for $j = 0, \cdots, K$, the vector $\widehat{\lambda}_j^{(N)}$ provides the estimators $\widehat{H}_j^{(N)}$ of H_j^* and $\widehat{\sigma}_j^{(N)}$ of σ_j^* by the relation $\widehat{\lambda}_j^{(N)} = \begin{pmatrix} -(2\widehat{H}_j^{(N)}+1) \\ \log\left((\widehat{\sigma}_j^{(N)})^2\right) + \log K_{\widehat{H}_j^{(N)}}(\psi) \end{pmatrix}$. For a given $T \in \mathcal{A}_K^{(N)}$, each $\widehat{\lambda}_j^{(N)}$ is obtained from a linear regression of (Y_i) onto (X_i) for $i = t_j+1, \cdots, t_{j+1}-\tau_N$. Thus, with $\widehat{T} = (\widehat{t}_j)_{0 \le j \le K+1}$ obtained from the minimization in T of $Q^{(N)}(T, \widehat{\Lambda})$, we define the different estimators of the change frequencies as

$$\widehat{\omega}_j^{(N)} = \alpha\, f_{\widehat{t}_j^{(N)}} = \alpha \cdot \frac{f_{min}}{\beta} \left(\frac{f_{max}}{f_{min}} \frac{\beta}{\alpha} \right)^{\frac{\widehat{t}_j^{(N)}}{a_N}} \quad \text{for } j = 1, \cdots, K. \tag{20}$$

We have the following convergence :

Proposition 4.1 *Let X_ρ verify Assumptions (C) and (B_K) with a known K, $(X_{\Delta_N}, \cdots, X_{N\Delta_N})$ be a discretized path, and ψ verify Assumption (A1). Let Δ_N be such as $N\Delta_N \to \infty$ and $N(\Delta_N)^3 \to 0$ when $N \to \infty$. Assume that $(\widehat{H}_i^{(N)}, \widehat{\sigma}_i^{(N)}) \in \mathcal{K}$ for all $i = 0, \cdots, K$. Then for all $\varepsilon > 0$, there exists $0 < C < \infty$ such as for all large N,*

$$\mathbb{P}\left((N\Delta_N)^{1/4} \left| \widehat{\omega}_j^{(N)} - \omega_j^* \right| \ge C \right) \le \varepsilon \ \text{ for } j = 1, \cdots, K. \tag{21}$$

Remark 4.2 *The proof of this proposition shows a more general result,* i.e. *for $(p, q) \in [3/4, 1] \times [0, 1]$, for $\varepsilon > 0$, there exists $C > 0$ such as*

$$\mathbb{P}\left(a_N^{1-p} \left| \widehat{\omega}_j^{(N)} - \omega_j^* \right| \ge C \right) \le \varepsilon \ \text{ for } j = 1, \cdots, K$$

with $a_N = (N\Delta_N)^q$. For numerical considerations and convergence rate of the following estimators of the parameters, we are going to fix now on $p = 3/4$ and $q = 1$ and then $a_N = N\Delta_N$.

For $j = 0, \cdots, K$, the natural estimates of H_j^* and σ_j^{2*} are given by the regression of (Y_i) onto $(\log f_i)$ for $i \in \{\widehat{t}_j^{(N)}, \cdots, \widehat{t}_{j+1}^{(N)} - \tau_N\}$. But the probability that $[\widehat{t}_j^{(N)}, \widehat{t}_{j+1}^{(N)} - \tau_N] \subset [t_j^*, t_{j+1}^* - \tau_N]$ does not increase fast enough to 1 as $N \to \infty$, in order to obtain a sufficiently fast convergence rate for these estimators. We address this difficulty as follows. We fix an integer number $m \ge 3$ and for $j = 0, \cdots, K$, we consider $[\tilde{U}_j^{(N)}, \tilde{V}_j^{(N)}]$ an interval strictly included in $[\widehat{t}_j^{(N)}, \widehat{t}_{j+1}^{(N)} - \tau_N]$, such as

$$\tilde{U}_j^{(N)} = \widehat{t}_j^{(N)} + \left[\frac{\widehat{t}_{j+1}^{(N)} - \widehat{t}_j^{(N)} - \tau_N}{m+1} \right] \ \text{ and } \ \tilde{V}_j^{(N)} = \widehat{t}_j^{(N)} + m \left[\frac{\widehat{t}_{j+1}^{(N)} - \widehat{t}_j^{(N)} - \tau_N}{m+1} \right]. \tag{22}$$

Then we estimate the parameters from a regression onto m points uniformly distributed in $[\tilde{U}_j^{(N)}, \tilde{V}_j^{(N)}]$; it provides the following estimator $\tilde{\lambda}_j^{(N)}$ from a regression of (Y_i) onto (X_i) for $i \in \{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\} = \Big\{ \tilde{U}_j^{(N)} + (k-1) \Big[\frac{\widehat{t}_{j+1}^{(N)}}{\ }$

By this way, define

$$\begin{aligned} \tilde{\lambda}_j^{(N)} &= \Big(-(2\tilde{H}_j^{(N)}+1), \log \tilde{\sigma^2}_j^{(N)} + \log K_{\tilde{H}_j^{(N)}}(\psi) \Big)' \\ &= \Big((\tilde{X}_j^{(N)})' \tilde{X}_j^{(N)} \Big)^{-1} (\tilde{X}_j^{(N)})' \tilde{Y}_j^{(N)} \ \text{ with } \ \begin{cases} \tilde{X}_j^{(N)} = (\log f_i \ , \ 1)_{i \in \{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\}} \\ \tilde{Y}_j^{(N)} = (Y_i)_{i \in \{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\}} \end{cases} , \end{aligned}$$

and for all $k=1,\cdots,m$, define $g_0^*(k)=\dfrac{f_{min}}{\beta}\left(\dfrac{\omega_1^*}{f_{min}}\right)^{k/(m+1)}$, $g_K^*(k)=\dfrac{\omega_K^*}{\alpha}\left(\dfrac{f_{max}}{f_{min}}\right)^{k/(m+1)}$ and $g_j^*(k)=\dfrac{\omega_j^*}{\alpha}\left(\dfrac{\alpha\omega_{j+1}^*}{\beta\omega_j^*}\right)^{k/(m+1)}$ for all $j\in\{1,\cdots,K-1\}$, .

We get the following central limit theorems for the corresponding estimators $(\tilde{H}_j^{(N)},\tilde{\sigma^2}_j^{(N)})$:

Proposition 4.2 *Under the same assumptions as in Proposition 4.1, for all $j=0,\cdots,K$,*

$$(N\Delta_N)^{1/2}\left(\tilde{\lambda}_j^{(N)}-\lambda_j^*\right)\quad\xrightarrow[N\to\infty]{\mathcal{D}}\quad\mathcal{N}(0,\Gamma_1^{\lambda_j^*})\tag{23}$$

where $\Gamma_1^{\lambda_j^*}=\left(X_j^{*'}X_j^*\right)^{-1}X_j^*\Sigma_j^*X_j^{*'}\left(X_j^{*'}X_j^*\right)^{-1}$, *with* $X_j^*=\left(\log g_j^*(k)\ ,\ 1\right)_{1\le k\le m}$ *and* $\Sigma_j^*=(s_{kl}^{*j})_{1\le k,l\le m}$ *the following matrix :*

$$s_{kl}^{*j}=2\cdot\left(g_j^*(k)g_j^*(l)\right)^{2H_j^*}\cdot\frac{\displaystyle\int_{I\!R}\left(\int_{I\!R}\overline{\widehat{\psi}}\left(\frac{\xi}{g_j^*(k)}\right)\widehat{\psi}\left(\frac{\xi}{g_j^*(l)}\right)|\xi|^{-(2H_j^*+1)}e^{-iu\xi}d\xi\right)^2du}{\left(\displaystyle\int_{I\!R}\left|\widehat{\psi}(u)\right|^2|u|^{-(2H_j^*+1)}du\right)^2}.\tag{24}$$

Remark 4.3 *Another possible choice would be to consider the regression for all the available frequencies in the interval $[\tilde{U}_j^{(N)},\tilde{V}_j^{(N)}]$. The number of considered frequencies increases then with the rate $a_N=N\Delta_N$. However, it does not improve significantly the convergence since the remainders of the regression are very strongly dependent.*

4.2 Goodness of fit test

It is also possible to estimate parameters H_j^* and σ_j^* from an feasible (or estimated) generalized least squares estimation (for more details, see Amemiya, chap. 6.3, 1985). Indeed, we can identify the asymptotic covariance matrix Σ_j^* for $j=0,\cdots,K$: this matrix has the form $\Sigma_j^*=\Sigma(H_j^*,\omega_j^*,\omega_{j+1}^*)$ and, from the previous limit theorems, $\widehat{\Sigma}_j^{(N)}=\Sigma(\tilde{H_j}^{(N)},\widehat{\omega}_j^{(N)},\widehat{\omega}_{j+1}^{(N)})$ converges in probability to Σ_j^*. Thus, it is possible to determine an estimation $\underline{\lambda}_j^{(N)}$ of λ_j^* with a feasible generalized least squares (F.G.L.S.) regression *i.e.* by minimizing

$$\|\ \tilde{Y}_j^{(N)}-\tilde{X}_j^{(N)}\lambda\ \|^2_{\widehat{\Sigma}_j^{(N)}}=(\tilde{Y}_j^{(N)}-\tilde{X}_j^{(N)}\lambda)'\left(\widehat{\Sigma}_j^{(N)}\right)^{-1}(\tilde{Y}_j^{(N)}-\tilde{X}_j^{(N)}\lambda).$$

First, we give asymptotic behavior of $\underline{\lambda}_j^{(N)}=\left\{\begin{array}{c}\left(-(2\underline{H}_j^{(N)}+1),\log\underline{\sigma}^{2}{}_j^{(N)}+\log K_{\underline{H}_j^{(N)}}(\psi)\right)'\\ \left((\tilde{X}_j^{(N)})'\left(\widehat{\Sigma}_j^{(N)}\right)^{-1}\tilde{X}_j^{(N)}\right)^{-1}(\tilde{X}_j^{(N)})'\left(\widehat{\Sigma}_j^{(N)}\right)^{-1}\tilde{Y}_j^{(N)}\end{array}\right.$.

Proposition 4.3 *Under the same assumptions as in Proposition 4.2, for all $j=0,\cdots,K$,*

$$(N\Delta_N)^{1/2}\left(\underline{\lambda}_j^{(N)}-\lambda_j^*\right)\quad\xrightarrow[N\to\infty]{\mathcal{D}}\quad\mathcal{N}(0,\Gamma_2^{\lambda_j^*})\tag{25}$$

with $\Gamma_2^{\lambda_j^*}=\left(X_j^{*'}\left(\Sigma_j^*\right)^{-1}X_j^*\right)^{-1}$.

For $j=0,\cdots,K$, the vectors $\tilde{Y}_j^{(N)}$ and $\tilde{X}_j^{(N)}\underline{\lambda}_j^{(N)}$ are two different estimators of the vector $\big(-(2H_j^*+1)\log f_i+\log\sigma_j^{2*}+\mathrm{lo}$
It suggests to define the following goodness of fit test. The test statistic is a distance $T^{(N)}$ between those both estimations for all $j=0,\cdots,K$:

$$T^{(N)}=(N\Delta_N)\cdot\left(\sum_{j=0}^{K}\|\ \tilde{Y}_j^{(N)}-\tilde{X}_j^{(N)}\underline{\lambda}_j^{(N)}\ \|^2_{\widehat{\Sigma}_j^{(N)}}\right).$$

This distance is the F.G.L.S. distance between points $(\log f_i, Y_i)_{i\in\{\tilde U_j^{(N)},\cdots,\tilde V_j^{(N)}\}}$ for $j = 0, \cdots, K$ and the $(K+1)$ F.G.L.S. regression lines. As a consequence, we get

Proposition 4.4 *Under assumptions of Proposition 4.1, we have*

$$T^{(N)} \xrightarrow[N\to\infty]{\mathcal{D}} \chi^2((K+1)(m-2)). \tag{26}$$

Remark 4.4 *Proposition 4.4 may be explained with heuristic arguments. Remainders are turned white, thus it is only natural for the sum of the second regression remainder squares to asymptotically form a* χ^2 *process. The number of freedom degrees is* $(K+1)(m-2)$ *because one loses two freedom degrees after the twice estimation of the* $(K+1)$ *vectors* λ_j^* *(we also show that these vectors are asymptotically independent).*

4.3 Estimation of the number of frequency change

Throughout the previous study, the number of frequency change, K, is assumed to be known. But the previous test provides a way for estimating K. In fact, it can be recursively done by beginning with $K = 0$ and continuing till the assumption "X_ρ is a (M_K)-f.B.m." is accepted. The following applications in biomechanics provide different examples of the power of discrimination of such a procedure. However, this estimation of the number of frequency change must be carefully applied : from numerical and heuristic arguments, it does not seem reasonable to work with $K > 2$.

4.4 Procedure of identification and discussion about the choice of parameters

Thus, for identifying a (M_K)-multiscale fractional (with K unknown) from a time series $(X_0, X_{\Delta_N}, \cdots, X_{N\Delta_N})$ we suggest the following procedure:

1. Begin with $K = 0$.
2. Choose a mother wavelet ψ (and thus α and β), a frequency band $[f_{min}, f_{max}]$ and m (see below for these different choices).
3. Compute the different frequencies $(f_i)_{0\leq i\leq a_N}$.
4. Compute the vector $(Y_i)_{0\leq i\leq a_N} = (\log J_N(1/f_i))_{0\leq i\leq a_N}$.
5. Minimize $Q^{(N)}(T, \Lambda)$ and thus compute the different values of $\widehat{\omega}_j^{(N)}$ for $j = 1, \cdots, K$.
6. Compute the different regression moments $\{\tilde U_j^{(N)}, \cdots, \tilde V_j^{(N)}\}$ and then the estimators $\tilde\lambda_j^{(N)}$ (for $j = 0, \cdots, K$).
7. Compute the different matrix $\widehat{\Sigma}_j^{(N)}$ and then $\underline{\lambda}_j^{(N)}$ (for $j = 0, \cdots, K$).
8. Compute $T^{(N)}$ and compare its value to the 95%-quantile of a $\chi^2((K+1)(m-2))$. If the test is rejected then go back to step 2. with $K = K + 1$.

How to chose the function ψ and the parameters f_{min}, f_{max} and m ?

1. **Choice of ψ :** The mother wavelet ψ has to verify Assumptions (A1) but as we say previously it is not mandatory to associate this function to orthogonality properties. However, the Lemarié-Meyer wavelet is a natural choice with good numerical properties of asymptotic decreasing but a too large ratio β/α which implies a too large transition zone of frequencies. The function ψ can

also be deduced from an arbitrary construction of its Fourier transform $\widehat{\psi}$; for instance, we propose $\widehat{\psi}_1(\lambda) = \exp\left(\frac{-1}{(|\lambda|-\alpha)(\beta-|\lambda|)}\right) \mathbf{1}_{\alpha\leq|\lambda|\leq\beta}$ and the function ψ_2 built from a translation of the Fourier transform of the Lemarié-Meyer function to $[-2\pi,-\pi]\cup[\pi,2\pi]$ (thus the ratio is now $\beta/\alpha=2$). The results obtained from those functions ψ_1 and ψ_2 are essentially the same than with the Lemarié-Meyer mother function, they appear more precise for the detection of frequency changes ω_j^* (because $\log\beta/\alpha$ and thus the transition band, could be as small as wanted) and less precise for the estimation of parameters H_j^* (because ψ_1 and ψ_2 are not concentrated as well around 0).

2. **Choice of f_{min} and f_{max} :** (we assume here that the frequencies are given in the inverse of $(X_1, X_2\cdots)$ time unity). The choice of f_{min} and f_{max} is first driven by the selection of a frequency band inside which the process has to be studied; the inspected frequency band is then $[\frac{f_{min}}{\beta}, \frac{f_{min}}{\alpha}]$. Secondly, $N\times\frac{f_{min}}{\beta}$ should be large enough for computing $I_N(\frac{\beta}{f_{min}})$ in (9). Formally one only needs to have $N\times\frac{f_{min}}{\beta}\geq 1$ but numerically $N\times\frac{f_{min}}{\beta}\geq 10$ seems to be necessary to use correctly the central limit theorem. Finally, the discretization problem implies that f_{max} cannot be too large for providing a good estimation of $d_{X_\rho}(\frac{\alpha}{f_{max}}, k\Delta_N)$ by $e_{X_\rho}(\frac{\alpha}{f_{max}}, k\Delta_N)$. In practice $\frac{f_{max}}{\alpha}\leq\frac{1}{\Delta_N}$ appears as a minimal condition.

3. **Choice of m :** Formally, m could be chosen such as $3\leq m<\min_j(t_{j+1}^*-\tau_N-t_j^*)$. Theoretically, the larger the m, the closer to 1 the power of the test. But numerical considerations imply that if m is too large then the different matrix $\widehat{\Sigma}_j^{(N)}$ are extremely correlated and the quality of the test is very dependent to the quality of the different estimations of $\widehat{\lambda}_j^*$. As a consequence, we chose $5\leq m\leq 10$.

5 Applications in Biomechanics

We apply our statistics to different trajectories with the following parameters :

- $N=6000$ and $\Delta_N=1/20$.
- The choice of the frequency band is $f_{min}=0.8$ and $f_{max}=12.5$ which corresponds to the band $[0.16, 2.5]$ in Hertz and a detection frequency band $[0.02, 1.2]$ Hz (with the Lemarié-Meyer wavelet).
- $m=5$.

First, we study the Y-trajectory for the $0cm$ clearance and $0degree$ angle. Here there is the double logarithm plotting ($\log J_N(f_k)$ by $\log f_k$) of this trajectory :

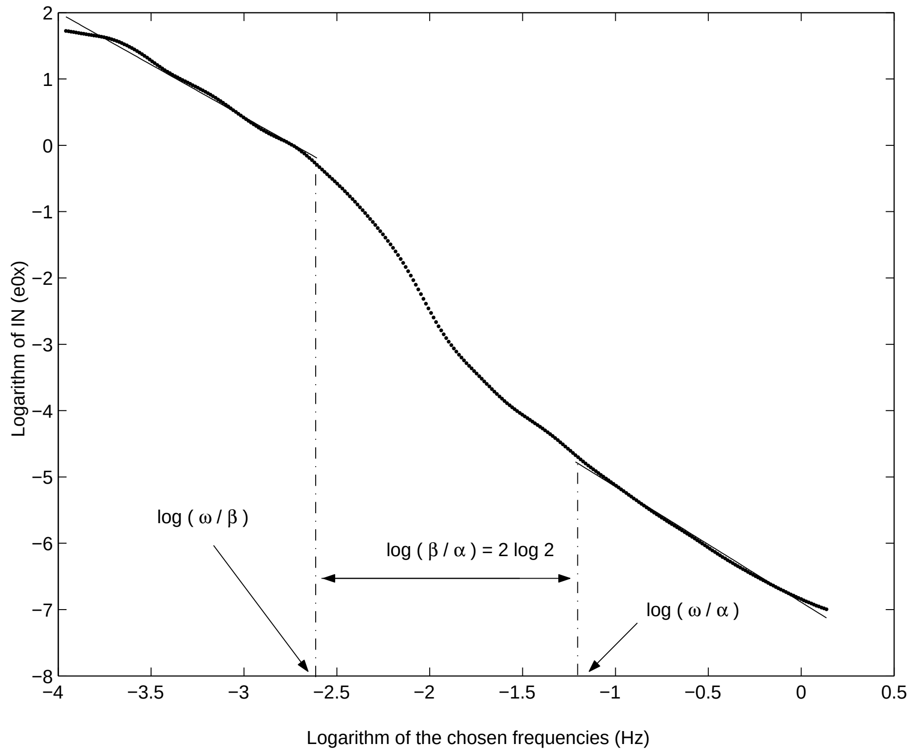


Figure 4 : The double logarithm plotting of the variance of wavelet coefficients of Y-trajectory ($0cm$, $0degree$) and and its corresponding regression lines

In figure 4, we observe two different bands of linearity $\simeq [0.02, 0.08]$ Hz ($= [\frac{f_{min}}{\beta}, \frac{\widehat{\omega}_1^{(N)}}{\beta}]$) and $\simeq [0.30, 1.2]$ Hz ($= [\frac{\widehat{\omega}_1^{(N)}}{\alpha}, \frac{f_{max}}{\alpha}]$) for the Y-trajectory, those two bands are spaced by a frequency band of length $\simeq 2 \log 2$ as expected by theoretical results. The estimation of the frequency change (from our algorithm) is

$$\widehat{\omega}_1^{(N)} \simeq 0.61 \ Hz.$$

We also observe for this graph the beginning of a change of behavior for low and high frequencies (see the previous explanations). For the Y-trajectory, we obtain the following different estimators of H_0^* and H_1^* :

$$\begin{array}{lll} \widehat{H}_0^{(N)} \simeq 0.29 & \tilde{H}_0^{(N)} \simeq 0.32 & \underline{H}_0^{(N)} \simeq 0.32 \\ \widehat{H}_1^{(N)} \simeq 0.37 & \tilde{H}_1^{(N)} \simeq 0.39 & \underline{H}_1^{(N)} \simeq 0.39 \end{array}$$

Finally, the value of the test statistic is :

$$T^{(N)} \simeq 3.7 < \chi^2_{95\%}(6) \simeq 12.6,$$

and thus the hypothesis “the Y-trajectory is a (M_1)-m.f.B.m.” is accepted. Moreover, the hypothesis “the Y-trajectory is a (M_0)-m.f.B.m.”, is rejected because then $T^{(N)} \simeq 42.4$ with $\widehat{H}^{(N)} \simeq 0.73$, $\tilde{H}^{(N)} \simeq 0.82$ and $\underline{H}^{(N)} \simeq 0.76$. The modelling by a (M_1)-m.f.B.m. rather than by a f.B.m. seems completely justified.

The following figure represents the double logarithm plotting for a trajectory of a f.B.m obtained from a simulation with the circulant matrix algorithm, with a Hurst parameter $H = 0.3$:

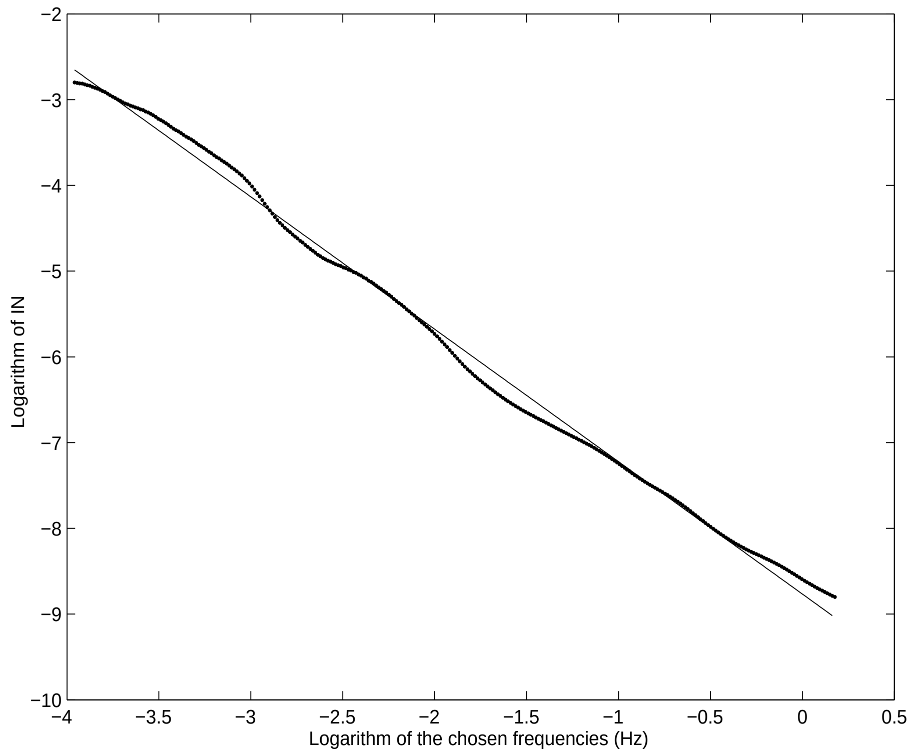


Figure 5 : The double logarithm plotting of the variance of wavelet coefficients of Y-trajectory (0 cm 0 degree) and its corresponding regression lines

In figure 5, we observe the linearity of the graph for the f.B.m trajectory for frequencies in the band $\simeq [0.02, 1.2]$ Hz. Moreover, the estimation of H is $\widehat{H}^{(N)} \simeq 0.27$, close to the theoretical value $H^* = 0.3$ and the hypothesis "the Y-trajectory is a f.B.m." is accepted by the test (as a consequence, for this trajectory, the modelling is without frequency change and $\widehat{H}^{(N)}$ provides an estimator of H^*).

We also study this Y-trajectory with two different mother "wavelets" (as described previously) : ψ_1 such as $\widehat{\psi}_1(\lambda) = \exp\left(-\frac{1}{(|\lambda|-5)(10-|\lambda|)}\right) \mathbf{1}_{5\leq|\lambda|\leq 10}$ (thus $\alpha = 5$ and $\beta = 10$) and ψ_2 such as $\widehat{\psi}_2(\lambda) = \widehat{\psi}(|\lambda| - \frac{4\pi}{3})$ (thus $\alpha = 2\pi$ and $\beta = 4\pi$) with ψ the Lemarié-Meyer mother wavelet. The following figure shows the results of this study :

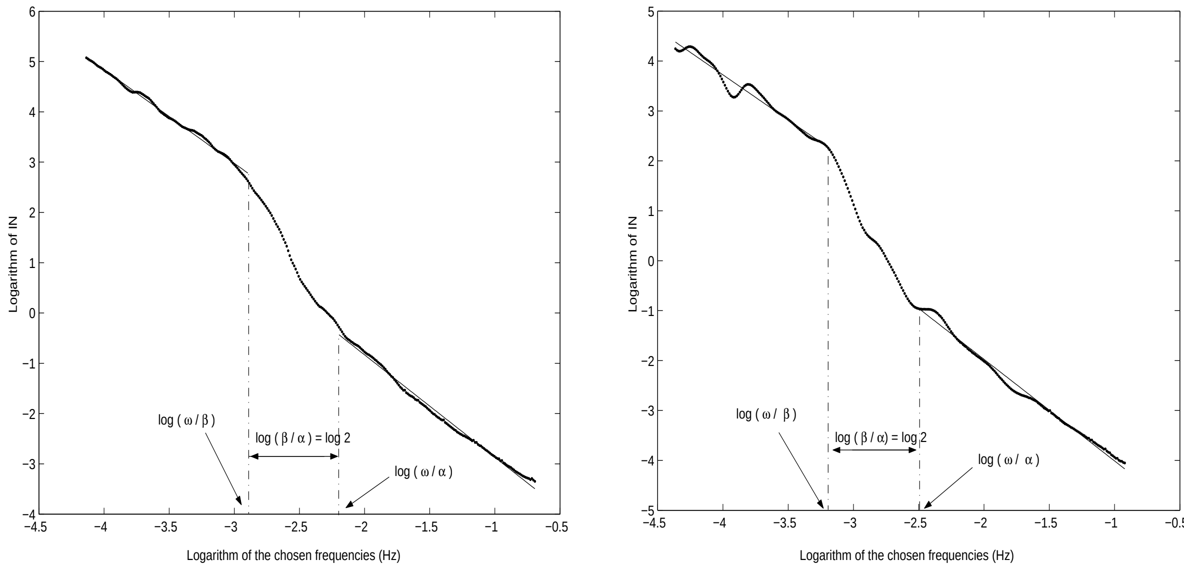
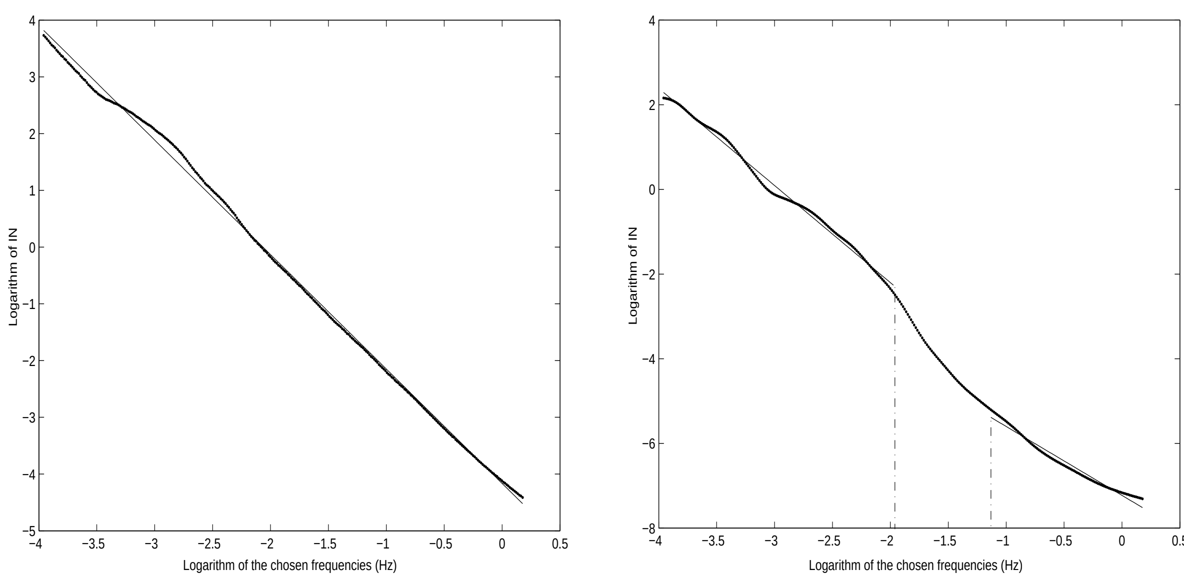


Figure 6 : The double logarithm plotting of the variance of wavelet coefficients of the Y-trajectory and its corresponding regression lines (left ψ_1, right, ψ_2)

The estimations of ω_1^* are $\widehat{\omega}_1^{(N)}(1) \simeq 0.55\ Hz$ with ψ_1 and $\widehat{\omega}_1^{(N)}(2) \simeq 0.52\ Hz$ with ψ_2, very close to the previous estimation $\widehat{\omega}_1^{(N)} \simeq 0.61\ Hz$ obtained with the Lemarié-Meyer mother wavelet ψ. On the other hand, the estimations of H_0^* and H_1^* are a little bit different from those obtained with ψ : with ψ_1, $\underline{H}_0^{(N)} \simeq 0.42$ and $\underline{H}_1^{(N)} \simeq 0.51$, with ψ_2, $\underline{H}_0^{(N)} \simeq 0.43$ and $\underline{H}_1^{(N)} \simeq 0.49$. It appears that when the transition band is small , the estimation of $\log \mathcal{I}_1$ is less precise. So, ψ_1 could be prefer for the estimation of ω_1^* and the Lemarié-Meyer mother wavelet ψ for estimating parameters H_0^* and H_1^*.

Finally, Figure 7 provides a comparison of the previous graphs and, first, the graph of the X-trajectory (for the 0 cm clearance and 0 degree angle), and second, the graphs of Y-trajectory with different clearances and angles between the feet, respectively $(2cm, 0\,degree)$, $(10cm, 15\,degree)$ and $(20cm, 40\,degree)$ (the mother wavelet is ψ_1) :



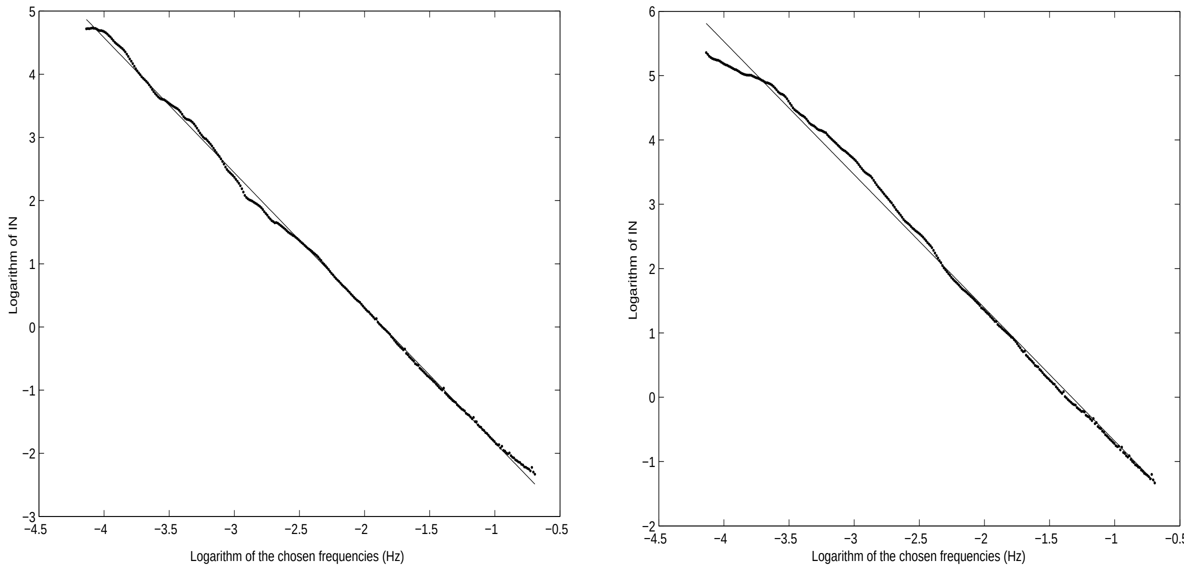


Figure 7 : The double logarithm plotting of the variance of wavelet coefficients (up and left, X-trajectory ($0cm$, $0\,degree$), up and right, Y-trajectory ($2cm$, $0\,degree$), down and left Y-trajectory ($10cm$, $15\,degree$) and down and right Y-trajectory ($20cm, 40\,degree$))

We observe the "perfect" linearity of the X-trajectory log-log plot and find $\widehat{H} \simeq 0.51$: the X-trajectory seems to be a classical Brownian motion trajectory. The test justifies the (M_1)-m.f.B.m. modelling for the Y-trajectory $(2cm, 0^0)$ with $T^{(N)} \simeq 6.7 < \chi^2_{95\%}(6) \simeq 12.6$) and reject the hypothesis "the Y-trajectory $(2cm, 0^0)$ is a f.B.m " (then $\widehat{H}^{(N)} \simeq 0.78$ and $T^{(N)} \simeq 28.3 > \chi^2_{95\%}(3) \simeq 7.81$); we find a frequency change $\widehat{\omega}_1^{(N)} \simeq 1.01$ Hz and $\underline{H}_0^{(N)} \simeq 0.60$ (for low frequencies), $\underline{H}_1^{(N)} \simeq 0.15$ (for high frequencies). Both other Y-trajectory, respectively, ($10cm$, $15\,degree$) and ($20cm$, $40\,degree$), accept the f.B.m. modelling with, respectively, $\tilde{H}^{(N)} \simeq 0.55$ and $\tilde{H}^{(N)} \simeq 0.53$.

In conclusion, all these results allow us a to give new interpretations on the upright position. First, the X-trajectory (the fore-aft direction) seems to be a white noise. The upright position appears as auto-stabilized in this direction. For the medio-lateral direction (Y-trajectory), when the clearance and the angle between the feet are small, two different mechanical behavior take place at the same time. The same auto-stabilization that for the fore-aft direction for high frequencies, and another type of biomechanical control for low frequencies. The change of behavior is between 0.5 and 1 Hz. But if the clearance or the angle between the feet increase, the second type of behavior (for low frequencies) seems to disappear, because the upright position is better stabilized. A good knowledge of these behaviors and their can allow a better detection of certain pathologies and to help in their cure.

A Proofs

A.1 Proofs of section 3

Proof. [**Property 3.1**] First, we prove that for all $m \in \mathbb{N}$, and b_N such as $(T_N - b_N \Delta_N) \to \infty$ and

$b_N\Delta_N \to \infty$ when $N \to \infty$,

$$\mathbb{E}\left(d^2_{X_\rho}(a,b_N\Delta_N)\right)=\mathbb{E}\left(\frac{1}{\sqrt{a}}\int_0^{T_N}\psi(\frac{t}{a}-b_N\Delta_N)X_\rho(t)dt\right)^2+\mathcal{O}\left(\frac{1}{T_N-b_N\Delta_N}+\frac{1}{b_N\Delta_N}\right). \qquad (27)$$

We have:

$$\begin{aligned}\mathbb{E}\left(d^2_{X_\rho}(a,b_N\Delta_N)\right) &= \mathbb{E}\left(\frac{1}{\sqrt{a}}\int_0^{T_N}\psi(\frac{t}{a}-b_N\Delta_N)X_\rho(t)dt\right)^2+..\\ &..\frac{2}{a}\mathbb{E}\left(\int_{\mathbb{R}}\int_{T_N-ab_N\Delta_N}^{\infty}\psi(\frac{u}{a})\psi(\frac{u'}{a})X_\rho(u+ab_N\Delta_N)X_\rho(u'+ab_N\Delta_N)dudu'\right)+..\\ &..\frac{2}{a}\mathbb{E}\left(\int_{\mathbb{R}}\int_{-\infty}^{ab_N\Delta_N}\psi(\frac{u}{a})\psi(\frac{u'}{a})X_\rho(u+ab_N\Delta_N)X_\rho(u'+ab_N\Delta_N)dudu'\right).\end{aligned}$$

But for all $m \in \mathbb{N}$, for all $u \in \mathbb{R}$,

$$\begin{aligned}&\int_{T_N-ab_N\Delta_N}^{\infty}\left|\psi(\frac{u'}{a})\mathbb{E}\left(X_\rho(u+ab_N\Delta_N)X_\rho(u'+ab_N\Delta_N)\right)\right|du' \leq\\ &\qquad C^2|u+ab_N\Delta_N|\int_{T_N-ab_N\Delta_N}^{\infty}u'^{m+1}\left|\psi(\frac{u'}{a})\right|\frac{1}{u'^m}\frac{|u'+ab_N\Delta_N|}{u'}du,\end{aligned}$$

because it exists $C>0$ such as $|\mathbb{E}X_\rho(t)X_\rho(t')| \leq C|tt'|$for $|t|>1$ and $|t'|>1$. Thus,

$$\begin{aligned}\mathbb{E}\left(\int_{\mathbb{R}}\int_{T_N-ab_N\Delta_N}^{\infty}\psi(\frac{u}{a})\psi(\frac{u'}{a})X_\rho(u+ab_N\Delta_N)X_\rho(u'+ab_N\Delta_N)dudu'\right) &\leq\\ \leq C'\frac{1}{(T_N-ab_N\Delta_N)^m}\int_{\mathbb{R}}\left|\psi(\frac{u}{a})\right||u+ab_N\Delta_N|du&\\ \leq C''\frac{1}{(T_N-ab_N\Delta_N)^m}&,\end{aligned}$$

when $T_N-ab_N\Delta_N \to \infty$, from the first condition of Assumption (A1), and with $C'>0$ and $C''>0$. The second part of the integral (between $-\infty$ and $-ab_N\Delta_N$) is obtained from the same trick. Thus, (27) is proved. We just have now to compared the integral and the sum. We use the relation (27) and:

$$\mathbb{E}\left(\frac{1}{\sqrt{a}}\int_0^{T_N}\psi(\frac{t}{a}-k\Delta_N)X_\rho(t)dt\right)^2=\mathbb{E}\left(\frac{\Delta_N}{\sqrt{a}}\sum_{p=0}^{N}\psi(\frac{p\Delta_N}{a}-k\Delta_N)X_\rho(p\Delta_N)\right)^2+\mathcal{O}(\Delta_N),$$

from a classical comparison of a Riemann sum and integral and because the function $\mathbb{E}X_\rho(t)X_\rho(t')$ is $\mathcal{C}^1$ on $\mathbb{R}^2$ except on the diagonal line $t=t'$. It finishes the proof of the first part of Property 3.1.

The second part of Property 3.1 is a consequence of the first part with a good choice of b_N and $m \in \mathbb{N}$. ■

A.2 Proofs of section 4

Proof. **[Proposition 4.1]**

In this proof, we generalize the number of chosen frequencies by considering $a_N=(N\Delta_N)^q$, **with** $q>0$.

For a given N, denote $T^* = (t_0^* = 0, t_1^*, \cdots, t_K^*, t_{K+1}^* = a_N)$ such as :

$$f_{t_j^*} < \frac{\omega_j^*}{\alpha} \leq f_{t_j^*+1}, \quad \text{for all} \quad j = 1, \dots, K$$

and for $T = (0, t_1, \cdots, t_K, a_N) \in \mathcal{A}_K^{(N)}$, we denote $Z_i^{(N)} = \sqrt{N\Delta_N}(Y_i - \log \mathcal{I}_1(1/f_i))$, $Y_{]t_j,t_{j+1}]} = (Y_{t_j+1}, \cdots, Y_{t_{j+1}-\tau_N})'$, $X_{]t_j,t_{j+1}]} = (\log f_{t_j+i}, 1)_{1\leq i\leq(t_{j+1}-t_j)}$, $Z^{(N)}_{]t_j,t_{j+1}]} = (Z^{(N)}_{t_j+1}, \cdots, Z^{(N)}_{t_{j+1}-\tau_N})'$.

First step : We would like to prove : $\widehat{\omega}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \omega_j^*$ for all $j = 1, \dots, K$.

Denote $Q_*^{(N)} = Q^{(N)}(t^*, \widehat{\Lambda}(t^*))$ where $\widehat{\Lambda}(t^*)$ is obtained from a linear regression of (Y_i) on $(\log f_i)$ for $i = t_j^*+1, \cdots, t_{j+1}^*-\tau_N$. Let $\varepsilon > 0$ and $\| T - T' \|_\infty = \max_{j\in\{1,\cdots,K\}} |t_j - t_j'|$ for $T = (0, t_1, \cdots, t_K, a_N) \in \mathcal{A}_K^{(N)}$ and $T' = (0, t_1', \cdots, t_K', a_N) \in \mathcal{A}_K^{(N)}$. Then, we get,

$$\mathbb{P}\left(\| \widehat{T} - t^* \|_\infty \geq \varepsilon a_N\right) \leq \mathbb{P}\left(\min_{T\in V_{\varepsilon a_N}} Q^{(N)}(T, \widehat{\Lambda}(T)) \leq Q_*^{(N)}\right),$$

where $V_{\varepsilon a_N} = \left\{T \in \mathcal{A}_K^{(N)}, \| T - t^* \|_\infty \geq \varepsilon a_N\right\}$. We want to show that for all $T \in V_{\varepsilon a_N}$, $Q_*^{(N)} = o(Q^{(N)}(T, \widehat{\Lambda}(T)))$. In fact,

$$\begin{aligned} Q_*^{(N)} &= \frac{1}{N\Delta_N}\sum_{j=0}^{K+1}(Z^{(N)}_{]t_j^*,t_{j+1}^*]})'\left[Id - X_{]t_j^*,t_{j+1}^*]}\left(X'_{]t_j^*,t_{j+1}^*]}X_{]t_j^*,t_{j+1}^*]}\right)^{-1}X'_{]t_j^*,t_{j+1}^*]}\right]Z^{(N)}_{]t_j^*,t_{j+1}^*]} \\ &\leq \frac{1}{N\Delta_N}\sum_{j=0}^{K+1}(Z^{(N)}_{]t_j^*,t_{j+1}^*]})'Z^{(N)}_{]t_j^*,t_{j+1}^*]} \\ &\leq \frac{1}{N\Delta_N}(Z^{(N)}_{[1,a_N]})'Z^{(N)}_{[1,a_N]}. \end{aligned}$$

From Proposition 3.1, we deduce

$$\frac{1}{a_N}(Z^{(N)}_{[1,a_N]})'Z^{(N)}_{[1,a_N]} \xrightarrow[N\to\infty]{\mathcal{D}} I_Z = \int_0^1 Z^2\left(\frac{\beta}{f_{min}}\left(\frac{\alpha f_{min}}{\beta f_{max}}\right)^u\right)du, \tag{28}$$

which is a positive and $\mathbb{L}^\infty$ random variable because Z is a continuous Gaussian process. Afterward, for a sequence $(\psi_k)_k \in \mathbb{R}^{\mathbb{N}}$ and a sequence of random variables $(\xi_k)_{k\in\mathbb{N}}$, we will write $\xi_N = \mathcal{O}_P(\psi_N)$ as $N \to \infty$, if for all $\varepsilon > 0$, there exists $c > 0$, such as ,

$$P\Big(|\xi_N| \leq c\cdot\psi_N\Big) \geq 1 - \varepsilon,$$

for all sufficiently large N. Here, we obtain :

$$Q_*^{(N)} = \mathcal{O}_P\left(\frac{a_N}{N\Delta_N}\right). \tag{29}$$

Now, let $T \in V_{\varepsilon a_N}$, we want a lower bound of $Q^{(N)}(T, \widehat{\Lambda}(T))$. We use the following decomposition

$$\begin{aligned} Q^{(N)}(T, \widehat{\Lambda}(T)) &= \sum_{j=0}^{K+1}\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left[Y_i - \log \mathcal{I}_1(1/f_i)\right]^2 + \left[X_i\widehat{\lambda}_k - \log \mathcal{I}_1(1/f_i)\right]^2 + \\ &\qquad 2\left[Y_i - \log \mathcal{I}_1(1/f_i)\right]\times\left[X_i\widehat{\lambda}_k - \log \mathcal{I}_1(1/f_i)\right] \\ &= Q_1 + Q_2 + Q_3. \end{aligned}$$

Then :

1. Since $Q_1 = \frac{1}{N\Delta_N} \sum_{j=0}^{K+1} (Z^{(N)}_{]t_j,t_{j+1}]})' Z^{(N)}_{]t_j,t_{j+1}]}$, as previously we get

$$Q_1 = \mathcal{O}_P\left(\frac{a_N}{N\Delta_N}\right). \tag{30}$$

2. Let $\underline{\tau} = \left(\log\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)\right)^{-1} \min_{j=1,\cdots,K} \left\{\log\left(\frac{\alpha\omega^*_{j+1}}{\beta\omega^*_j}\right)\right\}$. Then, for all $j \in \{0,1,\cdots,K\}$, $t^*_{j+1} - \tau_N \geq t^*_j + \underline{\tau} a_N$. Since $T \in V_{\varepsilon a_N}$, we have $\eta = \min\{\varepsilon, \underline{\tau}, \log(\beta/\alpha)\} > 0$ and there exists an integer $j \in \{0,\cdots,K+1\}$ for which there are no estimated abrupt change in the interval $[t^*_j - \eta a_N, t^*_j]$ or $[t^*_j - \tau_N, t^*_j - \tau_N + \eta a_N]$. Thus there exists $k \in \{0,\cdots,K+1\}$ verifying $[t^*_j - \eta a_N, t^*_j] \subset [t_k, t_{k+1} - \tau_N]$ (we follow here a similar proof than Bai and Perron in Lemma 2, p 69) and

$$\begin{aligned} Q_2 &\geq \sum_{i=t^*_j-\eta a_N+1}^{t^*_j} |X_i\widehat{\lambda}_k - \log \mathcal{I}_1(1/f_i)|^2 \\ &\geq \sum_{i=t^*_j-\eta a_N+1}^{t^*_j} \left| A(\widehat{H}^{(N)}_k, \widehat{\sigma}^{(N)}_k) + \frac{i}{a_N}\cdot B(\widehat{H}^{(N)}_k, \widehat{\sigma}^{(N)}_k) - g\left(\frac{i}{a_N}\right)\right|^2, \end{aligned} \tag{31}$$

with :

- $A(H,\sigma) = \log\left(\sigma^2 \cdot K_H(\psi)\right) - (2H+1)\cdot\log\left(\frac{f_{min}}{\beta}\right)$ for all $(H,\sigma) \in \mathcal{K}$;
- $B(H,\sigma) = -(2H+1)\cdot\log\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)$ for all $(H,\sigma)\in\mathcal{K}$;
- $g\left(\frac{i}{a_N}\right) = \log\left(\mathcal{I}_1(1/f_i)\right) = \log\left(\mathcal{I}_1\left(\frac{\beta}{f_{min}}\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)^{-i/a_N}\right)\right)$.

Since for all $(H,\sigma)\in\mathcal{K}$, the function $x \mapsto L_{(H,\sigma)}(x) = \Big(A(H,\sigma) + x\cdot B(H,\sigma) - g(x)\Big)^2$ is an infinitely derivable function on $I\!R$, we know from the theory of Riemann sums that :

$$\begin{aligned} u_N(H,\sigma) = \frac{1}{a_N}\sum_{i=t^*_j-\eta a_N+1}^{t^*_j} \left| A(H,\sigma) + \frac{i}{a_N}\cdot B(H,\sigma) - g\left(\frac{i}{a_N}\right)\right|^2 \\ \underset{N\to\infty}{\longrightarrow} u(H,\sigma) = \int_{s^*_j-\eta}^{s^*_j} \left(A(H,\sigma) + x\cdot B(H,\sigma) - g\left(x\right)\right)^2 \, dx, \end{aligned}$$

with $s^*_j = \log\left(\frac{\omega^*_j}{f_{min}}\right)\left(\log\left(\frac{\alpha f_{max}}{\beta f_{min}}\right)\right)^{-1} = \lim_{N\to\infty} \frac{t^*_j}{a_N}$. Moreover, the sequence $(u_N(H,\sigma))_N$ converges uniformly to $u(H,\sigma)$ because for N large enough

$$\begin{aligned} \sup_{(H,\sigma)\in\mathcal{K}} |u_N(H,\sigma) - u(H,\sigma)| &\leq \left(\frac{1}{a_N^2} + \eta\left|s^*_j - \frac{t^*_j}{a_N}\right|\right)\cdot \sup_{(H,\sigma)\in\mathcal{K}} \left\{ \sup_{0\leq x\leq (s^*_K+1)} \left|\frac{\partial L_{(H,\sigma)}}{\partial x}(x)\right|\right\} \\ &\underset{N\to\infty}{\longrightarrow} 0, \end{aligned}$$

since $\mathcal{K}$ is a compact set of $[0,1]\times]0,\infty[$ and thus $\sup_{(H,\sigma)\in\mathcal{K}}\left\{\sup_{0\leq x\leq(s^*_K+1)}\left|\frac{\partial L_{(H,\sigma)}}{\partial x}(x)\right|\right\} < \infty$. As a consequence, from (31) and since we assumed that $(\widehat{H}^{(N)}_i, \widehat{\sigma}^{(N)}_i) \in \mathcal{K}$ for all $i = 0,\cdots,K$, for some sufficiently small, fixed $\xi > 0$ and for all sufficiently large N,

$$Q_2 \geq a_N\left(\int_{s^*_j-\eta}^{s^*_j} \Big(A(\widehat{H}^{(N)}_k, \widehat{\sigma}^{(N)}_k) + x\cdot B(\widehat{H}^{(N)}_k, \widehat{\sigma}^{(N)}_k) - g\left(x\right)\Big)^2 \, dx - \xi\right). \tag{32}$$

But it is impossible that there exists $(a,b) \in \mathbb{R}^2$ such as $g(x) = a + b \cdot x$ for all $x \in [s_j^* - \eta, s_j^*]$, *i.e.*, $\mathcal{I}_1\Big(c_1 \cdot e^{c_2 \cdot x}\Big) = e^a \cdot e^{b \cdot x}$ for all $x \in [s_j^* - \eta, s_j^*]$ with $c_1 = \dfrac{\beta}{f_{min}}$, $c_2 = \log\left(\dfrac{\alpha f_{min}}{\beta f_{max}}\right)$, which can also be written as :

$$\mathcal{I}_1(x) = a_1 \cdot x^{b_1} \quad \text{for all} \quad x \in [\alpha/\omega_j^*, \alpha/\omega_j^* + \eta'], \tag{33}$$

with $\eta' > 0$ and $(a_1, b_1) \in \mathbb{R}^2$. Indeed, assume now (33) is true. But, for all $x \in [\alpha/\omega_j^*, \alpha/\omega_j^* + \eta']$,

$$\mathcal{I}_1(x) = 2\left(\sigma_{j-1}^{*2} \cdot x^{2H_{j-1}^*+1} \int_\alpha^{x \cdot \omega_j^*} \frac{|\widehat{\psi}(u)|^2}{u^{2H_{j-1}^*+1}}\,du + \sigma_j^{*2} \cdot x^{2H_j^*+1} \int_{x\cdot\omega_j^*}^{\beta} \frac{|\widehat{\psi}(u)|^2}{u^{2H_j^*+1}}\,du\right).$$

Then $\dfrac{\partial^n \mathcal{I}_1}{\partial x^n}(\alpha/\omega_j^*) = a_1 \cdot \dfrac{\partial^n x_1^b}{\partial x^n}(\alpha/\omega_j^*)$ for $n = 0, 1$, what implies that $b_1 = (2H_j^* + 1)$ and $a_1 = 2\sigma_j^{*2} K_{H_j^*}(\psi)$ (here, we use the equality $\widehat{\psi}(\alpha) = 0$). Thus, for all $x \in [\alpha/\omega_j^*, \alpha/\omega_j^* + \eta']$,

$$\begin{aligned} \sigma_{j-1}^{*2} \cdot x^{2H_{j-1}^*+1} \int_\alpha^{x\cdot\omega_j^*} \frac{|\widehat{\psi}(u)|^2}{u^{2H_{j-1}^*+1}}\,du &= \sigma_j^{*2} \cdot x^{2H_j^*+1} \int_\alpha^{x\cdot\omega_j^*} \frac{|\widehat{\psi}(u)|^2}{u^{2H_j^*+1}}\,du, \\ \Longrightarrow \int_{\alpha/x}^{\omega_j^*} |\widehat{\psi}(x\cdot y)|^2 &\left(\frac{\sigma_{j-1}^{*2}}{y^{2H_{j-1}^*+1}} - \frac{\sigma_j^{*2}}{y^{2H_j^*+1}}\right) dy = 0, \end{aligned}$$

and hence $\begin{cases} \sigma_{j-1}^{*2} = \sigma_j^{*2} \\ H_{j-1}^* = H_j^* \end{cases}$. But this condition is impossible from Assumption (B_K) and consequently there is no $(a,b) \in \mathbb{R}^2$ such as $g(x) = a + b \cdot x$ for all $x \in [s_j^* - \eta, s_j^*]$.

The function g belongs to the Hilbert space $\mathbb{L}^2([s_j^* - \eta, s_j^*]; dx)$. Since $\mathcal{L} = \{A + B \cdot x,\ x \in [s_j^* - \eta, s_j^*],\ (A,B) \in \mathbb{R}^2\}$ is a closed linear subspace of $\mathbb{L}^2([s_j^* - \eta, s_j^*]; dx)$, there exits a distance between g and $\mathcal{L}$ in $\mathbb{L}^2([s_j^* - \eta, s_j^*]; dx)$, *i.e.* there exists $(\tilde{A}, \tilde{B}) \in \mathbb{R}^2$ such as

$$\int_{s_j^*-\eta}^{s_j^*} \Big(\tilde{A} + \tilde{B} \cdot x - g(x)\Big)^2 dx = \inf_{(A,B)\in\mathbb{R}^2} \int_{s_j^*-\eta}^{s_j^*} \left(A + B\cdot x - g(x)\right)^2 dx = C > 0,$$

because $g \notin \mathcal{L}$. Then, by choosing ξ such as $0 < \xi < C/2$, the inequality (32) implies :

$$Q_2 \geq \frac{C}{2} \cdot a_N \tag{34}$$

for all sufficiently large N, with C a real positive number only depending on η, s_j^*, H_{j-1}^*, H_j^*, σ_{j-1}^*, σ_j^* and ψ.

3. The previous evaluations of Q_1 and Q_2 provide an upper bound of Q_3.We get

$$\begin{aligned} Q_3 &\leq 2\,(Q_1)^{1/2} \left(\sum_{k=0}^{K+1} \sum_{i=t_k+1}^{t_{k+1}-\tau_N} (X_i \widehat{\lambda}_k - \log \mathcal{I}_1(1/f_i))^2\right)^{1/2} \\ &\leq 2\,(Q_1)^{1/2} \times \left(a_N \cdot \sup_{f_{min} \leq f \leq f_{max}} \left\{2 \sup_{\lambda \in \mathcal{K}} \{(\log f, 1)\cdot\lambda)^2 + 2\log^2 \mathcal{I}_1(1/f)\right\}\right)^{1/2}, \\ &= \mathcal{O}_P\left(\frac{a_N}{\sqrt{N\Delta_N}}\right). \end{aligned} \tag{35}$$

We deduce from (30), (34) and (35) that $Q_1 = o(Q_2)$ and $Q_3 = o(Q_2)$, which implies $\mathbb{P}\left(\min_{T \in V_{\varepsilon a_N}} Q^{(N)}(T, \widehat{\Lambda}(T)) \geq \dfrac{C}{4} \cdot a_N\right) \underset{N\to\infty}{\longrightarrow} 1$ and thus

$$\lim_{N\to\infty} \mathbb{P}\left(\|\, \widehat{T} - T^* \,\|_\infty \geq \varepsilon a_N\right) = 0 \implies \widehat{\omega}_i^{(N)} \underset{N\to\infty}{\overset{\mathcal{P}}{\longrightarrow}} \omega_i^*.$$

Second step : For $j = 1, \cdots, K$, we want to prove that if $3/4 \leq p \leq 1$ and $0 \leq q \leq 1$, for all $\varepsilon > 0$, there exists $0 < C < \infty$ such as for sufficiently large N, $\mathbb{P}\left(a_N^{1-p}\left|\widehat{\omega}_j^{(N)} - \omega_j^*\right| \geq C\right) \leq \varepsilon$.

Mutatis mutandis, we follow the same method as in the proof of the convergence in probability. Now, let $0 < p < 1$, $0 < \eta = \dfrac{1}{2}\min\{\underline{\tau}, \log(\beta/\alpha)\}$ and consider $\min_{T \in W^\eta_{Ca_N^p}} Q^{(N)}(T, \widehat{\Lambda}(T))$ with

$$W^\eta_{Ca_N^p} = \left\{T \in \mathcal{A}_K^{(N)}, Ca_N^p \leq \| T - t^* \|_\infty \leq \eta a_N\right\}.$$

Then, as previously, for $T \in W^\eta_{Ca_N^p}$ and N large enough, it exists $j \in \{1, \cdots, K\}$ such as

$$t_j + Ca_N^p \leq t_j^* < t_{j+1} - \tau_N \tag{36}$$

(the following proof is even if one considers the alternative $t_j^* \leq t_j - Ca_N^p$). Then

$$\begin{aligned} Q^{(N)}(T, \widehat{\Lambda}(T)) \quad &\geq \quad \sum_{i=t_j^*+1}^{t_{j+1}-\tau_N} (Y_i - \log \mathcal{I}_1(1/f_i))^2 + (X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i))^2 + \\ &\qquad\qquad +2(Y_i - \log \mathcal{I}_1(1/f_i))(X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i)) \\ &\geq \quad Q_1' + Q_2' + Q_3'. \end{aligned}$$

1. First, we have again,

$$Q_1' = \mathcal{O}_P\left(\frac{a_N}{N\Delta_N}\right). \tag{37}$$

2. Secondly, $Q_2' = \displaystyle\sum_{i=t_j^*+1}^{t_{j+1}-\tau_N} (X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i))^2$. But we know $\log \mathcal{I}_1(1/f_i) = X_i\lambda_j^*$ for $i \in \{t_j^* + 1, \cdots, t_{j+1} - \tau_N\}$. Moreover, for $a_i = 1/f_i$, $i \in \{t_j + 1, \cdots, t_j^*\}$ and N large enough, $a_i \simeq \alpha/\omega_i^*$, and

$$\mathcal{I}_1(a_i) = \mathcal{I}_1\left(\frac{\alpha}{\omega_j^*}\right) + \left(a_i - \frac{\alpha}{\omega_j^*}\right)\mathcal{I}_1'\left(\frac{\alpha}{\omega_j^*}\right) + \mathcal{O}\left(a_i - \frac{\alpha}{\omega_j^*}\right)^2.$$

But $\mathcal{I}_1(a_i) = 2\left(\sigma_{j-1}^{*2} a_i^{2H_{j-1}^*+1}\displaystyle\int_\alpha^{a_i\omega_j^*} \frac{|\widehat{\psi}(u)|^2}{u^{2H_{j-1}^*+1}}\,du + \sigma_j^{*2} a_i^{2H_j^*+1}\int_{a_i\omega_j^*}^\beta \frac{|\widehat{\psi}(u)|^2}{u^{2H_j^*+1}}\,du\right)$ and $\mathcal{I}_1'\left(\dfrac{\alpha}{\omega_j^*}\right) = 2\sigma_j^{*2} K_{H_j^*}(\psi)(2H_j^*+1)\left(\dfrac{\alpha}{\omega_j^*}\right)^{2H_j^*}$; thus for $i \in \{t_j + 1, \cdots, t_j^*\}$,

$$\log \mathcal{I}_1(1/f_i) = X_i\lambda_j^* + \left[(2H_j^*+1)\frac{f_{min}}{\beta}\log\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)\right] \cdot \left(\frac{t_j^* - i}{a_N}\right) + \mathcal{O}\left(\frac{t_j^* - i}{a_N}\right)^2. \tag{38}$$

Then, with $\widehat{\lambda}_j = (\widehat{a}_j, \widehat{b}_j)'$, one gets for $i \in \{t_j^* + 1, \cdots, t_{j+1} - \tau_N\}$,

$$\left(X_i\widehat{\lambda}_j - \log \mathcal{I}_1(1/f_i)\right) = (\log f_i - \overline{logf})(\widehat{a}_j - a_j^*) + \overline{Z}, \tag{39}$$

$\overline{XXX}$ indicates the empirical mean of XXX between $t_j + 1$ and $t_{j+1} - \tau_N$. Thus,

$$Q_2' \quad \geq \quad \sum_{i=t_j^*+1}^{t_{j+1}-\tau_N}\left(\left(\log f_i - \overline{\log f}\right)(\widehat{a}_j - a_j^*) + \frac{1}{\sqrt{N\Delta_N}}\overline{Z}\right)^2. \tag{40}$$

We also have :

$$\begin{aligned}\widehat{a}_j &= \frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(Y_i-\overline{Y}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}\\ &= \frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(\log\mathcal{I}_1(1/f_i)+\frac{1}{\sqrt{N\Delta_N}}Z_i^{(N)}-\overline{\log\mathcal{I}_1}-\frac{1}{\sqrt{N\Delta_N}}\overline{Z}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2},\end{aligned}$$

and thus,

$$\widehat{a}_j-a_j^*=\frac{\displaystyle\sum_{i=t_j+1}^{t_j^*}\left(\log f_i-\overline{\log f}\right)\left(\log\mathcal{I}_1(1/f_i)-X_i'\lambda_j^*\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2} + \frac{1}{\sqrt{N\Delta_N}}\frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(Z_i^{(N)}-\overline{Z}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}. \tag{41}$$

From the definition of $(\log f_i)$,

$$\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2\simeq\left[\frac{1}{12}\log\left(\frac{\beta f_{max}}{\alpha f_{min}}\right)\right](t_{j+1}-\tau_N-t_j)=\mathcal{O}(a_N). \tag{42}$$

Expansions (42) and (38) imply there exist two constants $C_1>0$ and $C_2>0$ such as for N large enough :

$$C_1\left(\frac{t_j^*-t_j}{a_N}\right)^2\leq\left|\frac{\displaystyle\sum_{i=t_j+1}^{t_j^*}\left(\log f_i-\overline{\log f}\right)\left(\log\mathcal{I}_1(1/f_i)-X_i'\lambda_j^*\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}\right|\leq C_2\left(\frac{t_j^*-t_j}{a_N}\right)^2.$$

Moreover

$$\frac{1}{\sqrt{N\Delta_N}}\frac{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)\left(Z_i^{(N)}-\overline{Z}\right)}{\displaystyle\sum_{i=t_j+1}^{t_{j+1}-\tau_N}\left(\log f_i-\overline{\log f}\right)^2}=\mathcal{O}_P\left(\frac{1}{\sqrt{N\Delta_N}}\right).$$

Thus, we deduce from (41) that :

$$C_1\left(\frac{t_j^*-t_j}{a_N}\right)^2+\mathcal{O}_P\left(\frac{1}{\sqrt{N\Delta_N}}\right)\leq\left|\widehat{a}_j-a_j^*\right|.$$

As a consequence, for (p,q) such as $4q(1-p) \leq 1$ (for instance, $p = 3/4$ and $q = 1$), then $\left(\dfrac{t_j^* - t_j}{a_N}\right)^2 \cdot \sqrt{N\Delta_N} \geq C^2$, and thus for all $\varepsilon > 0$, for N sufficiently large, we can chose $C > 0$ such as :

$$I\!\!P\left(\frac{C_1^2}{2}\left(\log f_i - \overline{\log f}\right)^2\left(\frac{t_j^* - t_j}{a_N}\right)^4 \leq \left(\left(\log f_i - \overline{\log f}\right)(\widehat{a}_j - a_j^*)\right)^2\right) \geq 1 - \varepsilon. \tag{43}$$

Now, from (40), (43) and with $I\!\!P(t_{j+1} - \tau_N - t_j^* \geq \frac{\eta}{2}\, a_N) \underset{N\to\infty}{\longrightarrow} 1$, for $(p,q) \in [3/4,1] \times [0,1]$, for all $\varepsilon > 0$, for N sufficiently large, we can also chose $C > 0$ such as :

$$\begin{aligned} I\!\!P\left(\frac{C_1^2}{4}\left(\frac{t_j^* - t_j}{a_N}\right)^4 \cdot \sum_{i=t_j^*+1}^{t_{j+1}-\tau_N}\left(\log f_i - \overline{\log f}\right)^2 \leq Q_2'\right) &\geq 1 - \varepsilon, \\ \Longrightarrow \quad I\!\!P\left(C^4 \cdot C_2 \cdot a_N^{4p-3} \leq Q_2'\right) &\geq 1 - \varepsilon, \end{aligned} \tag{44}$$

with $C_2 > 0$ a real number not depending on C, N and ε.

3. Finally, from the classical bound of Q_3', we obtain,

$$Q_3' \leq 2 \cdot (Q_2')^{1/2} \cdot (Q_1')^{1/2}.$$

But, following a similar method as previously, from (43 one can find a upper-bound for Q_2', *i.e.* for $(p,q) \in [3/4,1] \times [0,1]$, for all $\varepsilon > 0$, for N sufficiently large, we can also chose $C > 0$ such as :

$$I\!\!P\left(Q_2' \leq C^4 \cdot C_3 \cdot a_N^{4p-3}\right) \quad \geq \quad 1 - \varepsilon,$$

with $C_3 > 0$ a real number not depending on C, N and ε. Thus, for $(p,q) \in [3/4,1] \times [0,1]$, for all $\varepsilon > 0$, we can also chose $C > 0$ such as :

$$I\!\!P\left(Q_3' \leq C^2 \cdot C_4 \cdot \frac{a_N^{2p-2}}{\sqrt{N\Delta_N}}\right) \quad \geq \quad 1 - \varepsilon, \tag{45}$$

with $C_4 > 0$ a real number not depending on C and N.

Now, from (37), (44) and (45), one deduces that for $(p,q) \in [3/4,1] \times [0,1]$, for all $\varepsilon > 0$, for N sufficiently large, we can chose $C > 0$ sufficiently large such as :

$$I\!\!P\left(\min_{T \in W^\eta_{Ca_N^p}} Q^{(N)}(T, \widehat{\Lambda}(T)) \geq C^4 \cdot \frac{C_2}{2} \cdot a_N^{4p-3}\right) \geq 1 - \varepsilon.$$

and thus like $Q_*^{(N)} = \mathcal{O}_P\left(\dfrac{a_N}{N\Delta_N}\right)$ from (29),

$$I\!\!P\left(\min_{T \in W^\eta_{Ca_N^p}} Q^{(N)}(T, \widehat{\Lambda}(T)) \leq Q_*^{(N)}\right) \leq \varepsilon,$$

that leads to $I\!\!P\left(a_N^{1-p}\left|\widehat{\omega}_j^{(N)} - \omega_j^*\right| \geq C\right) \leq \varepsilon$ for sufficiently large C and N. ■

Proof. [**Proposition 4.2**] From Proposition 4.1, we deduce that $\forall j = 0, \cdots, K$,

$$I\!\!P\left([\tilde{U}_j^{(N)}, \tilde{V}_j^{(N)}] \subset [t_j^*, t_{j+1}^* - \tau_N]\right) \underset{N\to\infty}{\longrightarrow} 1.$$

Denote $A_j^{(N)}$ the event $[\tilde U_j^{(N)},\tilde V_j^{(N)}]\subset[t_j^*,t_{j+1}^*-\tau_N]$. Then, $\forall j=0,\cdots,K$ and $\forall(x,y)\in I\!R^2$,

$$\begin{aligned}&I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\right)\\&\quad=I\!P\left(A_j^{(N)}\right)\times I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)+\\&\qquad+I\!P\left(\overline{A_j^{(N)}}\right)\times I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid\overline{A_j^{(N)}}\right).\end{aligned}$$

Now, since $I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid\overline{A_j^{(N)}}\right)\leq 1$ and $I\!P\left(\overline{A_j^{(N)}}\right)=1-I\!P\left(A_j^{(N)}\right)$, we obtain :

$$\begin{aligned}&I\!P\left(A_j^{(N)}\right)\cdot I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)\\&\quad\leq I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\right)\\&\quad\leq I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)+1-I\!P\left(A_j^{(N)}\right).\end{aligned}\tag{46}$$

Since $\widehat\omega_j^{(N)}\xrightarrow[N\to\infty]{\mathcal P}\omega_j^*$ and $\widehat\omega_{j+1}^{(N)}\xrightarrow[N\to\infty]{\mathcal P}\omega_{j+1}^*$, therefore $(\widehat\omega_j^{(N)},\widehat\omega_{j+1}^{(N)})\xrightarrow[N\to\infty]{\mathcal P}(\omega_j^*,\omega_{j+1}^*)$, we have $(f_k)_{k\in\{\tilde U_j^{(N)},\cdots,\tilde V_j^{(N)}\}}\xrightarrow[N\to\infty]{\mathcal P}$ and $\tilde X_j^{(N)}\xrightarrow[N\to\infty]{\mathcal P}X_j^*$. Thus, from Proposition 3.2 and central limit theorem (17), for all $(x_k)_{1\leq k\leq m}\in I\!R^m$, we have

$$I\!P\left(\sqrt{N\Delta_N}\left(\tilde Y_j^{(N)}-\tilde X_j^{(N)}\lambda_j^*\right)\in\prod_{k=1}^m]-\infty,x_k]\ \mid A_j^{(N)}\right)-I\!P\left(\tilde Z_j\in\prod_{k=1}^m]-\infty,x_k]\ \mid A_j^{(N)}\right)\xrightarrow[N\to\infty]{}0,$$

with $\tilde Z_j\overset{\mathcal D}{\sim}\mathcal N_m(0,\Sigma_j^*)$ and $\Sigma_j^*=\left(\mathrm{cov}\left(Z\Big(\frac{1}{g_j^*(k)}\Big),Z\Big(\frac{1}{g_j^*(l)}\Big)\right)\right)_{1\leq k,l\leq m}$ (it explains the expression (24) of Σ_j^*). From the equality $\tilde\lambda_j^{(N)}=\left((\tilde X_j^{(N)})'\tilde X_j^{(N)}\right)^{-1}(\tilde X_j^{(N)})'\tilde Y_j^{(N)}$, we deduce that for all $(x,y)\in I\!R^2$, with $\tilde\xi_j\overset{\mathcal D}{\sim}\mathcal N_2(0,\Gamma_1^{\lambda_j^*})$ and $\Gamma_1^{\lambda_j^*}=\left(X_j^{*\prime}X_j^*\right)^{-1}X_j^*\Sigma_j^*X_j^{*\prime}\left(X_j^{*\prime}X_j^*\right)^{-1}$,

$$I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)-I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)\xrightarrow[N\to\infty]{}0.\tag{47}$$

We also have :

$$\begin{aligned}&I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\right)+I\!P\left(A_j^{(N)}\right)-1\leq\\&\quad\leq I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\mid A_j^{(N)}\right)\leq\frac{I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\right)}{I\!P\left(A_j^{(N)}\right)}.\end{aligned}\tag{48}$$

Now, as $I\!P\left(A_j^{(N)}\right)\xrightarrow[N\to\infty]{}1$, from (46), (47) and (48), we deduce that for all $(x,y)\in I\!R^2$:

$$I\!P\left(\sqrt{N\Delta_N}\left(\tilde\lambda_j^{(N)}-\lambda_j^*\right)\in]-\infty,x]\times]-\infty,y]\right)\xrightarrow[N\to\infty]{}I\!P\left(\tilde\xi_j\in]-\infty,x]\times]-\infty,y]\right),$$

that achieves the proof. ■

Proof. [**Proposition 4.3**] First, from the expression of each s_{kl} given in (24) and with $\mathcal M_m(I\!R)$ the set of real m-by-m matrix, the function $\Sigma:(H,u,v)\mapsto\Sigma(H,u,v)\in\mathcal M_m(I\!R)$ is a continuous (and therefore measurable) function of (H,u,v) for H in a compact set included in $]0,1[$ and $(u,v)\in]f_{min},f_{max}[^2$. For all $j=0,\cdots,K$, we have :

1. from Assumptions (B_K) and (C), $(\tilde{H}_j^{(N)}, \tilde{\sigma}_j^{(N)}) \in \mathcal{K}$ and $(\widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)}) \in]f_{min}, f_{max}[^2$;

2. from (21) and (23), $\tilde{H}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} H_j^*$, $\widehat{\omega}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \omega_j^*$, $\widehat{\omega}_{j+1}^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \omega_{j+1}^*$ and therefore

$$(\tilde{H}_j^{(N)}, \widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)}) \xrightarrow[N\to\infty]{\mathcal{P}} (H_j^*, \omega_j^*, \omega_{j+1}^*).$$

As a consequence, $\widehat{\Sigma}_j^{(N)} = \Sigma(\tilde{H}_j^{(N)}, \widehat{\omega}_j^{(N)}, \widehat{\omega}_{j+1}^{(N)}) \xrightarrow[N\to\infty]{\mathcal{P}} \Sigma_j^*$, for all $j = 0, \cdots, K$, and since $\Sigma(H, u, v)$ is an invertible covariance matrix for all $(H, u, v) \in]0, 1[\times]f_{min}, f_{max}[^2$,

$$\left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \xrightarrow[N\to\infty]{\mathcal{P}} \left(\Sigma_j^*\right)^{-1}, \quad \text{for all} \quad j = 0, \cdots, K. \tag{49}$$

Secondly, denote $\begin{cases} \tilde{M}_j^{(N)} = \left((\tilde{X}_j^{(N)})' \left(\Sigma_j^*\right)^{-1} \tilde{X}_j^{(N)}\right)^{-1} (\tilde{X}_j^{(N)})' \left(\Sigma_j^*\right)^{-1} \\ \widehat{M}_j^{(N)} = \left((\tilde{X}_j^{(N)})' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \tilde{X}_j^{(N)}\right)^{-1} (\tilde{X}_j^{(N)})' \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \end{cases}$.

The 2-by-m matrix $\tilde{M}_j^{(N)}$ verifies :

$$\tilde{\lambda}_j^{(N)} = \tilde{M}_j^{(N)} \tilde{Y}_j^{(N)} = \lambda_j^* + \frac{1}{\sqrt{N\Delta_N}} \tilde{M}_j^{(N)} \tilde{Z}_j^{(N)}$$

with $\tilde{Z}_j^{(N)} = \left(Z^{(N)}(1/f_i)\right)_{i \in \{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\}}$ and $\tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{D}} \tilde{Z}_j = \left(Z(1/g_j^*(k))\right)_{1\le k\le m}$ from the central limit theorem (17). In the same way,

$$\underline{\lambda}_j^{(N)} = \widehat{M}_j^{(N)} \tilde{Y}_j^{(N)} = \lambda_j^* + \frac{1}{\sqrt{N\Delta_N}} \widehat{M}_j^{(N)} \tilde{Z}_j^{(N)}.$$

From (49), we obtain $\widehat{M}_j^{(N)} - \tilde{M}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} 0$, and thus,

$$\sqrt{N\Delta_N}\left(\underline{\lambda}_j^{(N)} - \lambda_j^*\right) - \tilde{M}_j^{(N)} \tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} 0,$$

with $\tilde{M}_j^{(N)} \tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_2(0, \Gamma_2^{\lambda_j^*})$ (the same covariance matrix as that obtained with a generalized least squares estimation), and this implies Proposition 4.3. ■

Proof. [**Proposition 4.4**] For each $j = 0, \cdots, K$, one first show that

$$N\Delta_N \cdot \| \tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)} \underline{\lambda}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} \xrightarrow[N\to\infty]{\mathcal{D}} \chi^2(m-2). \tag{50}$$

Indeed, $\| \tilde{Y}_j^{(N)} - \tilde{X}_j^{(N)} \underline{\lambda}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} = \| \widehat{P}_{j\perp}^{(N)} \tilde{Y}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} = \frac{1}{N\Delta_N} \| \widehat{P}_{j\perp}^{(N)} \tilde{Z}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}}$ where $\widehat{P}_{j\perp}^{(N)} = I_m - \tilde{X}_j^{(N)} \widehat{M}_j^{(N)}$ is the matrix of the orthogonal projector in $I\!R^m$ on the orthogonal of V_j, where $V_j = \{\tilde{X}_j^{(N)} \lambda, \ \lambda \in I\!R^2\}$ is the 2-dimensional subspace of $I\!R^m$ generated by $\tilde{X}_j^{(N)}$ (here the notion of orthogonality is based on the inner product $< u, v >_{\widehat{\Sigma}_j^{(N)}} = u' \cdot \left(\widehat{\Sigma}_j^{(N)}\right)^{-1} \cdot v$ for $u, v \in I\!R^m$). From the previous proofs, we know :

- $\widehat{\Sigma}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \Sigma_j^*$, $\tilde{X}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} X_j^*$ and therefore $\widehat{P}_{j\perp}^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} P_{j\perp}^*$ where $P_{j\perp}^* = \left(I_m - X_j^* \left(X_j^{*'} (\Sigma_j^*)^{-1} X_j^*\right)^{-1} X_j^{*'} (\Sigma_j^*)^{-1}\right)$ is the matrix of an orthogonal projector on a $(m-2)$-dimensional subspace of $I\!R^m$;

- $< u, v >_{\widehat{\Sigma}_j^{(N)}} \xrightarrow[N\to\infty]{\mathcal{P}} < u, v >_{\Sigma_j^*}$ for $u, v \in I\!R^m$;
- $\tilde{Z}_j^{(N)} \xrightarrow[N\to\infty]{\mathcal{P}} \tilde{Z}_j$ with $\tilde{Z}_j \overset{\mathcal{D}}{\sim} \mathcal{N}_m(0, \Sigma_j^*)$.

Consequently, $\| \widehat{P}_{j\perp}^{(N)} \tilde{Z}_j^{(N)} \|^2_{\widehat{\Sigma}_j^{(N)}} \xrightarrow[N\to\infty]{\mathcal{D}} \| P^*_{j\perp} \tilde{Z}_j \|^2_{\Sigma_j^*}$. From Cochran's Theorem, we know $\| P^*_{j\perp} \tilde{Z}_j \|^2_{\Sigma_j^*} \overset{\mathcal{D}}{\sim} \chi^2(m-2)$ and therefore (50) is proved.

Moreover, with the notations of Proposition 3.2, if $\log f \geq \log f' + \log \beta/\alpha$ then $\mathrm{cov}\,(Z(1/f), Z(1/f')) = 0$. But for all $(i,j) \in \{0, \cdots, K\}^2$, $i \neq j$, $\forall k \in \{\tilde{U}_i^{(N)}, \cdots, \tilde{V}_i^{(N)}\}$ and $\forall k' \in \{\tilde{U}_j^{(N)}, \cdots, \tilde{V}_j^{(N)}\}$, $|\log f_k - \log f_{k'}| \geq \log \beta/\alpha$. Thus, we deduce that the different $\underline{\lambda}_j^{(N)}$ are asymptotically Gaussian and independent. It provides Proposition 4.4. ■

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