Uniform limit theorems for the periodogram of weakly dependent time series and their applications to Whittle's estimate

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Abstract

We prove uniform convergence results such as a law of large numbers and a central limit theorem for the integrated periodogram of a weak dependent time series. Those probabilistic results are used for Whittle's parametric estimation. Using a general weakly dependent frame, we derive results for a large variety of models; with causal weak dependence, we consider examples as GARCH(p,q), ARCH(∞) or, more generally, bilinear models. Non-causal weak dependence is also considered yielding for instance the new case of a non-causal linear or ARCH(∞) model.

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Running title: Uniform limit theorems for the periodogram and Whittle estimate.

1 Introduction

Recall first that the periodogram is an estimate of the spectral density for a stationary sequence; unfortunately its variance does not converge to 0 hence this is not a consistent estimate of a spectral density. However, once integrated with respect to some $\mathbb{L}^2$ function, its behaviour becomes quite smoother and can allow an estimation of the spectral density. Moreover, a special case of the integrated periodogram is the Whittle's contrast. This provides an approximation of the likelihood allowing then parametric estimation for stationary time series.

A first part of this paper (Section 2) is aimed to get a uniform large number law (see Theorem 1) and a uniform central limit theorem (see Theorem 2) for the integrated periodogram of a time series. The uniformity is considered on some Sobolev class. We also prove uniform *a.s.* results for the law of large numbers of empirical covariances, as in Doukhan & León (1989). In addition to the usual cases of Gaussian, linear, or strongly mixing processes, our results hold for causal weakly dependence processes (see Dedecker & Doukhan, 2003) under weak conditions. The case of non-causal weak dependent time series, introduced in Doukhan & Louhichi, 1999, requires a different treatment (see Appendix 4). Thus, a minimization of the rate of convergence for general functional central limit theorems (see Theorem 7) is obtained for these time series; we essentially assume a condition linking the moment assumption with the decay rate of weak dependence. A Lindeberg blocking method is used to prove this very general result.

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We prove asymptotic normality of Whittle's estimate of parameters for weak dependent time series (see Section 3) by using these uniform central limit theorems. The usual conditions on the regularity of the spectral density are required, except with regard to the regularity of the spectral density in frequency which is weakened thanks to the uniformity in our central limit theorem. Those general results for parametric estimation are new; they extend Hannan, 1973, and Rosenblatt, 1985.

We give several examples of time series verifying the asymptotic normality of Whittle's estimate. A first class of examples includes causal time series like GARCH(p,q), ARCH(∞) or, more generally, bilinear processes $(X_k)_{k\in\mathbb{Z}}$ defined by

$$X_k = \xi_k\Big(a_0 + \sum_{j=1}^{\infty} a_j X_{k-j}\Big) + c_0 + \sum_{j=1}^{\infty} c_j X_{k-j},$$

with where $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random variables with zero mean and such that $\mathbb{E}(|\xi_0|^m) < \infty$ with $m > 4$ and a_j, c_j, $j \in \mathbb{N}$ are real coefficients verifying certain conditions. Nevertheless, if the study of the case of bilinear models is a new one, we have to remark that the cases of GARCH(p,q) and causal ARCH(∞) were already treated by Giraitis and Robinson, 2001, whom have obtained better conditions. The second class of examples includes non causal time series as two-sided linear and ARCH(∞) (introduced in Doukhan *et al.*, 2005) processes, respectively defined by

$$\begin{aligned} X_k &= \sum_{j\neq 0} a_j \xi_{k-j}, \qquad \text{and} \\ X_k &= \xi_k\Big(a_0 + \sum_{j\neq 0} a_j X_{k-j}\Big), \end{aligned}$$

with $(\xi_k)_{k\in\mathbb{Z}}$ i.i.d. random variables with zero mean, and a_j, $j \in \mathbb{N}$ real coefficients verifying certain conditions. The present use of this general weak dependent frame is thus not always optimal but it yields a unified treatment for the asymptotic properties of the Whittle estimate as well as the uniform limit results.

2 Uniform limit theorems for the periodogram

2.1 Notations and assumptions

Let $X = (X_k)_{k\in\mathbb{Z}}$ be a centered stationary time series with real values and such that $\mathbb{E}\left(X_0^4\right) < +\infty$. Denote $(R_s)_s$ the covariogram of X, such that :

$$R_s = \operatorname{Cov}(X_0, X_s) = \mathbb{E}(X_0 X_s) \quad \text{for} \quad s \in \mathbb{Z},$$

and the fourth cumulants of X, $(\kappa_4(X_0, X_i, X_j, X_k))_{i,j,k}$ such that $(\forall (i,j,k) \in \mathbb{Z}^3)$:

$$\begin{aligned} \kappa_4(X_0, X_i, X_j, X_k) &= \mathbb{E}X_0X_iX_jX_k \\ &- \mathbb{E}X_0X_i\mathbb{E}X_jX_k - \mathbb{E}X_0X_j\mathbb{E}X_iX_k - \mathbb{E}X_0X_k\mathbb{E}X_iX_j. \end{aligned}$$

Moreover, we will use the following assumption on X :

Assumption M : X is such that :

$$\gamma = \sum_{\ell\in\mathbb{Z}} R_\ell^2 < \infty \quad \text{and} \quad \kappa_4 = \sum_{i,j,k} |\kappa_4(X_0, X_i, X_j, X_k)| < \infty. \tag{1}$$

Let $g\ :[-\pi,\pi[\to\mathbb{R}$ a 2π-periodic function such that $g\in\mathbb{L}^2([-\pi,\pi[)$ (*i.e.* $\int_{-\pi}^{\pi}|g(\lambda)|^2\,d\lambda<\infty$). For X verifying assumption M, we define :

$$\begin{aligned}
I_n(\lambda) &= \frac{1}{2\pi\cdot n}\left|\sum_{k=1}^{n}X_k e^{-ik\lambda}\right|^2\\
I_n(g) &= \int_{-\pi}^{\pi} g(\lambda)I_n(\lambda)\,d\lambda,\\
\text{and}\ \ I(g) &= \int_{-\pi}^{\pi} g(\lambda)f(\lambda)\,d\lambda,
\end{aligned}$$

with f the spectral density of X (that exists and is in $\mathbb{L}^2([-\pi,\pi[)$ from Assumption M) defined by :

$$f(\lambda)=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}R_k\,e^{ik\lambda}\quad\text{for }\lambda\in[-\pi,\pi[.$$

Recall that

$$I_n(\lambda)=\frac{1}{2\pi}\sum_{|s|<n}\widehat{R}_n(s)e^{-is\lambda}\quad\text{with}\quad\widehat{R}_n(s)=\frac{1}{n}\sum_{j=1\vee(1-s)}^{(n-s)\wedge n}X_jX_{j+s}$$

and the summation contains $n-|s|$ terms, hence this estimate of R_s is biased. We intend to work in a Sobolev space $\mathcal{H}$ of locally $\mathbb{L}^2$ and $2\pi-$periodic functions defined from a non-negative sequence $(s_\ell)_{\ell\in\mathbb{Z}}$ such that :

$$\forall\ell\in\mathbb{Z}: s_\ell\le\frac{1}{|\ell|},\qquad s_\ell\ge s_{\pm(|\ell|+1)},\quad\text{and}\quad\sum_{\ell\in\mathbb{Z}}s_\ell<\infty.$$

Then, with g a 2π-periodic function such that $g\in\mathbb{L}^2([-\pi,\pi[)$ and the representation $g(\lambda)=\sum_{\ell\in\mathbb{Z}}g_\ell\,e^{i\ell\lambda}$, denote

$$\mathcal{H}=\{g\in\mathbb{L}^2([-\pi,\pi[)\,/\,\|g\|_{\mathcal{H}}<\infty\}\quad\text{with}\quad\|g\|^2_{\mathcal{H}}=\sum_{\ell\in\mathbb{Z}}s_\ell^{-1}|g_\ell|^2.$$

This space $\mathcal{H}$ is included in the space $C^\star$ of continuous and $2\pi-$periodic functions. In this case, we have $\|g\|_\infty=\sup_{[-\pi,\pi[}|g|\le\sqrt{c}\cdot\|g\|_{\mathcal{H}}$ with

$$c=\sum_{\ell\in\mathbb{Z}}s_\ell. \tag{2}$$

As usual $\mathcal{H}'$ denotes the dual of $\mathcal{H}$; it is defined from the identity $\|T\|_{\mathcal{H}'}=\sup_{\|g\|_{\mathcal{H}}\le1}|T(g)|$, hence if $T\in\mathcal{H}'$:

$$\|T\|^2_{\mathcal{H}'}=\sup_{\|g\|_{\mathcal{H}}\le1}|T(g)|^2=\sum_{\ell\in\mathbb{Z}}s_\ell|T(e_\ell)|^2,$$

with $e_\ell(\lambda)=e^{i\ell\lambda}$ for $\ell\in\mathbb{Z}$. We study the behavior of I_n-I in the function space $\mathcal{H}$ or equivalently in the Hilbert space $\mathcal{H}'$.

2.2 Uniform Law of Large Numbers

We did not find a reference for this result which seems quite standard. This is why we develop a Law of Large Numbers (LLN) for the integrated periodogram $(I_n(g))_g$. An important feature is that the results are only stated here in terms of cumulant sums; thus we need no additional mixing assumption. Afterward, we shall use the random variables $(Y_{j,k})_{j,k\in\mathbb{Z}}$ such that :

$$Y_{j,k}=X_jX_{j+k}-R_k,\quad\text{for all}\quad(j,k)\in\mathbb{Z}^2,$$

and thus $\mathbb{E}\,Y_{j,k}=0$. Now we use the lemma

Lemma 1 *If X verifies Assumption M, then :*

$$n \cdot \max_{\ell \geq 0} \left(Var\left(\widehat{R}_n(\ell)\right)\right) \leq \kappa_4 + 2\gamma.$$

Proof of lemma 1. To prove this result, we use the identity

$$\mathrm{Cov}\,(Y_{0,\ell}, Y_{j,\ell}) = \kappa_4(X_0, X_\ell, X_j, X_{j+\ell}) + R_j^2 + R_{j+\ell}R_{j-\ell}$$

and deduce from the stationarity of $(Y_{j,\ell})_{j\in\mathbb{Z}}$ when ℓ is a fixed integer :

$$\begin{aligned} n \cdot \mathrm{Var}\left(\widehat{R}_n(\ell)\right) &\leq \frac{1}{n} \sum_{j=1\vee(1-\ell)}^{(n-\ell)\wedge n} \sum_{j'=1\vee(1-\ell)}^{(n-\ell)\wedge n} |\mathrm{Cov}\,(Y_{j,\ell}, Y_{j',\ell})| \\ &\leq \sum_{j\in\mathbb{Z}} |\mathrm{Cov}\,(Y_{0,\ell}, Y_{j,\ell})| \\ &\leq \sum_j \left(|\kappa_4(X_0, X_\ell, X_j, X_{j+\ell})| + 2R_j^2\right) \\ &\leq \kappa_4 + 2\gamma, \end{aligned}$$

with Cauchy-Schwarz inequality for ℓ^2-sequences. ∎

Then, we can show the following lemma :

Lemma 2 *If X verifies Assumption M, then :*

$$\mathbb{E}\|I_n - I\|^2_{\mathcal{H}'} \leq \frac{3}{n}\Big(\gamma + c(\kappa_4 + 2\gamma)\Big).$$

Proof. Let $g(\lambda) = \sum_{\ell\in\mathbb{Z}} g_\ell\, e^{i\ell\lambda} \in \mathcal{H}$. As in Doukhan & León (1989), we use the decomposition :

$I_n(g) - I(g) = -T_1(g) - T_2(g) + T_3(g)$ with

$$\begin{aligned} T_1(g) &= \sum_{|\ell|\geq n} R_\ell\, g_\ell, \\ T_2(g) &= \frac{1}{n}\sum_{|\ell|<n} |\ell|\, R_\ell\, g_\ell, \\ \text{and}\quad T_3(g) &= \sum_{|\ell|<n} (\widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell))\, g_\ell. \end{aligned} \tag{3}$$

We also remark that $T_3(g) = I_n(g) - \mathbb{E}I_n(g)$. Thus, we obtain the inequality :

$$\mathbb{E}\|I_n - I\|^2_{\mathcal{H}'} \leq 3(\|T_1\|^2_{\mathcal{H}'} + \|T_2\|^2_{\mathcal{H}'} + \mathbb{E}\|T_3\|^2_{\mathcal{H}'}).$$

Cauchy-Schwarz inequality yields, with $|\ell|_+ = 1 \vee |\ell|$,

$$\begin{aligned} \|T_1\|^2_{\mathcal{H}'} &\leq \sum_{|\ell|\geq n} s_\ell\, R_\ell^2 \leq s_n \sum_{|\ell|\geq n} R_\ell^2 \leq \frac{1}{n}\sum_{|\ell|\geq n} R_\ell^2, \\ \|T_2\|^2_{\mathcal{H}'} &\leq \frac{1}{n^2}\sum_{|\ell|<n} |\ell|^2\, s_\ell\, R_\ell^2 \leq \frac{1}{n}\sum_{|\ell|<n} |\ell|\, s_\ell\, R_\ell^2 \leq \frac{1}{n}\sum_{|\ell|<n} R_\ell^2. \end{aligned}$$

Hence : $\|T_1\|^2_{\mathcal{H}'} + \|T_2\|^2_{\mathcal{H}'} \leq \dfrac{\gamma}{n}$. Lemma 1 entails

$$\begin{aligned} \|T_3\|^2_{\mathcal{H}'} &\leq \sum_{|\ell|<n} s_\ell\, (\widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell))^2, \\ \mathbb{E}\|T_3\|^2_{\mathcal{H}'} &\leq \sum_{|\ell|<n} s_\ell\, \mathrm{Var}\,(\widehat{R}_n(\ell)) \leq \frac{1}{n}\sum_{|\ell|<n} s_\ell\,(\kappa_4 + 2\gamma) \leq \frac{c(\kappa_4 + 2\gamma)}{n}, \end{aligned}$$

with c defined in (2). We combine those results to deduce lemma 2. ∎

Theorem 1 (Uniform SLLN) *If X verifies Assumption M, then :*

$$\|I_n - I\|_{\mathcal{H}'} \xrightarrow[n\to\infty]{a.s.} 0.$$

Proof of Theorem 1. We prove this strong law of large numbers from a weak $\mathbb{L}^2$-LLN and lemma 2. The scheme of proof is analogue to the one in the standard strong LLN. Set $t > 0$. First, we know that for all random variables X and Y, we have $\mathbb{P}(X + Y \geq 2t) \leq \mathbb{P}(X \geq t) + \mathbb{P}(Y \geq t)$. Thus :

$$\begin{aligned}
\mathbb{P}\left(\max_{n\geq N}\|I_n - I\|_{\mathcal{H}'} \geq 2t\right) &\leq \sum_{k=[\sqrt{N}]}^{\infty} \mathbb{P}(\|I_{k^2} - I\|_{\mathcal{H}'} \geq t) \\
&+ \sum_{k=[\sqrt{N}]}^{\infty} \mathbb{P}\left(\max_{k^2\leq n<(k+1)^2}\|I_n - I_{k^2}\|_{\mathcal{H}'} \geq t\right) \\
&\leq A_N + B_N. \qquad (4)
\end{aligned}$$

From Bienaymé-Tchebychev inequality, Lemma 2 implies that :

$$A_N \leq \frac{C_1}{t^2} \cdot \sum_{k\geq\sqrt{N}} \frac{1}{k^2}, \qquad (5)$$

with $C_1 \in \mathbb{R}_+$. Now set $\widetilde{R}_n(\ell) = \widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell)$. The fluctuation term B_N is more involved and its bound is based on the same type of decomposition than (3), because for $k^2 < n$:

$$I_n(g) - I_{k^2}(g) = -T_1'(g) + T_2'(g) - T_3'(g),$$

$$\begin{aligned}
\text{with now } T_1'(g) &= \sum_{k^2\leq|\ell|<n} R_\ell\, g_l, \\
T_2'(g) &= \frac{1}{n}\sum_{k^2\leq|\ell|<n} |\ell|\, R_\ell\, g_\ell, \\
\text{and } T_3'(g) &= \sum_{|\ell|<k^2} \widetilde{R}_{k^2}(\ell)\, g_\ell - \sum_{|\ell|<n} \widetilde{R}_n(\ell)\, g_\ell.
\end{aligned}$$

Hence, we obtain as previously : $\|T_1'\|^2_{\mathcal{H}'} + \|T_2'\|^2_{\mathcal{H}'} \leq \dfrac{\gamma}{k^2}$.

Set $J_k = \max_{k^2\leq n<(k+1)^2} \|I_n - I_{k^2}\|_{\mathcal{H}'}$ and $T_k^* = \max_{k^2\leq n<(k+1)^2} \|T_3'\|_{\mathcal{H}'}$. Then,

$$B_N \leq \sum_{k\geq\sqrt{N}} b_k, \quad \text{with} \quad b_k = \mathbb{P}(J_k \geq t) \leq \frac{\mathbb{E}(J_k^2)}{t^2}.$$

Now

$$\mathbb{E}(J_k^2) \leq 3(\|T_1'\|^2_{\mathcal{H}'} + \|T_2'\|^2_{\mathcal{H}'} + \mathbb{E}\|T_k^*\|^2_{\mathcal{H}'}) \leq \frac{3\gamma}{k^2} + 3 \cdot \mathbb{E}\|T_k^*\|^2_{\mathcal{H}'}.$$

Then, for $k^2 \leq n < (k+1)^2$ and $\ell \in \mathbb{Z}$,

$$\begin{aligned}
\widetilde{R}_n(\ell) &= \frac{k^2}{n}\widetilde{R}_{k^2}(\ell) + \Delta_{\ell,n,k} \\
\Delta_{\ell,n,k} &= \frac{1}{n}\sum_{(k^2\wedge(k^2-\ell))<h}^{n\wedge(n-\ell)} (X_h X_{h+\ell} - R_\ell) = \frac{1}{n}\sum_{(k^2\wedge(k^2-\ell))<h}^{n\wedge(n-\ell)} Y_{h,\ell}.
\end{aligned}$$

Remark that $\widetilde{R}_{k^2}(\ell)=0$ if $k^2 \le |\ell| \le n$ and thus $\widetilde{R}_n(\ell)=\Delta_{\ell,n,k}$ in such a case. Also note that

$$\begin{aligned} \Delta^*_{\ell,k} &= \max_{k^2\le n<(k+1)^2} |\Delta_{\ell,n,k}| \le \frac{1}{k^2} \sum_{(k^2\wedge(k^2-\ell))<h}^{(k^2+2k)\wedge((k^2+2k)-\ell)} |Y_{h,\ell}| \\ \text{and thus } \ \mathbb{E}(\Delta^*_{\ell,k})^2 &\le \frac{1}{k^4}(2k)^2\cdot \max_{(h,\ell)\in\mathbb{Z}^2} \big(\mathbb{E}(|Y_{h,\ell}|^2)\big) \\ &\le \frac{4}{k^2}\,\mathbb{E}(|X_0|^4). \end{aligned}$$

Write

$$\begin{aligned} T'_3(g) &= \sum_{|\ell|<k^2} \widetilde{R}_{k^2}(\ell)\left(1-\frac{k^2}{n}\right) g_\ell - \sum_{|\ell|<n} \Delta_{\ell,n,k}\, g_\ell \\ |T^*_k(g)| &\le \frac{2}{k}\sum_{|\ell|<k^2} |\widetilde{R}_{k^2}(\ell)\, g_\ell| + \sum_{|\ell|<(k+1)^2} \Delta^*_{\ell,k}\,|g_\ell|, \end{aligned}$$

and we thus deduce

$$\mathbb{E}\|T^*_k\|^2_{\mathcal{H}'} \le 2c\cdot\left(\frac{4}{k^2}\max_{\ell\in\mathbb{Z}}\Big(\mathrm{Var}\,(\widehat{R}_{k^2}(\ell))\Big) + \max_{\ell\in\mathbb{Z}}\big(\mathbb{E}(\Delta^*_{\ell,k})^2\big)\right) \le \frac{c\cdot A}{k^2}$$

for a constant $A>0$ depending on $\mathbb{E}|X_0|^4$, κ_4, and γ only (but not on the space $\mathcal{H}$). Hence $b_k \le 3(\gamma + A\cdot c)/(k^2t^2)$ is a summable series and

$$B_N \le \frac{C_2}{t^2}\cdot\sum_{k\ge\sqrt{N}}\frac{1}{k^2}, \tag{6}$$

with $C_2>0$. From (4), (5) and (6), we deduce the theorem. ∎

2.3 Uniform Central Limit Theorem

A uniform Central Limit Theorem (CLT) results both from tightness and from the finite dimensional convergence. First, for $g\in\mathcal{H}$, we define :

$$Z_n(g) = \sqrt{n}\,(I_n(g)-I(g)) \quad \text{for} \quad n\in\mathbb{N}^*.$$

Now, we obtain :

Lemma 3 (Tightness) *If X verifies Assumption M and if $n\cdot s_n \underset{n\to\infty}{\longrightarrow} 0$, then the sequence of process $(Z_n)_{n\in\mathbb{N}^*}$ is tight in $\mathcal{H}'$.*

Proof. As in de Acosta (1970), we prove that the sequence is flatly concentrated, this means that

$$\mathbb{E}\Big(\|p_L Z_n\|^2_{\mathcal{H}'}\Big) \underset{L\to\infty}{\longrightarrow} 0,$$

where $p_L:\mathcal{H}'\to F_L$ denotes the orthogonal projection on the closed linear subspace $F'_L\subset\mathcal{H}'$ generated by $(e_\ell)_{|\ell|\ge L}$ with $e_\ell(\lambda)=e^{i\ell\lambda}$ (also $F_L\subset\mathcal{H}$ will denote the closed subspace generated by $(e_\ell)_{|\ell|\ge L}$). Then, for $L>0$,

$$\|p_L Z_n\|_{\mathcal{H}'} = \sup_{\|g\|_{\mathcal{H}}<1,\ g\in F_L} |Z_n(g)|. \tag{7}$$

Thus, for $g\in F_L$ and $\|g\|_{\mathcal{H}}<1$, using again the decomposition (3), we obtain

$$|Z_n(g)|^2 \le 3n\cdot(|T_1(g)|^2+|T_2(g)|^2+|T_3(g)|^2).$$

First, we have :

$$\begin{aligned} n\cdot\left(|T_1(g)|^2+|T_2(g)|^2\right) &\leq n\cdot a_{n\vee L}\sum_{|\ell|\geq n\vee L}R_\ell^2+\left(\mathbb{I}_{\{L<n\}}\cdot\frac{1}{n}\sum_{L\leq|\ell|<n}|\ell|^2\,s_\ell\,R_\ell^2\right)\\ &\leq \gamma\cdot\Big((L\cdot a_L)\cdot\mathbb{I}_{\{L\geq n\}}+((L\cdot a_L)\vee(n\cdot s_n))\cdot\mathbb{I}_{\{L<n\}}\Big). \end{aligned}$$

Thus, with the assumption $(n\cdot s_n)\underset{n\to\infty}{\longrightarrow}0$, we obtain :

$$\sup_{\|g\|_{\mathcal{H}}<1,\ g\in F_L} n\cdot\left(|T_1(g)|^2+|T_2(g)|^2\right)\underset{L\to\infty}{\longrightarrow}0. \tag{8}$$

Also note that

$$\begin{aligned} \sqrt{n}\,T_3(g) &= \sqrt{n}\sum_{|\ell|<n}(\widehat{R}_n(\ell)-\mathbb{E}\widehat{R}_n(\ell))g_\ell\\ n\,|T_3(g)|^2 &\leq n\sum_{L\leq|\ell|<n}s_\ell(\widehat{R}_n(\ell)-\mathbb{E}\widehat{R}_n(\ell))^2\\ \mathbb{E}\Big(\sup_{\|g\|_{\mathcal{H}}<1,\ g\in F_L}n|T_3(g)|^2\Big) &\leq \sum_{L\leq|\ell|<n}s_\ell\cdot\sup_\ell\Big(n\operatorname{Var}(\widehat{R}_n(\ell))\Big)\leq\sum_{|\ell|\geq L}s_\ell\,(\kappa_4+3\gamma). \end{aligned}$$

Since $\sum_{\mathbb{Z}}s_\ell<+\infty$, we deduce $\mathbb{E}\Big(\sup_{\|g\|_{\mathcal{H}}<1,\ g\in F_L}n|T_3(g)|^2\Big)\underset{L\to\infty}{\longrightarrow}0$. With (8) and (7), the proof is achieved. ∎

We will also use the following classical lemma :

Lemma 4 (limit variance) *If X verifies Assumption M and $(\ell,k)\in\mathbb{Z}^2$ be arbitrary integers, then*

$$n\cdot Cov(\widehat{R}_n(k),\widehat{R}_n(\ell))\underset{n\to\infty}{\longrightarrow}\sigma_{k,\ell}\ \text{with}$$
$$\sigma_{k,\ell}=\sum_{h\in\mathbb{Z}}\left(R_hR_{h+\ell-k}+R_{h+\ell}R_{h-k}+\kappa_4(X_0,X_h,X_k,X_{h+\ell})\right). \tag{9}$$

Proof. See for instance Rosenblatt (1985) [25], p. 58. ∎

Finally, a functional central limit theorem for $(Z_n(g))_g$ will thus result from the finite dimensional convergence of the process $(Z_n(g_1),\dots,Z_n(g_k))$. Since $g\mapsto Z_n(g)$ is a linear functional such a theorem follows from a central limit theorem for empirical covariances. In the sequel, for $\ell\in\mathbb{Z}$, we will point out by $\mathcal{M}_0^{(\ell)}$ a σ-algebra such as

$$\mathcal{M}_0^{(\ell)}\supset\sigma\left(Y_{k,\ell},\ k\leq 0\right)=\sigma\left(X_kX_{k+\ell},\ k\leq 0\right),$$

where $\sigma(Z_i,\ i\in I)$ represents the σ-algebra of Ω generated by $(Z_i)_{i\in I}$. The most classical example of such σ-algebra $\mathcal{M}_0^{(\ell)}$ is

$$\mathcal{B}_m=\sigma\left(X_k,\ k\leq m\right),$$

when $m\geq\ell$. Then :

Lemma 5 (CLT) *Let $(\ell_1,\dots,\ell_m)\in\mathbb{Z}^m$ be arbitrary integers ($m\in\mathbb{N}^*$). Let X verify Assumption M and be such as :*

$$\sum_{k\geq 0}\mathbb{E}\left|Y_{0,\ell_i}\,\mathbb{E}\left(Y_{k,\ell_i}\,|\,\mathcal{M}_0^{(\ell_i)}\right)\right|<\infty\quad\text{for all}\ \ i\in\{1,\dots,m\}. \tag{10}$$

Then, if $\Sigma=(\sigma_{\ell_i,\ell_j})_{1\leq i,j\leq m}$ defined in (9) is a nonsingular matrix,

$$\Big(\sqrt{n}\,(\widehat{R}_n(\ell_i)-\mathbb{E}\widehat{R}_n(\ell_i))\Big)_{1\leq i\leq m}\ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}\ \mathcal{N}_m(0,\Sigma). \tag{11}$$

Proof. If condition (10) is verified, then the projective criterion introduced in Dedecker and Rio (2000), *i.e.* $\mathbb{E}\left|\sum_{k\geq 0} Y_{0,\ell_i}\,\mathbb{E}\left(Y_{k,\ell_i}\,|\,\sigma(\,Y_{j,\ell_i},\ j\leq 0)\right)\right| < \infty$ for all $i \in \{1,\dots,m\}$ is also verified, and the central limit theorem can be stated for each ℓ_i. Therefore, by considering a sequence $(Z_j)_{j\in\mathbb{Z}}$ where Z_j is a linear combination of $(Y_{j,\ell_1},\dots,Y_{j,\ell_1})$, and by applying it the same theorem (the projective criterion is also verified by $(Z_j)_{j\in\mathbb{Z}}$), the multidimensional central limit theorem (11) can be established. The proof of an analogue non causal CLT is provided in Hall & Heyde (1980), theorem 5.4, page 136. Unfortunately this condition does not seem to be adapted to work out the forthcoming examples. ∎

Remark. If for each $i \in \{1,\dots,m\}$, $\sum_{k=0}^{\infty}\left(\mathbb{E}\Big(\mathbb{E}\left(Y_{k,\ell_i}\,|\,\mathcal{B}_0\right)\Big)^2\right)^{1/2} < \infty$, then (10) is verified. Thus, lemma 5 is a generalization of a result of Rosenblatt (1985, Theorem 3, p. 58).

Let us now define for any $\lambda,\mu,\nu\in\mathbb{R}$, the bispectral density

$$f_4(\lambda,\mu,\nu) = \frac{1}{(2\pi)^3}\sum_{h=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\kappa_4(X_0,X_h,X_k,X_\ell)e^{i(h\lambda+k\mu+\ell\nu)}$$

(the existence of f_4 provided from Assumption M, and more precisely from $\kappa_4<\infty$). Note that $I_n(g) - \mathbb{E}I_n(g) = \sum_{\ell\in\mathbb{Z}} g_\ell\left(\widehat{R}_n(\ell) - \mathbb{E}\left(\widehat{R}_n(\ell)\right)\right)$ allows to deduce the limiting covariance $\Gamma(g_1,g_2)$ of the process $(Z_n)_{n\geq 1}$, *i.e.*

$$\Gamma(g_1,g_2) = \frac{1}{\pi}\int_{-\pi}^{\pi} g_1(\lambda)g_2(\lambda)f^2(\lambda)\,d\lambda + 2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_1(\lambda)g_2(\mu)f_4(\lambda,-\mu,\mu)\,d\lambda\,d\mu. \tag{12}$$

Thus, the previous lemmas together imply :

Theorem 2 *Under assumptions of Lemma 5 and if $n\cdot s_n \underset{n\to\infty}{\longrightarrow} 0$, then yields the Uniform Central Limit Theorem (UCLT) :*

$$Z_n = \sqrt{n}\,(I_n - I) \underset{n\to\infty}{\longrightarrow} Z \quad \textit{in the space } \mathcal{H}', \tag{13}$$

with $(Z(g))_{g\in\mathcal{H}}$ the centered Gaussian process with covariance $\Gamma(g_1,g_2)$ defined in (12).

Proof. The expression of $\Gamma(g_1,g_2)$ and some details of the finite dimensional convergence of the process $(Z_n(g_1),\dots,Z_n(g_k))$ can be found in Rosenblatt (1985, Corollary 2, p. 61). The tightness proved in Lemma 3 allows to establish the functional central limit theorem. ∎

Remark. In many cases, the limiting variance is a non degenerated positive operator. It is such a case for instance if $(X_n)_{n\in\mathbb{Z}}$ is a stationary Gaussian process with spectral density $f\neq 0$ almost everywhere on $[-\pi,\pi[$.

2.4 Examples of processes verifying the Uniform Theorems

In this section we provide various examples of time series verifying the previous uniform limit theorems (ULLN and UCLT). Here, we will assuming that $n\cdot s_n \underset{n\to\infty}{\longrightarrow} 0$.

<u>**Causal linear processes**</u>

Corollary 1 *Let X be a linear and causal time series verifying $X_n = \sum_{k=0}^{\infty} a_k\,\xi_{n-k}$ for $n\in\mathbb{Z}$, with $a_k\in\mathbb{R}$ for $k\in\mathbb{Z}$ and with $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of centered independent identically distributed random variables such that $\mathbb{E}\xi_0^4<\infty$. Moreover if*

$$\sum_k k\,a_k^2 < \infty,$$

then the Uniform CLT (13) holds.

Proof. This result can be deduced from Rosenblatt (1985, p. 59). In fact, if X verifies conditions of Corollary 1, then for all $\ell \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \|\mathbb{E}\,(Y_{k,\ell}\,|\,\mathcal{B}_\ell)\|_2 < \infty$$

and thus (10) is also verified. ∎

Gaussian processes

Corollary 2 *If the sequence $(X_n)_{n\in\mathbb{Z}}$ is a centered stationary Gaussian process such as $\sum_k R_k^2 < \infty$, then the Uniform CLT (13) holds.*

Proof. We can always write for all $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left(|Y_{0,\ell}\mathbb{E}\,(Y_{k,\ell}\,|\,\mathcal{M}_0^{(\ell)})|\right) &\leq \left|\mathrm{Cov}\,(Y_{0,\ell}\,,\,Y_{k,\ell})\right| \\ &\leq \left|\mathbb{E}\,(X_0X_\ell X_k X_{k+\ell}) - R_\ell^2\right|.\end{aligned}$$

But $\mathbb{E}\,(X_0X_\ell X_k X_{k+\ell}) - R_\ell^2 = R_k^2 + R_{k+\ell}R_{k-\ell}$ for a centered stationary Gaussian process. Therefore,

$$\sum_{k\geq 0}\mathbb{E}\left(|Y_{0,\ell}\mathbb{E}\,(Y_{k,\ell}\,|\,\mathcal{M}_0^{(\ell)})|\right) \leq \sum_{k\in\mathbb{Z}} R_k^2 + \left(\sum_{k\in\mathbb{Z}} R_{k+\ell}^2\right)^{1/2}\cdot\left(\sum_{k\in\mathbb{Z}} R_{k-\ell}^2\right)^{1/2},$$

from the Cauchy-Schwarz inequality for ℓ^2 sequences. ∎

Strong mixing processes

Corollary 3 *Let $X = (X_n)_{n\in\mathbb{Z}}$ verify Assumption M. Assume that X is α'-mixing in the sense that :*

$$\begin{aligned}\alpha'_n &= \sup_{\ell\geq 0}\left\{\alpha\Big(\sigma(X_n, X_{n+\ell})\,,\,\mathcal{B}_0\Big)\right\} \underset{n\to\infty}{\longrightarrow} 0, \quad \textit{where, as usually,} \\ \alpha\Big(\mathcal{A}\,,\,\mathcal{B}\Big) &= \sup_{\substack{A\,\in\,\mathcal{A}\\ B\,\in\,\mathcal{B}}}\Big|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\Big| \quad \textit{for}\ \ \mathcal{A},\mathcal{B}\subset\Omega.\end{aligned}$$

Moreover, assume that $\int_0^1 \alpha'^{-1}(u)Q_{X_0}^4(u)\,du < \infty$ where Q_X denotes $|X|-$quantile function and $\alpha'^{-1}(u) = \inf\{k\in\mathbb{N},\ \alpha'_k \leq u\}$. Then the Uniform CLT (13) holds.

Remark. We quote that $\alpha'_n \geq \alpha_n = \alpha(\mathcal{B}^n, \mathcal{B}_0)$ where $\mathcal{B}^n = \sigma(X_i\,;\ i\geq n)$. Hence this condition is weaker that the standard mixing coefficient in Rosenblatt (1985). However, no simple counter example seems to be available.

Proof. From Rio (1994) inequality and the stationarity of X, for all $\ell, k \in \mathbb{N}$, we have :

$$\begin{aligned}\|Y_{0,\ell}\,\mathbb{E}\,(Y_{k+\ell,\ell}\ |\ \mathcal{B}_\ell)\|_1 &\leq 2\int_0^{\alpha(\sigma(Y_{k+\ell,\ell}),\mathcal{B}_\ell)} Q_{Y_{0,\ell}}(u)\,Q_{Y_{k+\ell,\ell}}(u)\,du \\ &\leq 2\int_0^{\alpha(\sigma(X_k,X_{k+\ell}),\mathcal{B}_0)} Q_{Y_{0,\ell}}^2(u)\,du.\end{aligned}$$

Therefore, for all $\ell, k \in \mathbb{N}$,

$$\|Y_{0,\ell}\,\mathbb{E}\,(Y_{k+\ell,\ell}\ |\ \mathcal{B}_\ell)\|_1 \leq 2\int_0^{\alpha'_k} Q_{Y_{0,\ell}}^2(u)\,du \quad \text{for all}\ \ k\geq 0.$$

Consequently, for all $\ell \in \mathbb{N}$

$$\begin{aligned}
\sum_{k\geq 0} \|Y_{0,\ell}\,\mathbb{E}\left(Y_{k+\ell,\ell} \mid \mathcal{B}_\ell\right)\|_1 &\leq 2\int_0^1 \left(\sum_{k\geq 0} \mathbb{1}_{u\leq \alpha_k}\right) Q^2_{Y_{0,\ell}}(u)\,du, \\
&\leq 2\int_0^1 \alpha'^{-1}(u) Q^2_{Y_{0,\ell}}(u)\,du.
\end{aligned}$$

But Lemme 2.1 in Rio (2000) provides :

$$\begin{aligned}
\int_0^1 \alpha'^{-1}(u) Q^2_{Y_{0,\ell}}(u)du &\leq \int_0^1 \alpha'^{-1}(u)\Big(Q_{X_0}(u)Q_{X_\ell}(u)+Q_{R_\ell}(u)\Big)^2\,du \\
&\leq \int_0^1 \alpha'^{-1}(u)(Q^2_{X_0}(u)+|R_\ell|)^2\,du,
\end{aligned}$$

and therefore if $\displaystyle\int_0^1 \alpha'^{-1}(u)Q^4_{X_0}(u)\,du<\infty$, then $\displaystyle\sum_{k\geq 0}\|Y_{0,\ell}\,\mathbb{E}\left(Y_{k,\ell} \mid \mathcal{B}_\ell\right)\|_1 < +\infty$. for all $\ell \in \mathbb{N}$. ∎

Remarks. **1.** In the hypothesis of Corollary 3 and more precisely in the definition of α', $\alpha\Big(\sigma(X_n, X_{n+\ell})\,,\,\mathcal{B}_0\Big)$ may be replaced by the sharper expression $\alpha\Big(\sigma(X_n\times X_{n+\ell})\,,\,\mathcal{B}_0\Big)$.
2. If X is α-mixing process in the usual sense, that is,

$$\alpha_n=\Big\{\alpha\Big(\sigma(X_k,\,k\geq n)\,,\,\mathcal{B}_0\Big)\Big\} \underset{n\to\infty}{\longrightarrow} 0,$$

then X is α'-mixing (for all $n\in\mathbb{N}$, $\alpha'_n\leq\alpha_n$). Therefore, if X is a strongly α-mixing process verifying Assumption M such as $\displaystyle\int_0^1 \alpha^{-1}(u)Q^4_{X_0}(u)\,du<\infty$ then the Uniform CLT (13) holds.

Causal weak dependent time series

Let $h:\mathbb{R}^u\to\mathbb{R}$ be an arbitrary function. We set :

$$\mathrm{Lip}\, h=\sup\left\{\frac{|\,h(y_1,\ldots,y_u)-h(x_1,\ldots,x_u)\,|}{|y_1-x_1|+\cdots+|y_u-x_u|}\quad\text{for }(y_1,\ldots,y_u)\neq(x_1,\ldots,x_u)\right\}.$$

Now, Λ denotes the set of functions $h:\mathbb{R}^u\to\mathbb{R}$ for some $u\in\mathbb{N}$ such that $\mathrm{Lip}\,h<\infty$. Denote also :

- $\Lambda_1=\{h\in\Lambda,\ \mathrm{Lip}\,h\leq 1\}$, and
- $\Lambda^{(1)}=\{h\in\Lambda,\ \|h\|_\infty\leq 1\}$.

Then :

Corollary 4 *Let $X=(X_n)_{n\in\mathbb{Z}}$ verify Assumption M. Assume that X is $\theta-$weakly dependent in the sense that it exists $(\theta_r)_{r\in\mathbb{N}}$ such as for all $r\in\mathbb{N}$, all function $f:\mathbb{R}^2\to\mathbb{R}$ satisfying $\|f\|_\infty\leq 1$, and all random variable $Z\in\mathcal{B}_0$ satisfying $\|Z\|_\infty<1$,*

$$|Cov(f(X_{j_1},X_{j_2}),Z)|\leq 2\cdot Lip\, f\cdot\theta_r\quad \textit{for all }\ j_1,j_2\geq r \tag{14}$$

(as in Dedecker and Doukhan, 2003). We also suppose that

$$\exists m>4,\ \textit{such that}\ \ \|X_0\|_m<\infty\quad\textit{and}\quad\sum_{k=0}^{\infty}\theta_k^{\frac{m-4}{m-1}}<\infty.$$

Then the Uniform CLT (13) holds.

Proof. We truncate the variables $X_j = f_M(X_j) + g_M(X_j)$ where, for $x \in \mathbb{R}$, we set $f_M(x) = (x \wedge M) \vee (-M)$ (then $f_M \in [-M, M]$) and $g_M(x) = x - f_M(x) = x \cdot \mathbb{1}_{|x| \geq M}$. Note that $\operatorname{Lip} f_M = 1$ but $\|f_M\|_\infty = M$.

Then, $Y_{k,\ell} = \Big(f_M(X_k)\, f_M(X_{k+\ell}) - R'_\ell\Big) + U_{k,\ell,M}$ with :

- $R'_\ell = \operatorname{Cov}\Big(f_M(X_k), f_M(X_{k+\ell})\Big)$;

- $U_{k,\ell,M} = g_M(X_k) f_M(X_{k+\ell}) + f_M(X_k) g_M(X_{k+\ell}) + g_M(X_k) g_M(X_{k+\ell}) + R'_\ell - R_\ell$.

Therefore, $\mathbb{E}(U_{k,\ell,M}) = 0$ and, for m such that $\|X_0\|_m < \infty$, we derive

$$\begin{aligned}
\|U_{k,\ell,M}\|_1 &\leq M\mathbb{E}\left|X_k \cdot \mathbb{1}_{|X_k|>M}\right| + M\mathbb{E}\left|X_{k+\ell} \cdot \mathbb{1}_{|X_{k+\ell}|>M}\right| + \mathbb{E}\left((X_k \cdot \mathbb{1}_{|X_k|>M})^2\right) + \left|\mathbb{E}\left(X_0 X_\ell - f_M(X_0) f_M(X_\ell)\right)\right| \\
&\leq 2M\|X_0\|_m \Big(\mathbb{P}(X_0 > M)\Big)^{1-1/m} + \|X_0\|_m^2 \Big(\mathbb{P}(X_0 > M)\Big)^{1-2/m} + \mathbb{E}\left|f_M(X_0) g_M(X_\ell)\right| + \\
&\qquad + \mathbb{E}\left|g_M(X_0) f_M(X_\ell)\right| + \mathbb{E}\left|g_M(X_0) g_M(X_\ell)\right| \\
&\leq 4M\|X_0\|_m \Big(\frac{\mathbb{E}\,|X_0|^m}{M^m}\Big)^{1-1/m} + 2\|X_0\|_m^2 \Big(\frac{\mathbb{E}\,|X_0|^m}{M^m}\Big)^{1-2/m} \\
&\leq 6 \cdot M^{2-m} \cdot \|X_0\|_m^m, \qquad (15)
\end{aligned}$$

from Hölder and Markov inequalities. By the same procedure, we also obtain :

$$\begin{aligned}
\|U_{k,\ell,M}\|_2^2 &\leq 6\Big(\mathbb{E}\,(g_M^2(X_k) f_M^2(X_{k+\ell}) + \mathbb{E}\,(f_M^2(X_k) g_M^2(X_{k+\ell}) + \mathbb{E}\,(g_M^2(X_k) g_M^2(X_{k+\ell})\Big) \\
&\leq 18 \cdot M^{4-m} \cdot \|X_0\|_m^m. \qquad (16)
\end{aligned}$$

Let h_M be the function such as $h_M(x, y) = f_M(x) f_M(y) - R'_\ell$ for all $(x, y) \in \mathbb{R}^2$. Note that $\|h_M\|_\infty \leq 2M^2$ and $\operatorname{Lip} h_M = M$. Moreover, for all random variable W in $\mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$,

$$\|\mathbb{E}\,(W \mid \mathcal{B}_\ell)\|_1 = \sup_{Z \in (\Omega, \mathcal{B}_\ell, \mathbb{P})\ ,\ \|Z\|_\infty \leq 1} \Big|\mathbb{E}\,(W \cdot Z)\Big|.$$

Therefore,

$$\begin{aligned}
\Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\left(h_M(X_k, X_{k+\ell}) \mid \mathcal{B}_\ell\right)\Big\|_1 &= \Big\|\mathbb{E}\,(h_M(X_0, X_\ell) \cdot h_M(X_k, X_{k+\ell}) \mid \mathcal{B}_\ell)\Big\|_1 \\
&= \sup_{Z \in (\Omega, \mathcal{B}_\ell, \mathbb{P})\ ,\ \|Z\|_\infty \leq 1} \Big|\mathbb{E}\left(Z \cdot h_M(X_0, X_\ell) \cdot h_M(X_k, X_{k+\ell})\right)\Big| \\
&\leq \sup_{Z' \in (\Omega, \mathcal{B}_\ell, \mathbb{P})\ ,\ \|Z'\|_\infty \leq 2M^2} \Big|\operatorname{Cov}\left(h_M(X_k, X_{k+\ell})\,,\, Z'\right)\Big|.
\end{aligned}$$

Consequently, from the definition (14) and the stationarity of X, for all $k \geq 0$,

$$\Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\left(h_M(X_k, X_{k+\ell}) \mid \mathcal{B}_\ell\right)\Big\|_1 \leq 2 \cdot M^3 \cdot \theta_{k-|\ell|}. \qquad (17)$$

Thus,

$$\begin{aligned}
\Big\|Y_{0,\ell} \cdot \mathbb{E}\left(Y_{k,\ell} \mid \mathcal{B}_\ell\right)\Big\|_1 &\leq \Big\|(U_{0,\ell,M} \cdot \mathbb{E}\left(U_{k,\ell,M} \mid \mathcal{B}_\ell\right)\Big\|_1 + \Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\left(U_{k,\ell,M} \mid \mathcal{B}_\ell\right)\Big\|_1 \\
&\quad + \Big\|U_{0,\ell,M} \cdot \mathbb{E}\left(h_M(X_k, X_{\ell+k}) \mid \mathcal{B}_\ell\right)\Big\|_1 + \Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\left(h_M(X_k, X_{k+\ell}) \mid \mathcal{B}_\ell\right)\Big\|_1 \\
&\leq \Big\|U_{k,\ell,M}\Big\|_2^2 + 4M^2 \Big\|U_{k,\ell,M}\Big\|_1 + \Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\left(h_M(X_k, X_{k+\ell}) \mid \mathcal{B}_\ell\right)\Big\|_1 \\
&\leq 34 \cdot M^{4-m} \cdot \|X_0\|_m^m + 2 \cdot M^3 \cdot \theta_{k-|\ell|},
\end{aligned}$$

from (15), (15) and (17). With the choice $M = \theta_{k-|\ell|}^{-1/(m-1)}$ we prove that if

$$\sum_{k=0}^{\infty} \theta_k^{1-3/(m-1)} < \infty, \qquad \|X_0\|_m < \infty$$

for some $m > 4$, together yield the Uniform CLT (13). ∎

Remark. The $\theta-$weakly dependence of X also holds with

$$\theta_r = \sup_{\{f/\mathrm{Lip} f \le 1\}} \|\mathbb{E}\left[f(X_{j_1}, X_{j_2}) \,|\, \mathcal{B}_0\right] - \mathbb{E}\left[f(X_{j_1}, X_{j_2})\right]\|_1.$$

2.5 Non-causal weak dependent time series

(The details of this topic is developed as an appendix in a more general frame).

The class $\Lambda^{(1)} = \mathbb{B}^\infty \cap \Lambda$ will be used together with the functions ψ_1 defined by

$$\psi_1(g_1, g_2, u, v) = u \cdot \mathrm{Lip}\,(g_1) + v \cdot \mathrm{Lip}\,(g_2).$$

where g_1, g_2 are two real functions of $\Lambda^{(1)}$ respectively defined on $\mathbb{R}^u$ and $\mathbb{R}^v$ $(u, v \in \mathbb{N}^*)$. In short, $\psi_1 = u \cdot \mathrm{Lip}\,(g_1) + v \cdot \mathrm{Lip}\,(g_2)$. If the sequence $(X_n)_{n\in\mathbb{Z}}$ is η-weakly dependent, there exists a sequence $\eta = (\eta_r)_{r\in\mathbb{N}}$ decreasing to zero at infinity such that for any u-tuple $(i_1, \ldots, i_u)$ and any v-tuple $(j_1, \ldots, j_v)$ with $i_1 \le \cdots \le i_u < i_u + r \le j_1 \le \cdots \le j_v$, one has

$$\left|\mathrm{Cov}\left(g_1(X_{i_1}, \ldots, X_{i_u}), g_2(X_{j_1}, \ldots, X_{j_v})\right)\right| \le \psi_1 \cdot \eta_r.$$

Here, using the tools developed in paragraph 4, we obtain, under certain assumptions on the time series, the Uniform CLT, and more precisely a convergence rate to the Gaussian measure :

Theorem 3 *Let $X = (X_n)_{n\in\mathbb{Z}}$ verify Assumption M. Assume that X is η-weakly dependent. Suppose also that*

$$\exists m > 4, \text{ such that } \|X_0\|_m < \infty \text{ and } \eta_n = \mathcal{O}(n^{-\alpha}) \text{ with } \alpha > \max\left(3\,;\, \frac{2m-1}{m-4}\right).$$

Suppose that the sequence $(s_n)_n$ is such as $s_n = \mathcal{O}(n^{-s})$ with $s > 1$. Then the Uniform CLT (13) holds. Moreover, for $\phi : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^3(\mathbb{R})$ function having bounded derivatives up to order 3, and for $g \in \mathcal{H}$:

$$\left|\mathbb{E}\left[\phi\Big(\sqrt{n}(I_n(g) - I(g))\Big) - \phi\big(\sigma(g)\cdot N\big)\right]\right| \le C \cdot n^{-\frac{t}{t+3}\left(\frac{\alpha(m-4)-2m+1}{2(m+1+\alpha\cdot m)}\right)}$$

where $C > 0$, $t = \left(\left(2\alpha\dfrac{m-2}{m-1} - 1\right) \wedge \left(\dfrac{s-1}{2}\right)\right)$, $N \sim \mathcal{N}(0,1)$ and $\sigma^2(g) = \Gamma(g,g)$ (see 12).

Corollary 5 *Under the same assumptions than in Theorem 3, for $\ell \in \mathbb{Z}$ and $\phi : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^3(\mathbb{R})$ function having bounded derivatives up to order 3,*

$$\left|\mathbb{E}\left[\phi\Big(\sqrt{n}(\widehat{R}_n(\ell) - R_\ell)\Big) - \phi\big(\sigma_\ell \cdot N\big)\right]\right| \le C \cdot n^{-\frac{\alpha(m-4-2m+1}{2(m+1+\alpha\cdot m)}},$$

with $C > 0$, $N \sim \mathcal{N}(0,1)$ and $\sigma_\ell^2 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \cos^2(\lambda\ell))f^2(\lambda)d\lambda + 2\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(\lambda\ell)\cos(\mu\ell) f_4(\lambda, -\mu, \mu) d\lambda d\mu.$

Remark. The convergence rate of both the functional central limit theorems 3 and 5 is provided from the Bernstein's blocks method, and should not be optimal. However, under not too restrictive conditions ($\alpha \to \infty$ and $s \to \infty$), the convergence rate of those theorems can be $n^{-\lambda}$ with $\lambda < 1/2$ as close to $1/2$ as one wants.

3 Applications to parametric estimation

Now we will apply the previous result to finite parameters estimates. Let $X = (X_n)_{n\in\mathbb{Z}}$ a time series verifying Assumption M. We also assume that the spectral density f of X can be written under the form :

$$f(\lambda) = f_{(\beta,\sigma^2)}(\lambda) = \sigma^2 \cdot g_\beta(\lambda) \quad \text{for all } \lambda \in [-\pi, \pi[, \tag{18}$$

that is, f depends on a finite number of unknown parameters, a variance term σ^2 and a $\mathbb{R}^p$-vector β, where $\beta = (\beta^{(1)}, \ldots, \beta^{(p)})$. Denote also σ^* and $\beta^* = (\beta^{(1)*}, \ldots, \beta^{(p)*})$ the true value of σ and β. As a consequence, for all $\lambda \in [-\pi, \pi[$, we will now denote $\sigma^{*2} g_{\beta^*}(\lambda)$ the spectral density of X. We will also assume that β and g_β satisfy some of the following conditions :

- Condition C1 : the true values σ^* and β^* are such as $\sigma^* > 0$ and β^* lies in a region $\mathcal{K} \subset \mathbb{R}^p$ where $\mathcal{K}$ is an open and relatively compact set.

- Condition C2 : if β_1, β_2 are distinct elements of $\mathcal{K}$, the set $\{\lambda \in [-\pi, \pi[, \; g_{\beta_1}(\lambda) \neq g_{\beta_2}(\lambda)\}$ has positive Lebesgue measure.

- Condition C3 : there is a normalization condition :
$$\int_{-\pi}^{\pi} \log(g_\beta(\lambda))\, d\lambda = 0 \quad \text{for all} \quad \beta \in \mathcal{K}.$$

- Condition C4 : for all $\beta \in \mathcal{K}$, the function $\lambda \mapsto g_\beta^{-1}(\lambda) = \dfrac{1}{g_\beta(\lambda)} \in \mathcal{H}$.

- Condition C5 : for all $\lambda \in [-\pi, \pi[$, the function $\beta \mapsto g_\beta^{-1}(\lambda)$ is continuous on $\mathcal{K}$.

- Condition C6 : for all $\lambda \in [-\pi, \pi[$, the function $\beta \mapsto g_\beta^{-1}(\lambda)$ is twice continuously differentiable on $\mathcal{K}$.

- Condition C7 : for all $\beta_0 \in \mathcal{K}$ and $(i, j) \in \{1, \ldots, p\}$, the function $\lambda \mapsto \left(\dfrac{\partial^2 g_\beta^{-1}}{\partial \beta^{(i)} \partial \beta^{(j)}}\right)_{\beta_0} (\lambda) \in \mathcal{H}$.

- Condition C8 : for all $\beta \in \mathcal{K}$, the function $\lambda \mapsto g_\beta(\lambda)$ is continuously differentiable on $[-\pi, \pi[$.

Let $(X_1, \ldots, X_n)$ be a known path of X. Define the Whittle maximum likelihood estimators of β^* and σ^{*2}, that are :

$$\begin{aligned} \widehat{\beta}_n &= \text{Argmin}_{\beta \in \mathcal{K}} \left\{ \frac{1}{2\pi} I_n(g_\beta^{-1}) \right\} = \text{Argmin}_{\beta \in \mathcal{K}} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_n(\lambda)}{g_\beta(\lambda)}\, d\lambda \right\}, \\ \text{and} & \\ \widehat{\sigma}_n^2 &= I_n(g_{\widehat{\beta}_n}^{-1}). \end{aligned}$$

In the following paragraphs, we will show the strong consistency of the estimators $\widehat{\beta}_n$ and $\widehat{\sigma}_n^2$.

3.1 Asymptotic properties of the Whittle parametric estimators

Theorem 4 *Let X verify the assumptions of Theorem 2. Under Conditions C1-5, then*

$$\widehat{\beta}_n \xrightarrow[n \to \infty]{a.s.} \beta^* \quad \textit{and} \quad \widehat{\sigma}_n^2 \xrightarrow[n \to \infty]{a.s.} \sigma^{*2}.$$

Proof. From Theorem 1 and Conditions C4 and C5 (the function $\beta \mapsto g_\beta^{-1}$ is also uniformly continuous on $\mathcal{K}$ because $\mathcal{K}$ is a relatively compact set), with probability 1,

$$\lim_{n \to \infty} I_n(g_\beta^{-1}) = I(g_\beta^{-1}),$$

uniformly in β on $\mathcal{K}$. From Condition C2, we know that

$$I(g_\beta^{-1}) > \sigma^{*2} = I(g_{\beta^*}^{-1}) \quad \text{for all} \quad \beta \neq \beta^*$$

(see Lemma 2, in Hannan, 1973). Therefore (see the proof of Theorem 1 in Hannan, 1973), $\widehat{\beta}_n = \text{Argmin}_{\beta \in \mathcal{K}} \left\{ I_n(g_\beta^{-1}) \right\}$ converges almost surely to β^* and $I_n(g_{\widehat{\beta}_n}^{-1})$ converges to σ^{*2}. ∎

Remarks on the conditions C1-5 The C1-3 conditions are usual and can be found for example in Rosenblatt (1985) for mixing time series or in Fox and Taqqu (1986) for strong dependence times series. The condition C5 is weaker than the condition of differentiability generally required. The condition C4 is not usual and is totally connected with the uniformity of the limit theorems.

Theorem 5 *Let X verify the assumptions of Theorem 2. Under Conditions C1-7 and if the matrix $W^* = (w^*_{ij})_{1\le i,j\le p}$, with*

$$w^*_{ij} = \int_{-\pi}^{\pi} g^2_{\beta^*}(\lambda)\cdot\Big(\frac{\partial g_\beta^{-1}}{\partial\beta^{(i)}}\Big)_{\beta^*}(\lambda)\cdot\Big(\frac{\partial g_\beta^{-1}}{\partial\beta^{(j)}}\Big)_{\beta^*}(\lambda)\,d\lambda$$

is nonsingular, then

$$\sqrt{n}(\widehat{\beta}_n-\beta^*) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_p\Big(0\,,\,(\sigma^*)^{-4}\cdot(W^*)^{-1}\cdot Q^*\cdot(W^*)^{-1}\Big), \tag{19}$$

with the matrix $Q^* = (q^*_{ij})_{1\le i,j\le p}$ such as :

$$q^*_{ij} = 2\pi\left(\sigma^{*4}w^*_{ij} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda,\mu,-\mu)\Big(\frac{\partial g_\beta^{-1}}{\partial\beta^{(i)}}\Big)_{\beta^*}(\lambda)\Big(\frac{\partial g_\beta^{-1}}{\partial\beta^{(j)}}\Big)_{\beta^*}(\mu)\,d\lambda\,d\mu\right).$$

Proof. Let $U_n(\beta) = I_n(g_\beta^{-1})$. From Conditions 2 and 6, $\beta\mapsto U_n(\beta)$ exists and is twice differentiable on $\mathcal{K}$. Denote $\frac{\partial}{\partial\beta}U_n(\beta)$ the vector $\Big(\frac{\partial}{\partial\beta^{(i)}}U_n(\beta)\Big)_{1\le i\le p}$ and $\frac{\partial^2}{\partial\beta^2}U_n(\beta)$ the $(p\times p)$ matrix $\Big(\frac{\partial^2}{\partial\beta^{(i)}\partial\beta^{(j)}}U_n(\beta)\Big)_{1\le i,j\le p}$. According to the mean value theorem,

$$\frac{\partial}{\partial\beta}U_n(\widehat{\beta}_n) = \frac{\partial}{\partial\beta}U_n(\beta^*) + \frac{\partial^2}{\partial\beta^2}U_n(\overline{\beta})(\widehat{\beta}_n-\beta^*),$$

where $\|\overline{\beta}-\beta^*\|_p < \|\widehat{\beta}_n-\beta^*\|_p$ (with $\|.\|_p$ the euclidian norm in $\mathbb{R}^p$). Since $\widehat{\beta}_n$ minimizes $\beta\mapsto U_n(\beta)$, it follows that

$$\frac{\partial}{\partial\beta}U_n(\beta^*) = \Big[-\frac{\partial^2}{\partial\beta^2}U_n(\overline{\beta})\Big](\widehat{\beta}_n-\beta^*). \tag{20}$$

But, from Theorem 4, $\widehat{\beta}_n \xrightarrow[n\to\infty]{a.s.} \beta^*$ and then $\overline{\beta} \xrightarrow[n\to\infty]{a.s.} \beta^*$. Consequently, from Condition C7 and Theorem 1 (Uniform Law of Large Number),

$$\frac{\partial^2}{\partial\beta^2}U_n(\overline{\beta}) \xrightarrow[n\to\infty]{a.s.} \Big(\int_{-\pi}^{\pi}\frac{\partial^2}{\partial\beta^{(i)}\partial\beta^{(j)}}g^{-1}_{\beta^*}(\lambda)\cdot\sigma^{*2}g_{\beta^*}(\lambda)\,d\lambda\Big)_{1\le i,j\le p} = \sigma^{*2}W^*,$$

(see Lemma 3 of Fox and Taqqu, 1986). Moreover, from Theorem 2 and Condition 6,

$$\sqrt{n}\Big(\frac{\partial}{\partial\beta}U_n(\beta^*) - \frac{\partial}{\partial\beta}I(g^{-1}_{\beta^*})\Big) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_p(0,Q^*),$$

$$\text{and thus}\quad \sqrt{n}\frac{\partial}{\partial\beta}U_n(\beta^*) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_p(0,Q^*),$$

because $\frac{\partial}{\partial\beta}I(g^{-1}_{\beta^*}) = \int_{-\pi}^{\pi}\Big(\frac{\partial g_\beta^{-1}(\lambda)}{\partial\beta}\Big)_{\beta^*}\cdot\sigma^{*2}g_{\beta^*}(\lambda)\,d\lambda = \sigma^{*2}\frac{\partial}{\partial\beta}\Big(\int_{-\pi}^{\pi}\log(g_\beta^{-1}(\lambda))\,d\lambda\Big)_{\beta^*} = 0$ from Condition C2. Therefore, if the matrix W^* is nonsingular, from (20),

$$\sqrt{n}(\widehat{\beta}_n-\beta^*) \xrightarrow[n\to\infty]{\mathcal{D}} -(\sigma^*)^{-2}(W^*)^{-1}\cdot\mathcal{N}_p(0,Q^*),$$

and this completes the proof of Theorem 5. ∎

Theorem 6 *Let X verify the assumptions of Theorem 2. Under Conditions C1-8, then*

$$\sqrt{n}(\widehat{\sigma}_n^2-\sigma^*) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}\Big(0\,,\,2\sigma^{*4}+2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_4(\lambda,\mu,-\mu)g_{\beta^*}^{-1}(\lambda)g_{\beta^*}^{-1}(\mu)\,d\lambda\,d\mu\Big). \tag{21}$$

Moreover, $\sqrt{n}(\widehat{\sigma}_n^2-\sigma^*)$ and $\sqrt{n}(\widehat{\beta}_n-\beta^*)$ are jointly asymptotically normal with covariance :

$$\lim_{n\to\infty}\sqrt{n}\Big(Cov\,(\widehat{\sigma}_n^2\,,\,\widehat{\beta}_n^{(i)})\Big)_{1\le i\le p}=\big(\sigma^{*2}{\cdot}W^*\big)^{-1}{\cdot}\Big(2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_4(\lambda,\mu,-\mu)g_{\beta^*}^{-1}(\lambda)\Big(\frac{\partial}{\partial\beta^{(i)}}g_{\beta^*}^{-1}(\mu)\Big)\,d\lambda\,d\mu\Big)_{1\le i\le p}.$$

Proof. The Taylor's formula implies that :

$$U_n(\beta^*)=U_n(\widehat{\beta}_n)+(\beta^*-\widehat{\beta}_n)'\cdot\Big(\frac{\partial^2}{\partial\beta^2}U_n(\underline{\beta})\Big)\cdot(\beta^*-\widehat{\beta}_n),$$

with probability 1, and with $\|\underline{\beta}-\beta^*\|_p<\|\widehat{\beta}_n-\beta^*\|_p$. From previous Theorem 5, it follows

$$\sqrt{n}(U_n(\beta^*)-\sigma^{*2})=\sqrt{n}(U_n(\widehat{\beta}_n)-\sigma^{*2})+\mathcal{O}_p(n^{-1/2}).$$

Under condition C8 (that implies $\sum|s|\cdot R(s)<\infty$), we also have $\mathbb{E}\Big(U_n(\beta^*)\Big)=\sigma^{*2}+\mathcal{O}(\log n/n)$ (see for instance Rosenblatt, 1985). As a consequence, $\sqrt{n}\Big(U_n(\beta^*)-\mathbb{E}\Big(U_n(\beta^*)\Big)\Big)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}\Big(0,\Gamma(g_{\beta^*}^{-1},g_{\beta^*}^{-1})\Big)$ with $g_{\beta^*}^{-1}\in\mathcal{H}$. Therefore,

$$\sqrt{n}\Big(U_n(\widehat{\beta}_n)-\sigma^{*2}\Big)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}\Big(0,\Gamma(g_{\beta^*}^{-1},g_{\beta^*}^{-1})\Big),$$

and it implies property (21). The end of the proof of Theorem 6 follows the same arguments as in Rosenblatt (1985). ∎

3.2 Example of Whittle parametric estimates for different time series

GARCH and ARCH(∞) processes

The famous and from now on classical GARCH(q',q) model introduced by Engle (1982) and Bollerslev (1986) and is given by equations

$$X_k=\rho_k\cdot\xi_k\quad\text{with}\quad\rho_k^2=a_0+\sum_{j=1}^{q}a_jX_{k-j}^2+\sum_{j=1}^{q'}c_j\rho_{k-j}^2, \tag{22}$$

where $(q',q)\in\mathbb{N}^2$, $a_0>0$, $a_j\ge0$ and $c_j\ge0$ for $j\in\mathbb{N}$ and $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random variables with zero mean (for an excellent survey about ARCH modelling, see Giraitis *et al.*, 2005). Under some additional conditions, similarly as in the case of ARMA models, the GARCH model can be written as a particular case of ARCH(∞) model (introduced in Robinson, 1991) that verified :

$$X_k=\rho_k\cdot\xi_k\quad\text{with}\quad\rho_k^2=b_0+\sum_{j=1}^{\infty}b_jX_{k-j}^2, \tag{23}$$

with a sequence $(b_j)_j$ depending on the family (a_j) and (c_j). Different sufficient conditions can be provided for obtaining a m-order stationary solution to (22) or (23). However, if (X_k) is a solution of (22) or (23), then (X_k^2) can be written as a solution of a particular case of equation (24) (see Giraitis *et al.*, 2005). More precisely,

$$X_k^2=\varepsilon_k\Big(\gamma\cdot b_0+\gamma\sum_{j=1}^{\infty}b_jX_{k-j}^2\Big)+\lambda_1\cdot b_0+\lambda_1\sum_{j=1}^{\infty}b_jX_{k-j}^2\quad\text{for}\quad k\in\mathbb{Z},$$

with $\varepsilon_k = (\xi_k^2 - \lambda_1)/\gamma$ for $k \in \mathbb{Z}$, $\lambda_1 = \mathbb{E}\xi_0^2$ and $\gamma^2 = \mathrm{Var}\,(\xi_0^2)$. Notice that the time series $(Y_k)_{k\in\mathbb{Z}}$ defined by

$$Y_k = X_k^2 - \lambda_1 \cdot b_0 \cdot \big(1 - \lambda_1 \sum_{j=1}^{\infty} b_j\big)^{-1} \quad \text{for} \quad k \in \mathbb{Z},$$

verifies an equation (24) with parameter $c_0 = 0$ (as in Proposition 2). As a consequence, a sufficient condition for the stationarity of $(X_k^2)_{k\in\mathbb{Z}}$ verifying $\|X_0^2\|_m < \infty$ is

$$(\|\varepsilon_0\|_m + 1) \cdot \sum_{j=1}^{\infty} |b_j| < 1 \qquad \Longleftrightarrow \qquad \Big(\frac{\|\xi_0^2 - \lambda_1\|_m}{\gamma} + 1\Big) \cdot \sum_{j=1}^{\infty} |b_j| < 1.$$

The question of the Whittle estimation of the parameter of a stationary solution of (22) was studied by Zaffaroni and d'Italia (published in 2003) and improved by Giraitis and Robinson (2001). The recent book by Straumann (2005) also provides an up-to-date and complete overview to this question; in Chapter 4, he recalls Giraitis and Robinson's results while chapter 8 is devoted to the results in Mikosch and Straumann (2002), related to the case of heavy tailed processes which we do not consider in the present paper: in this paper the cases of intermediate moment conditions of order > 4 are investigated for the special case of GARCH(1,1) processes; the convergence rates are proved to be slower than the present ones, already obtained by Giraitis and Robinson in this GARCH case.

Notice that for both models (22) or (23), the spectral density is a constant. As a consequence, the idea of Whittle estimation in the GARCH case, that was pointed out by Bollerslev (1986), is based on the ARMA representation verified by $(X_k^2)_{k\in\mathbb{Z}}$. The spectral density of $(X_k^2)_{k\in\mathbb{Z}}$ where $(X_k)_{k\in\mathbb{Z}}$ is a stationary solution of (23), is :

$$f^{X^2}_{\beta,\sigma^2}(\lambda) = \frac{\sigma^2}{2\pi} \cdot \Big|1 - \sum_{j=1}^{\infty} b_j(\beta) \cdot e^{ij\lambda}\Big|^{-2},$$

with $\beta = (\beta^{(1)}, \ldots, \beta^{(p)}) \in \mathbb{R}^p$ such that $b_j = b_j(\beta)$ for $j \in \mathbb{N}$, and $\sigma^2 = \mathbb{E}(X_0^2 - \rho_0^2) = b_0^2(\beta) \cdot h(\lambda_1, \gamma, \sum_{j=1}^{\infty} b_j(\beta))$, where h is a positive real function.

Therefore, the previous results on Whittle estimate for bilinear time series imply the following results for ARCH(∞) models :

Proposition 1 *If X be a stationary ARCH(∞) time series following equation (23), such that it exists $m > 8$ verifying $\mathbb{E}(|\xi_0|^m) < \infty$, with the following condition of stationarity,*

$$\Big(\big(\frac{\|\xi_0^2 - \lambda_1\|_{m/2}}{\|\xi_0^2 - \lambda_1\|_2} + 1\big) \;\wedge\; \|\xi_0\|_m^2\Big) \cdot \sum_{j=1}^{\infty} |b_j| < 1, \qquad \textit{and :}$$

- *Geometric decay : $\forall j \in \mathbb{N}$, $0 \le b_j$ and $\exists \mu \in]0,1[$ such that $\sum b_j\, \mu^{-j} < 1$;*
- *Riemannian decay : $\forall j \in \mathbb{N}$, $c_j \ge 0$, $\exists \nu > \dfrac{2m-8}{m-8}$ such that $a_j = \mathcal{O}(j^{-\nu})$;*

then, under Conditions C1-7, the central limit theorems (19) and (21) are verified.

Corollary 6 *If it exists $m > 8$ such that X is a m-ordre stationary GARCH(q',q) time series verifying equation (22), then with $\beta = (a_1, \ldots, a_q, c_1, \ldots, c_{q'})$ and $\sigma^2 = a_0^2 \cdot h(\lambda_1, \gamma, \sum_{j=1}^{\infty} b_j(\beta))$.*

Proof. First, we recall θ-weak dependence property of the ARCH(∞) obtained in Doukhan *et al* (2005) and for then inducing θ-weak dependence property for $(X_k^2)_{k\in\mathbb{Z}}$. In the "Geometric decay" case, $\theta_r = \mathcal{O}(e^{-c\sqrt{r}})$ with $c > 0$. In the "Riemannian decay" case, with $\nu > 2$, $\theta_r = \mathcal{O}\big(r^{-nu+1}\big)$. Now, after applying the following lemma 9 for $h(x) = x^2$ (section 5), *i.e.* $a = 2$, we deduce that $(X_k^2)_{k\in\mathbb{Z}}$ is $\theta^{1-1/m}$-weak dependent time series. The result of Corollary 4 implies that 1/ in the "Geometric decay" case, $(X_k^2)_{k\in\mathbb{Z}}$

verifies the Uniform C.L.T., 2/ in the "Riemannian decay" case, $(X_k^2)_{k\in\mathbb{Z}}$ verifies the Uniform C.L.T. if $\quad (1-\nu)\cdot\big(1-\dfrac{2}{m}\big)\cdot\dfrac{m/2-4}{m/2-1}<-1$, *i.e.* $\nu>\dfrac{2m-8}{m-8}$. ∎

Remarks. In Giraitis and Robinson (2001), the obtained results in term of the asymptotic normality of the Whittle estimate are better, in the sense that : 1/ only the $m=8$ is required; 2/ the required conditions on the sequence (b_j) the in the general case of ARCH(∞) model are only $b_0>0$ and $b_j\geq 0$ for $j\in\mathbb{N}^*$ and the stationarity condition $\|\xi_0\|_m^2\cdot\sum_{j=1}^\infty|b_j|<1$. However, the method developed in Giraitis and Robinson (2001) for establishing the central limit theorem verified by the periodogram is essentially *ad hoc* and can not be used for non-causal or non linear time series.

Bilinear processes

Giraitis and Surgailis (2002) introduced and studied the following bilinear process define by :

$$X_k=\xi_k\Big(a_0+\sum_{j=1}^\infty a_jX_{k-j}\Big)+c_0+\sum_{j=1}^\infty c_jX_{k-j}, \tag{24}$$

where $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random variables with zero mean and such that $\|\xi_0\|_p<+\infty$ with $p\geq 1$, and a_j, c_j, $j\in\mathbb{N}$ are real coefficients. Assume $c_0=0$ and define the generating functions :

$$A(z)=\sum_{j=1}^\infty a_jz^j \qquad\qquad C(z)=\sum_{j=1}^\infty c_jz^j$$
$$G(z)=(1-C(z))^{-1}=\sum_{j=0}^\infty g_jz^j \qquad H(z)=A(z)G(z)=\sum_{j=1}^\infty h_jz^j.$$

Then, if $\|\xi_0\|_p\cdot\sum_{j=1}^\infty|h_j|<\infty$, for instance if $\|\xi_0\|_p\cdot\sum_{j=1}^\infty|a_j|+\sum_{j=1}^\infty|c_j|<1$, then there exists a unique centered stationary and ergodic solution X in $\mathbb{L}^p(\Omega,\mathcal{A},\mathbb{P})$ of equation (24) (see Doukhan *et al.*, 2004). Then its covariogram is defined by

$$R_k=\frac{a_0^2\cdot\|\xi_0\|_2}{1-\sum_{j=1}^\infty h_j^2}\sum_{j=0}^\infty g_j\,g_{j+k},$$

and verified $\sum_k|R_k|<\infty$. If we assume that there exists $\beta=(\beta^{(1)},\ldots,\beta^{(p)})$ such that for all $k\in\mathbb{Z}$, $a_k=a_k(\beta)$ and $c_k=c_k(\beta)$, the spectral density of X exists and verifies :

$$f(\lambda)=f_{(\beta,\sigma^2)}(\lambda)=\frac{a_0^2(\beta)\cdot\sigma^2}{2\pi\big(1-\sum_{j=1}^\infty h_j^2(\beta)\big)}\sum_{k=-\infty}^\infty\sum_{j=0}^\infty g_j(\beta)\,g_{j+k}(\beta)\,e^{-k\lambda},$$

with $\sigma^2=\|\xi_0\|_2^2$. Like in Doukhan *et al.* (2004), we consider three different cases of the convergence rate to zero of the sequences (a_k) and (c_k), and provide the following proposition using the previous results for causal weak dependent time series.

Proposition 2 *If X be a bilinear time series verifying equation (24) with $c_0=0$, $\mathbb{E}(|\xi_0|^m)<\infty$ with $m>4$ and such that $\|\xi_0\|_m\cdot\sum_{j=1}^\infty|a_j|+\sum_{j=1}^\infty|c_j|<1$ and :*

- *Finite case : $\exists J\in\mathbb{N}$ such that $\forall j>J$, $a_j=c_j=0$;*
- *Geometric decay : $\exists\mu\in]0,1[$ such that $\sum_j|c_j|\mu^{-j}\leq 1$ and $\forall j\in\mathbb{N}$, $0\leq a_j\leq\mu^j$;*
- *Riemannian decay : $\forall j\in\mathbb{N}$, $c_j\geq 0$, $\exists\mu>\dfrac{2m-5}{m-4}$ such that $a_j=\mathcal{O}(j^{-\mu})$ and $\exists\nu>0$ such that $\sum_j c_jj^{1+\nu}<\infty$, with $\nu>\dfrac{(m-1)\delta}{(m-4)\delta-(m-1)\log 2}$ where $\delta=\log\Big(1+\dfrac{1-\sum_j|c_j|}{\sum_j c_jj^{1+\nu}}\Big)$.*

Then, under Conditions C1-7, the central limit theorems (19) and (21) are verified.

Proof. The three different cases of the Proposition are studied in Doukhan *et al.* (2004) and the θ-weak dependence behavior is deduced for each case. Thus, in the "Finite" and the "Geometric decay" cases,

$\theta_r = \mathcal{O}(e^{-c\sqrt{r}})$ with $c > 0$, that implies the conditions required in Corollary 4 and therefore the conditions of Theorem 5 and 6.
In the "Riemannian decay" case, $\theta_r = \mathcal{O}\Big(\big(\frac{r}{\log r}\big)^d\Big)$ with $d = \max\Big(-(\mu-1)\,;\,-\frac{\mu\cdot\delta}{\delta+\mu\cdot\log 2}\Big)$. As a consequence, from Corollary 4, the Uniform CLT (13) is verified if $d\cdot\frac{m-1}{m-4} < -1$, that is true for the conditions on μ and ν fixed in Proposition 2. ∎

Non-causal (two-sided) linear processes

Let X be a non causal (two-sided) linear time series verifying :

$$X_k = \sum_{j=-\infty}^{\infty} a_j \xi_{k-j} \quad \text{for} \quad k \in \mathbb{Z},$$

with $(a_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(\xi_k)_{k\in\mathbb{Z}})$ a sequence of centered independent identically distributed random variables such that $\mathbb{E}(\xi_0^2) = \sigma^2 < \infty$ and $\mathbb{E}(|\xi_0|^m) < \infty$ with $m \geq 4$. We assume that there exists $\beta = (\beta^{(1)}, \ldots, \beta^{(p)})$ such that for all $k \in \mathbb{Z}$, $a_k = a_k(\beta)$. Moreover, we assume that (a_k) is such that $a_k = \mathcal{O}(|k|^{-a})$ with $a > 1$, and therefore the spectral density of X exists and verifies :

$$f(\lambda) = f_{(\beta,\sigma^2)}(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{k=-\infty}^{\infty} a_k(\beta) e^{-ik\lambda}\right|^2.$$

As a consequence the writing of f is in form suggested in (18). Then the results of the previous paragraph can be applied.

Proposition 3 *If X be a linear time series verifying :*

$$X_k = \sum_{j=-\infty}^{\infty} a_j \xi_{k-j} \quad \textit{for} \quad k \in \mathbb{Z},$$

with $(a_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(\xi_k)_{k\in\mathbb{Z}})$ a sequence of centered independent identically distributed random variables such that $\mathbb{E}(\xi_0^2) = \sigma^2 < \infty$ and $\mathbb{E}(|\xi_0|^m) < \infty$ with $m > 4$. We assume that (a_k) is such that :

$$a_k = \mathcal{O}(|k|^{-a}) \quad \textit{with} \quad a > \max\Big\{\frac{7}{2}\,;\,\frac{5m-6}{2(m-4)}\Big\}.$$

Then, under Conditions C1-7, the central limit theorems (19) and (21) are verified.

Proof. A η-weak dependence condition of non causal linear random fields could be found in Doukhan and Lang (2002, p. 3); under the previous assumptions, X is a η-weak dependent time series with the relation :

$$\eta_{2r}^2 = \mathcal{O}\big(\sum_{|k|>r} a_k^2\big) \quad \Longrightarrow \quad \eta_r = \mathcal{O}\big(\frac{1}{r^{a-1/2}}\big).$$

The Proposition 3 is then a consequence of Theorem 3. ∎

Remarks. 1/ The Condition C8 of Theorem (21) is automatically verified by the convergence rate of (a_k) and therefore is not required in Proposition 3;

2/ To our knowledge, the known results about asymptotic behavior of Whittle parametric estimation for non-gaussian linear processes are essentially devoted to one-sided (causal) linear processes (see for instance, Hannan, 1973, Hall and Heyde, 1980, Rosenblatt, 1985, Brockwell and Davis, 1988). In such a case, the conditions on (a_k) are Conditions C1-7, with : $\sum_k k\, a_k^2 < \infty$ (as in Corollary 1) for the UCLT and the existence of $\sum_k k\, a_k e^{-ik\lambda}$ for Condition 8. It is such a case if $m = 4$ and $a_k = \mathcal{O}(|k|^{-a})$ with $a > 2$.

3/ There exist very few results in the case of two-sided linear processes. In Rosenblatt (2000, p. 52) a condition for strong mixing property for two-sided linear processes is provided, but some restrictive conditions on the process are also required for obtaining a central limit theorem for Whittle estimators : the distribution of random variables ξ_k has to be absolutely continuous with respect to the Lebesgue measure with a bounded variation density, $m > 4 + 2\delta$ with $\delta > 0$ and a central limit theorem obtained with a tapered periodogram (under assumption also $\sum_{m=1}^{\infty} \alpha_{4,\infty}(m)^{\delta/(2+\delta)} < \infty$ where $\alpha_{4,\infty}(m) \geq \alpha_m$ denote a strong mixing coefficient define now with four points in the future instead of 2 for α'_m (the same remark following Corollary 3 still holds). The case of strongly dependent two-sided linear processes was also treated by Giraitis and Surgailis (1990) or Horvath and Shao (1999), however with more restrictive conditions than Conditions C1-7 and with $a_k = \mathcal{O}(|k|^{-a})$ for a fixed $-1 < a < 0$.

4/ In such case of linear processes, it is well known that : $\sqrt{n}(\widehat{\beta}_n - \beta^*) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_p\big(0\,,\,2\pi \cdot (W^*)^{-1}\big)$, $\widehat{\sigma}_n^2$ is a consistent estimate of σ^4 and therefore $\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^*) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\big(0\,,\sigma^{*4} \cdot \gamma_4\big)$, with γ_4 the fourth cumulant of the $(\xi_k)_{k\in\mathbb{Z}}$), and $\sqrt{n}(\widehat{\beta}_n - \beta^*)$ and $\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^*)$ are asymptotically normal.

Non-causal (two-sided) ARCH(∞) processes

The asymptotic normality of Whittle estimate may be obtained for a non-causal ARCH(∞) process. This class of times series is a natural generalization of causal ARCH(∞) process and was first introduced by Doukhan *et al.* (2005). A two-sided ARCH(∞) sequence $(X_k)_{k\in\mathbb{Z}}$ is defined by :

$$X_k = \xi_k\Big(a_0 + \sum_{j\neq 0} a_j X_{k-j}\Big), \quad \text{for } k \in \mathbb{Z}, \tag{25}$$

where $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random **bounded** variables with zero mean and $(a_k)_{k\in\mathbb{Z}}$ is a sequence of real numbers such that :

$$\lambda = \|\xi_0\|_\infty \cdot \sum_{j\neq 0} |a_j| < 1.$$

Such a condition implies the existence and the stationarity in $\mathbb{L}^k$ (for any $k \in]0,\infty]$) of a solution of (25). The following proposition specifies a behavior of the sequence $(a_k)_{k\in\mathbb{Z}}$ that implies a η-weak dependence of the times series and the normality of the Whittle estimate :

Proposition 4 *If X be a stationary non-causal ARCH(∞), i.e. a stationary solution of (25), such that $\|\xi_0\|_\infty \cdot \sum_{j\neq 0} |a_j| < 1$. We assume that he sequence $(a_k)_{k\in\mathbb{Z}}$ is such that :*

$$a_k = \mathcal{O}(|k|^{-a}) \quad \textit{with} \quad a > 4.$$

Then, under Conditions C1-7, the central limit theorems (19) and (21) are verified.

Proof. A η-weak dependence condition of non causal ARCH(∞) could be found in Doukhan *et al.* (2005) : under the previous assumptions, X is a η-weak dependent time series with the relation :

$$\eta_r = \mathcal{O}\Big(\sum_{2k<r} k \cdot \lambda^{k-1} \big(\sum_{|j|\geq r/k} |a_j|\big)\Big) \quad \Longrightarrow \quad \eta_r = \mathcal{O}\big(\frac{1}{r^{a-1}}\big).$$

The Proposition 4 is then a consequence of Theorem 3. ∎

Remarks. The condition on the sequence $(\xi_k)_{k\in\mathbb{Z}}$, *i.e.* i.i.d. random **bounded** variables, is restricting. However, if it is only a sufficient condition for the existence of a non causal ARCH(∞) process, it seems to be very close to be also a necessary condition (see Doukhan *et al.*, 2005).

4 Appendix: non-causal weak dependence

4.1 A general central limit theorem

We consider the following empirical mean,

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} h(x_k) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is a function and $(x_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{R}^d$ is a stationary centered sequence that verifying certain conditions. We will study the case where S_n verifies a central limit theorem,

$$S_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2) \quad \text{with} \quad \sigma^2 = \sum_{k=-\infty}^{\infty} \mathrm{Cov}\left(h(x_0), h(x_k)\right) < \infty$$

(in section 4.3, we shall exhibit a condition for this series to converge).

More precisely, the aim of the following subsections will be to precise conditions to obtain a decay rate to 0 of $\left|\Delta_n\right|$ when $\eta_r = \mathcal{O}\left(r^{-\alpha}\right)$ for some suitable $\alpha > 0$, with

$$\Delta_n = \mathbb{E}\left(\phi(S_n) - \phi(Z)\right), \tag{26}$$

for ϕ a $\mathcal{C}^3(\mathbb{R})$ function with bounded derivatives up to order 3, and $Z \sim \mathcal{N}(0,\sigma^2)$.

Assumptions A on the sequence $(x_n)_n$:

1. the sequence $(x_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{R}^d$ is endowed with the norm $|(u_1,\dots,u_d)| = \max\{|u_1|,\dots,|u_d|\}$;
2. there exists $m-$th order moments for $(x_n)_n$ with $m > 4$;
3. $(x_n)_{n\in\mathbb{Z}}$ is $\left(\eta, \Lambda^{(1)}, \psi_1\right)-$weakly dependent, that is for all g_1, g_2 functions of $\Lambda^{(1)}$ respectively defined on $\mathbb{R}^u$ and $\mathbb{R}^v$ $(u,v\in\mathbb{N}^*)$, there exists a sequence $\eta = (\eta_r)_{r\in\mathbb{N}}$ decreasing to zero at infinity such that for all u-tuple $(i_1,\dots,i_u)$ and all v-tuple $(j_1,\dots,j_v)$ with $i_1 \le \cdots \le i_u < i_u + r \le j_1 \le \cdots \le j_v$,
$$\left|\mathrm{Cov}\left(g_1(x_{i_1},\dots,x_{i_u}), g_2(x_{j_1},\dots,x_{j_v})\right)\right| \le \Big(u\cdot(\mathrm{Lip}\, g_1)\cdot\|g_2\|_\infty + v\cdot(\mathrm{Lip}\, g_2)\cdot\|g_1\|_\infty\Big)\eta_r;$$
4. the sequence $\eta = (\eta_r)_{r\in\mathbb{N}}$ verifies :
$$\eta_r = \mathcal{O}\left(r^{-\alpha}\right) \quad \text{with} \quad \alpha > 0. \tag{27}$$

Assumptions H on the function h :

1. $\mathbb{E}(h(x_0)) = 0$;
2. There exists $a \ge 1$ and $A = A(d) \ge 1$ such as for all $u, v \in \mathbb{R}^d$,
$$\begin{cases} |h(u)| & \le \quad A(|u|^a \vee 1) \\ |h(u) - h(v)| & \le \quad A\left(\left(|u|^{a-1} + |v|^{a-1}\right)\vee 1\right)|u-v| \end{cases}.$$

Examples. A natural example of such frame is related to the empirical covariance where $x_n = (X_n, X_{n+\ell})$ and $h(u_1,u_2) = u_1u_2 - R_\ell$. In this case $a = 2$. Extensions to the higher order spectral functions are straightforward with $a = d$ and $x_n = (X_0, X_{n+\ell_1},\dots,X_{n+\ell_{d-1}})$ and $h(u_1,\dots,u_d) = (u_1u_2\cdots u_d) - \mathbb{E}(X_0X_{n+\ell_1}\cdots X_{n+\ell_{d-1}})$. More general polynomials may also be considered as well as functions with polynomial growth (and derivatives with polynomial growth).

As a consequence of the assumptions on h and $(x_n)_{n\in\mathbb{Z}}$, and in view of using a Bernstein's blocks technique, we prove the following theorem :

Theorem 7 *Let h and $(x_n)_{n\in\mathbb{Z}}$ satisfy respectively assumptions* **H** *and* **A**, *with $m > 2a$. Then, for any $N \sim \mathcal{N}(0,1)$ random variable, for any $\phi \in \mathcal{C}^3(\mathbb{R})$ with bounded derivatives, if $\alpha > \max\Big(3\,;\,\dfrac{2m-1}{m-2a}\Big)$, there exists $c > 0$ such that :*

$$\Big|\mathbb{E}(\phi(S_n) - \phi(\sigma\cdot N))\Big| \le c\cdot n^{-\lambda} \quad \textit{with} \quad \lambda = \frac{\alpha(m-2a)-2m+1}{2(m+a-1+\alpha\cdot m)}, \tag{28}$$

and $\sigma^2 = \displaystyle\sum_{k=-\infty}^{\infty} Cov\,(h(x_0), h(x_k))$.

Remark. 1/ We consider here the estimate of Dudley $\Big|\mathbb{E}(\phi(S_n) - \phi(\sigma\cdot N))\Big|$. An interesting application of the majoration (28) consists on a measure of the distance between S_N and its Gaussian approximation. Indeed, the result may be specified and the bound may also be written with a constant $c = \big(\|\phi'\|_\infty + \|\phi'\|_\infty + \|\phi'\|_\infty\big)\cdot C$ for some C which does not depend on ϕ. This implies, arguing as in Doukhan (1994), that there exists some $c' > 0$ such that :

$$\sup_{t\in R} |\mathbb{P}(S_n \le t) - P(\sigma\cdot N \le t)| \le C'\cdot n^{-\lambda/4},$$

for $n\in\mathbb{N}$. Unfortunately, this rate is far from being optimal as stressed by Rio (2000) which obtains rates $n^{-\rho}$ for some $\rho < 1/3$ in the case of strongly mixing sequences.

2/ If α and m are large enough, then $\lambda \to \dfrac{1}{2}$. In such a case, the rate is not so far from the i.i.d. sequences's rate.

The following subsections are devoted to prove this result.

From now on, $c > 0$ denotes a constant which may vary from one line to the other.

4.2 Truncation

We now define a truncation in order to be able to use the previous dependence condition and make Lindeberg technique work. For $T > 1$, define $f_T(x) = (x \wedge T) \vee (-T)$ for $x \in \mathbb{R}$. Then Lip $f_T = 1$, $\|f_T\|_\infty = T$. For $(u_1, \ldots, u_d) \in \mathbb{R}^d$, we denote

$$F_T(u_1, \ldots, u_d) = (f_T(u_1), \ldots, f_T(u_d))$$

and

$$Y_i = h(x_i), \qquad Y_i^{(T)} = h(F_T(x_i)) - \mathbb{E}\big[h(F_T(x_i))\big], \quad E_i^{(T)} = Y_i - Y_i^{(T)} \tag{29}$$

Lemma 6 *Let h and $(x_n)_{n\in\mathbb{Z}}$ satisfy respectively assumptions* **H** *and* **A**, *with $m > 2a$. Then,*

- ***a)*** $\mathbb{E}\left[|E_0^{(T)}|\right] \le c\cdot A\cdot T^{a-m}$ *and* $\mathbb{E}\left[(E_0^{(T)})^2\right] \le c\cdot A^2\cdot T^{2a-m}$;
- ***b)*** *for all $i\in\mathbb{Z}$, $|Cov\,(Y_0^{(T)}, E_i^{(T)})| \le \mathbb{E}\,(|Y_0^{(T)}|, |E_i^{(T)}|) \le c\cdot A^2\cdot T^{2a-m}$;*
- ***c)*** *for all $i\in\mathbb{Z}$, $|Cov\,(Y_0^{(T)}, Y_i^{(T)})| \le c\cdot A^2\cdot T^{2a-1}\cdot\eta_i$.*

Proofs. First note that for $\gamma \ge 1$ such as $a\gamma \le m$, from assumptions on h,

$$\begin{aligned}\mathbb{E}\left[\Big|h(x_0) - h(F_T(x_0))\Big|^\gamma\right] &\le A^\gamma\cdot\mathbb{E}\left[\Big|(|x_0|^{a-1} + |F_T(x_0)|^{a-1})\cdot|x_0 - F_T(x_0)|\Big|^\gamma\right] \\ &\le (2A)^\gamma\cdot\mathbb{E}\left[|x_0|^{a\gamma}\cdot\mathbb{I}_{\{|x_0|\ge T\}}\right] \\ &\le (2A)^\gamma\cdot\mu\cdot T^{\gamma a-m} \quad \text{(Markov inequality)}.\end{aligned} \tag{30}$$

a) The assumptions on h lead to

$$\begin{aligned}
\mathbb{E}\left[|E_0^{(T)}|\right] &\leq \mathbb{E}\left[|h(F_T(x_0)) - h(x_0)|\right] + \mathbb{E}\left[|h(F_T(x_0))|\right] \\
&\leq 2\mathbb{E}\left[|h(F_T(x_0)) - h(x_0)|\right].
\end{aligned}$$

Now the relation (30) with $\gamma = 1$ leads to $\mathbb{E}\left[|h(F_T(x_0)) - h(x_0)|\right] \leq 2A \cdot \mu \cdot T^{a-m}$. Then,

$$\mathbb{E}\left[|Y_{T,0}|\right] \leq 4A \cdot \mu \cdot T^{a-m}.$$

By the same arguments,

$$\begin{aligned}
\mathbb{E}\left[(E_0^{(T)})^2\right] &\leq 4\mathbb{E}\left[(h(F_T(x_0)) - h(x_0))^2\right] \\
&\leq 16A^2 \cdot \mu \cdot T^{2a-m} \quad \text{(relation (30) with } \gamma = 2).
\end{aligned}$$

b) Analogously, relation (30) with Hölder inequality yields

$$\begin{aligned}
|\mathrm{Cov}\,(Y_0^{(T)}, E_i^{(T)})| &\leq \left\|Y_0^{(T)}\right\|_{m/a} \cdot \left(\mathbb{E}\left[\left|E_0^{(T)}\right|^{\frac{m}{m-a}}\right]\right)^{\frac{m-a}{m}} \quad \text{(Hölder inequality)} \\
&\leq 2\left\|h(F_T(x_0))\right\|_{m/a} \cdot 2\left(\mathbb{E}\left(|h(x_0) - h(F_T(x_0))|^{\frac{m}{m-a}}\right)\right)^{\frac{m-a}{m}} \\
&\leq 2A \cdot \left\||x_0|^a \vee 1\right\|_{m/a} \cdot 2A \cdot \mu^{\frac{m-a}{m}} \cdot \left(T^{-m(1-\frac{a}{m-a})}\right)^{\frac{m-a}{m}} \quad \text{(assumptions on } h) \\
&\leq c \cdot A^2 \cdot T^{2a-m}.
\end{aligned}$$

c) Let $h^{(T)}(u) = h(F_T(u)) - \mathbb{E}\left[h(F_T(x_0))\right]$ for $u \in \mathbb{R}^d$. From assumptions on h, it can easily be shown $\|h^{(T)}\|_\infty \leq 2A \cdot T^a$ and $\mathrm{Lip}\, h^{(T)} \leq 2A \cdot T^{a-1}$. Consequently, with the weak dependence inequality,

$$\begin{aligned}
|\mathrm{Cov}\,(Y_0^{(T)}, Y_i^{(T)})| &\leq |\mathrm{Cov}\,(h^{(T)}(x_0), h^{(T)}(x_i))| \\
&\leq 8A^2 \cdot T^{2a-1} \cdot \eta_i.
\end{aligned}$$

∎

4.3 Variances

When it exists, we put : $\sigma^2 = \sum_{i=-\infty}^{\infty} \mathrm{Cov}\,(h(x_0)), h(x_i)) = \sum_{i=-\infty}^{\infty} \mathrm{Cov}\,(Y_0, Y_i)$. Now, we are going to approximate σ^2 by $\frac{1}{p} \cdot \sigma_p^2$, where :

$$\sigma_p^2 = \mathrm{Var}\left(\sum_{i=1}^{p} Y_i\right)$$

Then, the two following results can be shown :

Lemma 7 *Let h and $(x_n)_{n\in\mathbb{Z}}$ satisfy respectively assumptions* **H** *and* **A**, *with $m > 2a$ and if*

$$\alpha > \frac{m-1}{m-2a} \quad \textit{that implies} \quad \sum_{i=1}^{\infty} \eta_i^{\frac{m-2a}{m-1}} < \infty, \quad \textit{then :} \tag{31}$$

a) *The series σ^2 converges;*

b) *There is a constant $c > 0$ such that*

$$\left|\sigma^2 - \frac{\sigma_p^2}{p}\right| \leq c \cdot \left(\frac{\log p}{p} + \left(\frac{1}{p}\right)^{\left(\frac{\alpha(m-2a)}{m-1} - 1\right)}\right). \tag{32}$$

Proof. **a)** We assume that $\eta_i \le 1$ for each $i \ge 0$. With $T_i \ge 1$ for $i \in \mathbb{Z}$, we write

$$\mathrm{Cov}\,(Y_0, Y_i) = \mathrm{Cov}\,(E_0^{(T_i)}, E_i^{(T_i)}) + \mathrm{Cov}\,(Y_0^{(T_i)}, E_i^{(T_i)}) + \mathrm{Cov}\,(Y_i^{(T_i)}, E_0^{(T_i)}) + \mathrm{Cov}\,(Y_0^{(T_i)}, Y_i^{(T_i)}).$$

From the previous lemma, $|\mathrm{Cov}\,(Y_0, Y_i)| \le c(T_i^{2a-m} + T_i^{2a-1} \cdot \eta_{|i|})$. Now, set $T_i^{2a-m} = T_i^{2a-1} \cdot \eta_{|i|}$, then $T_i = \eta_{|i|}^{-\frac{1}{m-1}} \ge 1$ and

$$|\mathrm{Cov}\,(Y_0, Y_i)| \le c \cdot \eta_i^{\frac{m-2a}{m-1}}. \tag{33}$$

As a consequence, $\displaystyle\sum_{i=-\infty}^{\infty} |\mathrm{Cov}\,(Y_0, Y_i)| \le c \cdot \sum_{i=-\infty}^{\infty} \eta_i^{\frac{m-2a}{m-1}}$ and σ^2 exists.

b) Decompose $\sigma^2 - \dfrac{\sigma_p^2}{p} = D_1 + D_2$ with $D_1 = \sum_{|i| \ge p} \mathrm{Cov}\,(Y_0, Y_i)$ and $D_2 = \frac{1}{p} \sum_{|i|<p} |i| \cdot \mathrm{Cov}\,(Y_0, Y_i)$. From assumption (31), we conclude as above with inequality (33), because :

- $\big|D_1\big| \le c \cdot \displaystyle\sum_{i \ge p} \eta_i^{\frac{m-2a}{m-1}} \le c \cdot \Big((\frac{1}{p})^{\left(\frac{\alpha(m-2a)}{m-1} - 1\right)} \Big)$, and
- $|D_2| \le \dfrac{c}{p} \cdot \displaystyle\sum_{|i|<p} |i| \cdot \eta_{|i|}^{\frac{m-2a}{m-1}} = c \cdot \Big(\frac{\log p}{p} + (\frac{1}{p})^{\left(\frac{\alpha(m-2a)}{m-1} - 1\right)} \Big)$, following the two different cases :

  $\alpha \ \ge 2 \cdot \dfrac{m-1}{m-2a}$ or $\dfrac{m-1}{m-2a} < \alpha < 2 \cdot \dfrac{m-1}{m-2a}$. ∎

4.4 A $(2+\delta)$ order moment inequality

For $p \in \mathbb{N}^*$, define : $W_p = \displaystyle\sum_{i=1}^{p} Y_i$. We now extend the idea in [12] to derive the following bound :

Lemma 8 *Let h and $(x_n)_{n \in \mathbb{Z}}$ satisfy respectively assumptions* **H** *and* **A***, with $m > 2a$. Then, if $\alpha > 3$, for all $0 < \delta < \dfrac{m-2a}{a}$, there exists a constant $c > 0$ such that :*

$$\mathbb{E}|W_p|^{2+\delta} \le c \cdot p^r \quad \textit{with} \quad \frac{2+\delta}{2} \le r = 2 + \delta - \frac{m - 2a - a \cdot \delta}{m-1} < 2 + \delta.$$

Proof. Let $\Delta = 2 + \delta$ and $m = a(2+\zeta)$. With inequality (30) and $W_p^{(T)} = \sum_{i=1}^{p} Y_i^{(T)}$, we obtain :

$$\|W_p\|_\Delta \le \|W_p^{(T)}\|_\Delta + p\|Y_0 - Y_0^{(T)}\|_\Delta \le \|W_p^{(T)}\|_\Delta + c \cdot p \cdot T^{a - \frac{m}{\Delta}}.$$

The Hölder inequality provides :

$$\mathbb{E}|W_p^{(T)}|^\Delta \le \left(\mathbb{E}|W_p^{(T)}|^2 \right)^{1-\delta/2} \left(\mathbb{E}|W_p^{(T)}|^4 \right)^{\delta/2}$$

Now from **c)** of Lemma 6, we obtain $\mathbb{E}|W_p^{(T)}|^2 \le c \cdot p \cdot T^{2a-1} \displaystyle\sum_{i=0}^{\infty} \eta_i$. Setting

$$C_{r,T} = \max_{u=1,2,3} \sup_{s_{u+1} - s_u = r} \left| \mathrm{Cov}\left(\prod_{i=1}^{u} Y_{s_i}^{(T)}, \prod_{i=u+1}^{4} Y_{s_i}^{(T)} \right) \right|$$

where this supremum is set over $s_1 \le s_2 \le s_3 \le s_4$, we obtain as in [9],

$$\mathbb{E}|W_p^{(T)}|^4 \le c \left(p \sum_{k=0}^{p-1} (k+1)^2 C_{k,T} + \left(p \cdot T^{2a-1} \sum_{i=0}^{\infty} \eta_i \right)^2 \right).$$

We quote that $C_{k,T} \le T^{4a-1}\eta_k$ to derive

$$\mathbb{E}|W_p^{(T)}|^4 \le c\left(p\cdot T^{4a-1} + \left(p\cdot T^{2a-1}\right)^2\right).$$

Thus, from previous inequalities and with $m = (2+\zeta)a$,

$$\begin{aligned}\mathbb{E}|W_p|^\Delta &\le c\left(p^\Delta \cdot T^{a\Delta-m} + \left(p\cdot T^{2a-1}\right)^{1-\delta/2} \times \left(p\cdot T^{4a-1} + p^2\cdot T^{4a-2}\right)^{\delta/2}\right)\\ &\le c\left(p^\Delta T^{a(\delta-\zeta)} + \left(p\cdot T^{2a-1}\right)^{\Delta/2} + p\cdot T^{a\Delta-1}\right).\end{aligned}$$

We now optimize this last inequality in p by setting $T = p^b$ with $b > 0$ and choosing b such that the right member is minimum. With the condition $\delta < a\zeta$, we first show that it is necessary to have $b < 1$ and the optimal b is obtained by balancing of $p^\Delta T^{a(\delta-\zeta)}$ and $p \cdot T^{a\Delta-1}$. This value of b is :

$$b = \frac{1+\delta}{m-1},$$

that verifies $b < 1$. We thus obtain

$$\mathbb{E}|W_p|^\Delta \le c\cdot p^{2+\delta-\frac{a(\zeta-\delta)}{m-1}},$$

that implies the result of the lemma. ∎

Remark Notice that $r = 2+\delta - \dfrac{m-2a-a\cdot\delta}{m-1} > \dfrac{1}{2}$, contrarily to the classical Marcinkiewicz-Zygmund inequalities.

4.5 Bernstein blocks : Proof of Theorem 7.

We now consider three sequences of positive integers $p = (p(n))_{n\in\mathbb{N}}$, $q = (q(n))_{n\in\mathbb{N}}$ and $k = (k(n))_{n\in\mathbb{N}}$ such that :

- $\lim_{n\to\infty} \dfrac{p(n)}{n} = \lim_{n\to\infty} \dfrac{q(n)}{p(n)} = 0;$
- $k(n) = \left[\dfrac{n}{p(n)+q(n)}\right]$ (thus $\lim_{n\to\infty} k(n) = \infty$).

In order to fix their dependence, these sequences are chosen as

$$p(n) = [n^\beta], \qquad q(n) = [n^\gamma], \qquad \text{with } 0<\gamma<\beta<1,$$

the exponents β and γ will be chosen below. We form the blocks $I_1, \dots, I_k$ and J such as :

$$\begin{aligned} I_j &= \Big\{(j-1)(p(n)+q(n))+1, \dots, (j-1)(p(n)+q(n))+p(n)\Big\} \text{ for } j = 1,\dots,k(n);\\ U_j &= \sum_{i\in I_j} Y_i, \quad \text{for } j=1,\dots,k(n).\end{aligned}$$

Then expression (26) is decomposed as :

$$\Delta_n = \sum_{\ell=1}^{3} \Delta_{\ell,n},$$

where we set, for a standard Gaussian $N \sim \mathcal{N}(0,1)$,

$$\begin{aligned}
\Delta_{1,n} &= \mathbb{E}\left(\phi(S_n) - \phi\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{k} U_j\right)\right), \\
\Delta_{2,n} &= \mathbb{E}\left(\phi\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{k} U_j\right) - \phi\left(N\sigma_p\sqrt{\frac{k}{n}}\right)\right), \\
\Delta_{3,n} &= \mathbb{E}\left(\phi\Big(N\sigma_p\sqrt{\frac{k}{n}}\Big) - \phi(\sigma N)\right).
\end{aligned}$$

Term $\Delta_{1,n}$**.** Using inequality (33) we derive with a Taylor expansion up to order 2 :

$$\begin{aligned}
|\Delta_{1,n}| &\le c \cdot \frac{k(n)\cdot q(n) + p(n)}{n}\,\frac{\|\phi''\|_\infty}{2}\sum_{i=0}^{\infty}\eta_i^{(m-2a)/(m-1)} \\
&\le c\cdot\left(n^{\beta-1} + n^{\gamma-\beta}\right). \qquad (34)
\end{aligned}$$

Term $\Delta_{3,n}$**.**

Now, Taylor formula implies :

$$\begin{cases} f\left(N\sigma_p\sqrt{\frac{k}{n}}\right) = \phi(0) + N\sigma_p\sqrt{\frac{k}{n}}\phi'(0) + \frac{1}{2}N^2\sigma_p^2\frac{k}{n}\phi''(V_1); \\ \phi\left(N\sigma\right) = \phi(0) + N\sigma\phi'(0) + \frac{1}{2}N^2\sigma^2\phi''(V_2), \end{cases}$$

with V_1 and V_2 two random variables. Then, with Lemma 7,

$$\begin{aligned}
\left|\Delta_{3,n}\right| &\le \|\phi''\|_\infty \cdot \left|\frac{k(n)}{n}\,\sigma_p^2 - \sigma^2\right| \\
&\le \|\phi''\|_\infty \cdot \left(\frac{p(n)\cdot k(n)}{n}\left|\sigma^2 - \frac{1}{p(n)}\sigma_p^2\right| + \frac{n - p(n)\cdot k(n)}{n}\sigma^2\right) \\
&\le c\cdot\left(\log(p(n))\cdot p^{-1}(n) + p(n)^{1-\frac{\alpha(m-2a)}{m-1}} + \frac{q(n)}{p(n)}\right)
\end{aligned}$$

$$\text{and therefore}\quad \left|\Delta_{3,n}\right| \le c\cdot\left(n^{-\beta}\cdot\log n + n^{\beta-\frac{\alpha\cdot\beta\cdot(m-2a)}{m-1}} + n^{\gamma-\beta}\right). \qquad (35)$$

Term $\Delta_{2,n}$**.** Let $(N_j)_{1\le j\le k(n)}$ be independent $\mathcal{N}\Big(0,\sigma_p^2\Big)$−Gaussian random variables, independent of the process $(x_i)_{i\in\mathbb{Z}}$ (such variables classically exist if the underlying probability space is rich enough).

We set $\phi_j(t) = \mathbb{E}\left(\phi\left(\frac{1}{\sqrt{n}}t + \frac{1}{\sqrt{n}}\sum_{j<i\le k} N_i\right)\right)$. In the sequel, for simplicity, empty sums are set equal to 0. Then :

$$\begin{aligned}
\Delta_{2,n} &= \mathbb{E}\left(\phi\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{k} U_j\right) - \phi\left(N\sigma_p\sqrt{\frac{k}{n}}\right)\right) \\
&= \mathbb{E}\left(\sum_{j=1}^{k}\phi\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{j} U_j + \frac{1}{\sqrt{n}}\sum_{i=j+1}^{k} N_i\Big) - \phi\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{j-1} U_j + \frac{1}{\sqrt{n}}\sum_{i=j}^{k} N_i\right)\right) \\
&= \sum_{j=1}^{k}\mathbb{E}\nu_{j,n},
\end{aligned}$$

with $\nu_{j,n} = \phi_j\left(Z_j + U_j\right) - \phi_j\left(Z_j + N_j\right)$ and $Z_j = \sum_{i=1}^{j-1} U_i$.

Moreover, $\|\phi_j^{(\ell)}\|_\infty \le n^{-\ell/2}\|\phi^{(\ell)}\|_\infty$ for $\ell = 0,1,2,3$. Making two distinct Taylor expansions (up to order 2 and 3 respectively) we obtains the two following expressions with some random variables L_j for $j = 1, 2, 3, 4$:

$$\begin{aligned}
\nu_{j,n} - \left[\phi_j'(Z_j)(U_j - N_j) + \frac{1}{2}\phi_j''(Z_j)(U_j^2 - N_j^2)\right] &= \frac{1}{6}\left(\phi_j^{(3)}(L_1)U_j^3 - \phi_j^{(3)}(L_2)N_j^3\right) \\
&= \frac{1}{2}\left[(\phi_j''(L_3) - \phi_j''(Z_j))U_j^2 \right. \\
&\quad \left. -(\phi_j''(L_4) - \phi_j''(Z_j))N_j^2\right] \\
\left|\nu_{j,n} - \left[\phi_j'(Z_j)(U_j - N_j) + \frac{1}{2}\phi_j''(Z_j)(U_j^2 - N_j^2)\right]\right| &\leq c\left(\frac{|U_j|^2}{n} \wedge \frac{|U_j|^3}{n^{3/2}} + \frac{|N_j|^2}{n} \wedge \frac{|N_j|^3}{n^{3/2}}\right) \\
&\leq \frac{c}{n^{1+\delta/2}}(|U_j|^{2+\delta} + |N_j|^{2+\delta})
\end{aligned}$$

because the sequence $(N_j)_j$ is independent of the sequence $(x_j)_j$, and thus independent of the sequence $(U_j)_j$, and with the two relations $\mathbb{E}U_j^2 = \sigma_p^2 = \mathbb{E}N_j^2$ and $s^2 \wedge s^3 \leq s^{2+\delta}$ with $\delta \in [0, 1]$ (that is valid for all $s \geq 0$). Now with the inequality $\mathbb{E}|N_j|^{2+\delta} = \left(\mathbb{E}U_j^2\right)^{1+\delta/2}\mathbb{E}|\mathcal{N}(0,1)|^{2+\delta} \leq c \cdot \mathbb{E}|U_j|^{2+\delta}$ we derive

$$|\mathbb{E}\nu_{j,n}| \leq |\mathrm{Cov}\,(\phi_j'(Z_j), U_j)| + \frac{1}{2} \cdot |\mathrm{Cov}\,(\phi_j''(Z_j), U_j^2)| + \frac{c}{n^{1+\delta/2}} \cdot \mathbb{E}|U_j|^{2+\delta}.$$

Thus, using lemma 8

$$\begin{aligned}
\left|\Delta_{2,n}\right| &\leq \sum_{j=1}^{k(n)}\left(C_j + C_j' + c \cdot n^{-1-\delta/2}p^r\right) \\
&\leq c \cdot n^{-\delta/2}p^{r-1} + \sum_{j=1}^{k(n)}(C_j + C_j'),
\end{aligned} \tag{36}$$

where

$$\left\{\begin{array}{lcl} C_j &=& \left|\mathrm{Cov}\,(\phi_j'(Z_j), U_j)\right|, \\ C_j' &=& \frac{1}{2}\left|\mathrm{Cov}\,(\phi_j''(Z_j), U_j^2)\right|. \end{array}\right.$$

Now, we can write the random variables U_j, U_j^2, $\phi_j'(Z_j)$, $\phi_j''(Z_j)$ as functions $G : (\mathbb{R}^d)^u \to \mathbb{R}$ of $x_{i_1}, \ldots, x_{i_u}$. The important characteristics of such G are driven by the following respective orders :

Random variable	Order w	$\\|G\\|_\infty$	$\mathrm{Lip}\, G$
$U_j^{(T)}$	$p(n)$	$\mathcal{O}\left(A \cdot p(n)T^a\right)$	$\mathcal{O}\left(A \cdot p(n)T^{a-1}\right)$
$(U_j^{(T)})^2$	$p(n)$	$\mathcal{O}\left(A^2 \cdot p(n)^2T^{2a}\right)$	$\mathcal{O}\left(A^2 \cdot p(n)^2T^{2a-1}\right)$
$\phi_j'(Z_j)$	$\leq n$	$\leq n^{-1/2}\\|\phi'\\|_\infty$	$\mathcal{O}\left(A \cdot T^{a-1}n^{-1}\right)$
$\phi_j''(Z_j)$	$\leq n$	$\leq n^{-1}\\|\phi''\\|_\infty$	$\mathcal{O}\left(A \cdot T^{a-1}n^{-3/2}\right)$

In order to use the weak dependence device for these two random variables C_j and C_j', we have to use truncation $U_j^{(T)}$ obtained by replacing Y_i's by $Y_i^{(T)}$ and then,

$$\left\{\begin{array}{lcl} C_j &\leq& C_j^{(T)} + c \cdot \|\phi'\|_\infty \cdot \dfrac{p(n)}{\sqrt{n}} \cdot \mathbb{E}|E_0^{(T)}| \ \text{ with } \ C_j^{(T)} = \left|\mathrm{Cov}\,(\phi_j'(Z_j), U_j^{(T)})\right|; \\ C_j' &\leq& C_j'^{(T)} + c \cdot \|\phi''\|_\infty \cdot \dfrac{p^2(n)}{n} \cdot \mathbb{E}\,|Y_0^2 - (Y_0^{(T)})^2| \ \text{ with } \ C_j'^{(T)} = \dfrac{1}{2}\left|\mathrm{Cov}\,(\phi_j''(Z_j), (U_j^{(T)})^2)\right|. \end{array}\right.$$

From the previous bounds, we obtain :

$$\begin{aligned}
C_j^{(T)} &\leq c \cdot A^2 \cdot \left(p(n) \cdot T^{2a-1} + \|\phi'\|_\infty \cdot p(n)^2 \cdot T^{a-1} \cdot n^{-1/2}\right) \cdot \eta_{q(n)}, \\
C_j'^{(T)} &\leq c \cdot A^3 \cdot \left(p(n)^2 \cdot T^{3a-1} \cdot n^{-1/2} + \|\phi''\|_\infty \cdot p(n)^3 \cdot T^{2a-1} \cdot n^{-1}\right) \cdot \eta_{q(n)},
\end{aligned}$$

For this, one should mention that if $s \in \mathbb{N}^*$, the function $G_T^{(s)}$ defined on $\mathbb{R}^{ds}$ as $G_T^{(s)}(u_1, \ldots, u_s) = \prod_{j=1}^s \Big(h(F_T(u_j)) - \mathbb{E}\big[h(F_T(x_0))\big]\Big)$ satisfies $\|G_T^{(s)}\|_\infty \leq T^{sa}$ and $\mathrm{Lip}\, G_T^{(s)} \leq c \cdot A^s \cdot T^{sa-1}$.

Thus,

$$\left\{ \begin{array}{lll} C_j & \leq & c \cdot A^3 \cdot \Big(\dfrac{p(n) \cdot}{\sqrt{n}} T^{a-m} + \Big(p(n) \cdot T^{2a-1} + \dfrac{p^2(n)}{\sqrt{n}} T^{a-1} \Big) \eta_{q(n)} \Big) ; \\ C'_j & \leq & c \cdot \cdot A^3 \Big(\dfrac{p^2(n)}{n} \cdot T^{2a-m} + \Big(\dfrac{p^2(n)}{\sqrt{n}} \cdot T^{3a-1} + \dfrac{p^3(n)}{n} \cdot T^{2a-1} \Big) \cdot \eta_{q(n)} \Big) , \end{array} \right. \tag{37}$$

from Lemma 29, relation (33), and because, from Lemma 29,

$$\mathbb{E}\, |Y_0^2 - (Y_0^{(T)})^2| \leq \mathbb{E}\, |E_0^{(T)}|^2 + 2\mathbb{E}\, (|Y_0^{(T)}|, |E_0^{(T)}|) \leq c \cdot T^{2a-m}.$$

Now, those bounds have to be minimized in n by choosing T as a function of n. Note that even if T may be chosen differently for the terms C_j and C'_j, this will be useless for our bounds. Remark that the aim is to write C_j and C'_j bounded by n^{-c} with the largest $c > 0$. As a consequence, we deduce from (37) that β will have to be the closer to 0. As a consequence, we assume $\beta < 1/2$ and thus C_j and C'_j are minimized by selecting $T = n^{\frac{\alpha\gamma - 1/2}{a+m-1}}$, that implies :

$$\left\{ \begin{array}{lll} C_j & \leq & c \cdot A^3 \cdot n^{\beta - 1/2 - (\alpha\gamma - 1/2)\left(\frac{m-a}{m+a-1}\right)} ; \\ C'_j & \leq & c \cdot A^3 \cdot n^{2\beta - 1 - (\alpha\gamma - 1/2)\left(\frac{m-2a}{m+a-1}\right)} , \end{array} \right.$$

under the conditions $\dfrac{1}{2\alpha} < \gamma < \beta < \dfrac{1}{2}$. Finally, from (36), we obtain the following bound :

$$\big|\Delta_{2,n}\big| \quad \leq \quad c \cdot A^3 \cdot \Big(n^{\beta(r-1) - \delta/2} + n^{1/2 - (\alpha\gamma - 1/2)\left(\frac{m-a}{m+a-1}\right)} + n^{\beta - (\alpha\gamma - 1/2)\left(\frac{m-2a}{m+a-1}\right)} \Big). \tag{38}$$

Therefore, inequalities (34), (35), (38) and condition $\dfrac{1}{2\alpha} < \gamma < \beta < \dfrac{1}{2}$ provide :

$$\big|\Delta_n\big| \quad \leq \quad c \cdot A^3 \cdot n^{\max(p_1, p_2, p_3, p_4, p_5)} \quad \text{with} \quad \left\{ \begin{array}{lll} p_1 & = & \beta \Big(1 - \alpha \cdot \frac{m-2a}{m-1} \Big) \\ p_2 & = & \gamma - \beta \\ p_3 & = & \beta \Big(1 + \delta - \Big(\frac{m-2a-a\delta}{m-1} \Big) \Big) - \delta/2 \\ p_4 & = & 1/2 - (\alpha\gamma - 1/2) \Big(\frac{m-a}{m+a-1} \Big) \\ p_5 & = & \beta - (\alpha\gamma - 1/2) \Big(\frac{m-2a}{m+a-1} \Big) \end{array} \right. . \tag{39}$$

We have the possibility to make varying δ, β, γ (with certain conditions) for :

1. obtaining conditions on α and m, such that it exists δ, β, γ verifying $\max(p_1, p_2, p_3, p_4, p_5) < 0$;
2. minimizing $\max(p_1, p_2, p_3, p_4, p_5)$ with an optimal choice of δ, β, γ under the previous conditions.

To solve 1., the condition $p_3 < 0$ implies $\beta < \dfrac{m-2a}{2(m-a)}$ with the optimal choice $\delta = m/a - 2$. Moreover, condition $p_4 < 0$, implies $\gamma > \dfrac{1}{2\alpha} \left(\dfrac{2m-1}{m-a} \right)$. As a consequence, $\max(p_1, p_2, p_3, p_4, p_5) < 0$ is verified when :

$$\frac{1}{2\alpha} \left(\frac{2m-1}{m-a} \right) < \gamma < \beta < \frac{m-2a}{2(m-a)} \qquad \Longrightarrow \qquad \alpha > \frac{2m-1}{m-2a}. \tag{40}$$

To solve 2., fist we show that only coefficients p_2, p_3 and p_4 have to be considered for the minimization (under conditions (40), coefficients p_1 and p_5 are smaller than p_2, p_3 and p_4). Then, the optimal choice for γ and δ is provided by the resolution of the system : $p_2 = p_3$ and $p_2 = p_4$, that implies to :

$$\beta_0 = \frac{m+2a-1+\alpha(m-2a)}{2(m+a-1+\alpha\cdot m)} \quad \text{and} \quad \gamma_0 = \frac{3m+2a-2}{2(m+a-1+\alpha\cdot m)},$$

and therefore, we obtain the optimal rate :

$$\big|\Delta_n\big| \quad \le \quad c\cdot A^3\cdot n^{-\lambda} \quad \text{with} \quad \lambda = \frac{\alpha(m-2a)-2m+1}{2(m+a-1+\alpha\cdot m)}. \ \blacksquare$$

4.6 Application to the periodogram and empirical covariance

Proof of Theorem 3 and Corollary 5 :

Set $g \in \mathcal{H}$ with $g(\lambda) = \sum_{\ell\in\mathbb{Z}} g_\ell e^{i\lambda\ell}$ and let $k \in \mathbb{N}^*$. Then define $g^{(k)}(\lambda) = \sum_{|\ell|<k} g_\ell e^{i\lambda\ell}$.

We can write :

$$\Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(I_n(g)-I(g))\Big)-\phi\Big(\sigma(g)\cdot N\Big)\Big]\Big| \le D^{(k)}_{1,n}+D^{(k)}_{2,n}+D^{(k)}_{3,n},$$

with

$$\begin{aligned}
D^{(k)}_{1,n} &= \Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(I_n(g^{(k)})-I(g^{(k)}))\Big)-\phi\Big(\sigma(g^{(k)})\cdot N\Big)\Big]\Big| \\
D^{(k)}_{2,n} &= \Big|\mathbb{E}\Big[\phi\Big(\sigma(g^{(k)})\cdot N\Big)-\phi\Big(\sigma(g)\cdot N\Big)\Big]\Big| \\
D^{(k)}_{3,n} &= \Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(I_n(g^{(k)})-I(g^{(k)}))\Big)-\phi\Big(\sqrt{n}(I_n(g)-I(g))\Big)\Big]\Big|.
\end{aligned}$$

Term $D^{(k)}_{1,n}$ **:** For $i=1,\ldots,n$, set $x_i = (X_{i+\ell})_{|\ell|<k}$ a stationary random vector in $\mathbb{R}^{2k-1}$. The function :

$$h(x_i) = \sum_{|\ell|<k} g_\ell(X_iX_{i+\ell}-R_\ell) \ \text{ for } i=1,\ldots,n,$$

satisfies the assumptions $\mathbf{H}$ (defined in the sequel) with $a=2$ and $A = A(2k-1) = 2k-1$. Define also :

$$S^{(k)}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n h(x_i) = \sqrt{n}(I_n(g^{(k)})-I(g^{(k)})).$$

By applying theorem 7 for this function h, one obtains :

$$D^{(k)}_{1,n} \le C_1\cdot k^3\cdot n^{-\lambda}, \tag{41}$$

with $C_1>0$ and $\lambda = \dfrac{\alpha(m-4)-2m+1}{2(m+1+\alpha\cdot m}$.

Term $D^{(k)}_{2,n}$ **:** With the same trick used for obtaining the bound of $\Delta_{4,n}$ in the previous proof, we have :

$$D^{(k)}_{2,n} \le \|\phi''\|_\infty\cdot\Big|\sigma^2(g)-\sigma^2(g^{(k)})\Big|.$$

But, from the expression 12, we deduce :

$$\Big|\sigma^2(g)-\sigma^2(g^{(k)})\Big| \le \Big|\frac{1}{\pi}\int_{-\pi}^{\pi}(g^2(\lambda)-(g^{(k)}(\lambda))^2)f^2(\lambda)\,d\lambda + \\ 2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(g(\lambda)g(\mu)-g^{(k)}(\lambda)g^{(k)}(\mu))f_4(\lambda,-\mu,\mu)d\lambda d\mu\Big|.$$

With $g\in\mathcal{H}$, we have :

$$\|g-g^{(k)}\|_\infty \le \Big(\sum_{|\ell|\ge k}s_\ell\Big)^{1/2}\Big(\sum_{|\ell|\ge k}s_\ell^{-1}g_\ell^2\Big)^{1/2} \le \Big(\sum_{|\ell|\ge k}s_\ell\Big)^{1/2}\|g\|_{\mathcal{H}}.$$

Consequently, with also $\|g+g^{(k)}\|_\infty \le 2\Big(\sum_{\ell\in\mathbb{Z}} s_\ell\Big)^{1/2}\cdot\|g\|_{\mathcal{H}}$, there exists $C_2>0$ such as :

$$D_{2,n}^{(k)} \le C_2\cdot(k^{\frac{1-s}{2}}), \tag{42}$$

with $s_\ell=\mathcal{O}(\ell^{-s})$ and $s>1$.

Term $D_{3,n}^{(k)}$ **:** First, from a Taylor development,

$$D_{3,n}^{(k)} \le \frac{1}{2}\cdot\|\phi''\|_\infty\cdot n\cdot\mathbb{E}\Big(I_n(g-g^{(k)})-I(g-g^{(k)})\Big)^2.$$

With the same decomposition as in the proof of lemma 2, one obtains :

$$\mathbb{E}\Big(I_n(g-g^{(k)})-I(g-g^{(k)})\Big)^2 \le 3\Big(\Big(\sum_{|\ell|\ge n} R_\ell g_\ell\Big)^2+\Big(\frac{1}{n}\sum_{k\le|\ell|<n}|\ell|R_\ell g_\ell\Big)^2+\Big\|\sum_{k\le|\ell|<n} g_\ell\Big(\widehat{R}_n(\ell)-\mathbb{E}(\widehat{R}_n(\ell))\Big)\Big\|_2^2\Big).$$

First,

$$\Big(\sum_{|\ell|\ge n} R_\ell g_\ell\Big)^2 \le s_n\cdot\sum_{|\ell|\ge n} R_\ell^2\cdot\sum_{|\ell|\ge n} s_\ell^{-1}g_\ell^2 \le \frac{1}{n}\cdot\|g\|_{\mathcal{H}}^2\cdot\sum_{|\ell|\ge n} R_\ell^2.$$

Using the weak dependence of $(X_i)_i$ and with the same trick as in the proof of lemma 7 and more precisely with inequality (33) adapted to the function $h(x)=x$ (therefore with $a=1$),

$$|R_\ell| \le c\cdot\eta_{|\ell|}^{\frac{m-2}{m-1}} \le c\cdot|\ell|^{-\alpha\,\frac{m-2}{m-1}},$$

from the rate $\eta_{|\ell|}=\mathcal{O}(|\ell|^{-\alpha})$ with $\alpha>3$. As a consequence,

$$\sum_{|\ell|\ge n} R_\ell^2 \le c\cdot n^{1-2\alpha\,\frac{m-2}{m-1}} \quad\text{and}\quad \Big(\sum_{|\ell|\ge n} R_\ell g_\ell\Big)^2 \le c\cdot n^{-2\alpha\,\frac{m-2}{m-1}}.$$

In the same way,

$$\Big(\frac{1}{n}\sum_{k\le|\ell|<n}|\ell|R_\ell g_\ell\Big)^2 \le \frac{1}{n}\cdot\|g\|_{\mathcal{H}}^2\cdot\sum_{k\le|\ell|<n} R_\ell^2 \le \frac{c}{n}\cdot k^{1-2\alpha\,\frac{m-2}{m-1}}.$$

Finally,

$$\begin{aligned}
\Big\|\sum_{k\le|\ell|<n} g_\ell\Big(\widehat{R}_n(\ell)-\mathbb{E}(\widehat{R}_n(\ell))\Big)\Big\|_2^2 &\le \Big(\sum_{k\le|\ell|<n}|g_\ell|\Big(\mathrm{Var}\,(\widehat{R}_n(\ell))\Big)^{1/2}\Big)^2\\
&\le \max_{\ell\in\mathbb{Z}}\Big(\mathrm{Var}\,(\widehat{R}_n(\ell))\Big)\cdot\|g\|_{\mathcal{H}}^2\cdot\sum_{k\le|\ell|<n} s_\ell\\
&\le \frac{1}{n}\cdot(\kappa_4+2\gamma)\cdot\|g\|_{\mathcal{H}}^2\cdot\sum_{k\le|\ell|<n} s_\ell \quad\text{from lemma 1.}
\end{aligned}$$

Finally, with $s_\ell=\mathcal{O}(\ell^{-s})$ and $s>1$, there exists $C_3>0$ such as :

$$D_{3,n}^{(k)} \le C_3\cdot(k^{1-2\alpha\,\frac{m-2}{m-1}}+k^{1-s}). \tag{43}$$

Now, with (41), (42) and (43), we deduce by considering $t=\Big(2\alpha\,\dfrac{m-2}{m-1}-1\Big)\wedge\dfrac{s-1}{2}$ and selecting k such as $k^{t+3}=n^\lambda$, that there exists $C>0$ such as :

$$\Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(I_n(g)-I(g))\Big)-\phi\Big(\sigma(g)\cdot N\Big)\Big]\Big| \le C\cdot n^{-\frac{t}{t+3}\,\lambda}. \ \blacksquare$$

Proof of Corollary 5. Only set $g(\lambda)=e^{i\ell\lambda}$ in Theorem 3. Since this function belongs to each space $\mathcal{H}_s$, this follows that the terms $D_{2,n}^{(k)}$ and $D_{3,n}^{(k)}$ both vanish and the result follows from the bound (41). $\blacksquare$

5 Appendix : a useful lemma

In several occurrences we need the following heredity lemma. Indeed, for ARCH models, the corresponding times series are weakly dependent but being orthogonal sequences their spectral density is constant thus meaningless and one better consider the squares process. More generally on may consider an instantaneous function of the initial process and we thus need to be in position to apply the results of the previous sections.

Lemma 9 *Assume that $(X_i)_{i\in\mathbb{Z}}$ is a stationary time series such that it exists $p>0$ verifying $\|X_0\|_p<\infty$. Let $(Y_i)_{i\in\mathbb{Z}}$ the stationary times series defined by $Y_i=h(X_i)$ for $i\in\mathbb{Z}$ and $h:\mathbb{R}\to\mathbb{R}$ such that $|h(x)|\le c\cdot|x|^a$ and $|h(x)-h(y)|\le c\cdot|x-y|\cdot(|x|^{a-1}+|y|^{a-1})$ for $(x,y)\in\mathbb{R}^2$ and $c>0$, $0<a<p$.*

- *If $(X_i)_{i\in\mathbb{Z}}$ is θ-weak dependent time series (in the sense of Corollary 4), then $(Y_i)_{i\in\mathbb{Z}}$ is a stationary θ^Y-weak dependent time series, such that $\forall r\in\mathbb{N}$, $\theta^Y_r=c\cdot\theta_r^{\frac{p-a}{p-1}}$ with constant $c>0$;*
- *If $(X_i)_{i\in\mathbb{Z}}$ is η-weak dependent time series, then $(Y_i)_{i\in\mathbb{Z}}$ is a η^Y-weak dependent time series, such that $\forall r\in\mathbb{N}$, $\eta^Y_r=c\cdot\eta_r^{\frac{p-a}{p-1}}$ with a constant $c>0$.*

Proof. Let $f:\mathbb{R}^u\to\mathbb{R}$ and $g:\mathbb{R}^v\to\mathbb{R}$ two real functions such that $\mathrm{Lip}\, f<\infty$, $\|f\|_\infty\le 1$, $\mathrm{Lip}\, g<\infty$, $\|g\|_\infty\le 1$. Denote $x^{(M)}=(x\wedge M)\vee(-M)$ for $x\in\mathbb{R}$. For simplicity we first assume that $v=2$. Let $i_1,\dots,i_u,j_1,\dots,j_v\in\mathbb{Z}^{u+v}$ such that $i_1,\dots,i_u\ge r$ and $j_1,\dots,j_v\le 0$ and denote $X_{\mathbf{i}}=(X_{i_1},\dots,X_{i_u})$ and $X_{\mathbf{j}}=(X_{j_1},\dots,X_{j_v})$. We then define functions $F:\mathbb{R}^u\to\mathbb{R}$ and $G:\mathbb{R}^v\to\mathbb{R}$ through the relations:

- $F(X_{\mathbf{i}})=f(h(X_{i_1}),\dots,h(X_{i_u}))$ and $F^{(M)}(X_{\mathbf{i}})=f(h(X_{i_1}^{(M)}),\dots,h(X_{i_u}^{(M)}))$;
- $G(X_{\mathbf{j}})=g(h(X_{j_1}),\dots,h(X_{j_v}))$ and $G^{(M)}(X_{\mathbf{j}})=g(h(X_{j_1}^{(M)}),\dots,h(X_{j_v}^{(M)}))$;

Then :

$$\begin{aligned}|\mathrm{Cov}\,(F(X_{\mathbf{i}}),G(X_{\mathbf{j}}))| &\le |\mathrm{Cov}\,(F(X_{\mathbf{i}}),G(X_{\mathbf{j}})-G^{(M)}(X_{\mathbf{j}}))|+|\mathrm{Cov}\,(F(X_{\mathbf{i}}),G^{(M)}(X_{\mathbf{j}}))|\\ &\le 2\mathbb{E}|G(X_{\mathbf{j}})-G^{(M)}(X_{\mathbf{j}}))|+2\mathbb{E}|F(X_{\mathbf{i}})-F^{(M)}(X_{\mathbf{i}})|+|\mathrm{Cov}\,(F^{(M)}(X_{\mathbf{i}}),G^{(M)}(X_{\mathbf{j}}))|\end{aligned}$$

The last relation comes from $\|f\|_\infty\le 1$. But we also have

$$\begin{aligned}\mathbb{E}|G(X_{\mathbf{j}})-G^{(M)}(X_{\mathbf{j}}))| &\le v\cdot\mathrm{Lip}\, g\cdot\mathbb{E}|h(X_0)-h(X_0^{(M)})|\\ &\le 2c\cdot v\cdot\mathrm{Lip}\, g\cdot\mathbb{E}\big(|X_0|^a\cdot\mathbb{1}_{|X_0|>M}\big)\qquad\text{(from the assumptions on } h),\\ &\le 2c\cdot v\cdot\mathrm{Lip}\, g\cdot\|X_0\|_p\cdot M^{a-p}\qquad\text{(from Markov inequality)}.\end{aligned}$$

The same thing holds for F. Moreover, the functions $F^{(M)}:\mathbb{R}^u\to\mathbb{R}$ and $G^{(M)}:\mathbb{R}^v\to\mathbb{R}$ verify $\mathrm{Lip}\, F^{(M)}=\mathrm{Lip}\, F^{(M)}=c\cdot M^{a-1}$, with $c>0$, and $\|F^{(M)}\|_\infty\le 1$, $\|G^{(M)}\|_\infty\le 1$. Thus, from the definition of the θ-weak dependence of X and the choice of $\mathbf{i},\mathbf{j}$, we obtain

$$\begin{aligned}\left|\mathrm{Cov}\,\big(F^{(M)}(X_{\mathbf{i}}),G^{(M)}(X_{\mathbf{j}})\big)\right| &\le cv\cdot\mathrm{Lip}\, f\cdot M^{a-1}\theta_r,\ \text{ if } u=2,\text{ under condition }\theta\\ &\le c(v\cdot\mathrm{Lip}\, f+u\cdot\mathrm{Lip}\, g)\cdot M^{a-1}\eta_r,\quad\text{under condition }\eta.\end{aligned}$$

Finally, we obtain respectively :

$$\begin{aligned}|\mathrm{Cov}\,(F(X_{\mathbf{i}}),G(X_{\mathbf{i}}))| &\le cv\cdot\mathrm{Lip}\, f\cdot\big(M^{a-1}\cdot\theta_r+M^{a-p}\big)\\ &\le c(v\cdot\mathrm{Lip}\, f+u\cdot\mathrm{Lip}\, g)\big(M^{a-1}\cdot\eta_r+M^{a-p}\big).\end{aligned}$$

By the optimal choice of $M=\theta_r^{1/(1-p)}$, we obtain :

$$\begin{aligned}|\mathrm{Cov}\,(f(Y_{\mathbf{i}}),g(Y_{\mathbf{j}}))| &\le cv\cdot\mathrm{Lip}\, f\cdot\theta_r^{\frac{p-a}{p-1}},\quad\text{or}\\ &\le c(v\cdot\mathrm{Lip}\, f+u\cdot\mathrm{Lip}\, g)\cdot\eta_r^{\frac{p-a}{p-1}}.\quad\blacksquare\end{aligned}$$

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