Curve crossings and specular points, d'après Longuet-Higgins.

Marie F. Kratz * José R. León †

Abstract

We use the Hermite expansion for the number of crossings of a differentiable curve by a stationary process to study the number of specular points of a curve and to understand its dynamical behavior.

1 Introduction

Longuet-Higgins in the fifties and the sixties (see [11] and [12]) developed a theoretical method that allows counting the number of specular points of a random surface when a ray of light hits it. His main motivation was the study of the light reflection on the sea surface aimed to determinate the height of the waves. By using elements of geometrical optics he showed that the specular points of the surface $W(t,x)$ are the crossings of $\partial_x W(t,x)$ to the curve $\psi(x) = -\kappa x$, where κ is a constant depending on the distances between the light source and the surface and between the observer and the surface. He also studied the speed in which these specular points move, obtaining the mean density of the speed. Finally he considered the time evolution of specular points, showing that the specular points evolving during a period of time, up to the curvature of the surface don't vanish. The creation or annihilation of specular points may be called a "twinkle": in those points the light reflection is more intense and there is a flash. Longuet-Higgins obtained also a formula that counts the number of such "twiklings".

It should be noticed that all the Longuet Higgins formulae were obtained in a heuristic form, nevertheless it is worth to mention that all of them are strictly exact.

In fact, these formulae are generalizations of the well known Rice formula ([13]) designed to count the mean number of roots of a random curve. The formula for the mean number of specular points is nothing more than an application of the formula of Cramer-Leadbetter ([7]) that counts the mean number of crossings of a differentiable curve. The formula for the speed of specular points is a consequence of a generalization of Rice formula to the bidimensional case, obtained by using the "co-area" formula ([6]). The mean number of "twiklings" can be rigourously computed by using the formula that counts the number of solutions of a system of random equations obtained by Cabaña ([6]) and more recently by Azaïs and Wschebor ([2]).

In this work we propose to go beyond, thanks to our method ([8], [9]) which allows to explain well the Longuet-Higgins discovery. We look for the representation into the Itô-Wiener Chaos (Hermite expansion) for the number of specular points and for the speed of these points. We are interested in obtaining variance results that allow to build confidence intervals, having in mind applications to sea modelling. We also start the study of "twiklings", by providing its mean number.

Recently there has been a renewed interest in applying the generalizations of the Rice formulae to explain some difficult phenomena in optics (see [3] and [4]). The present work could be considered as a first intent to tackle mathematically such type of problems.

The paper is organized as follows. In section 2, we get the expansion into Hermite polynomials for the number of crossings of a differentiable curve by another method than the one proposed by Slud,

*MAP5, UMR 8145, Université Paris V & SAMOS-MATISSE, UMR 8595, Université Paris I.

†Escuela de Matemática. Facultad de Ciencias. U.C.V. Venezuela.

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who obtained the same type of results ([14]). Indeed, instead of approaching the number of crossings of a differentiable curve by the number of crossings of a polygonal Gaussian process, we approach the process then the crossings by a smoothed-by-convolution process. As an application, section 3.1 provides the Hermite expansion as well as the variance of the number of specular points, whereas section 3.2 is devoted to the study of the speed of specular points. In the last section, we start studying the number of "twiklings".

2 Hermite expansion for the number of curve-crossings

Let $X = \{X_t, t \in I\!R\}$ be a centered stationary Gaussian process, variance one, with twice differentiable correlation function r given by $r(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda} F(d\lambda)$, where F is the spectral measure.
As in Cramér & Leadbetter ([7]), let us define the number of crossings of a differentiable function ψ by the process X, as the random variable

$$N_t^X(\psi) = card\{s \le t : X_s = \psi_s\}.$$

$N_t^X(\psi)$ can also be seen as the number $N_t^Y(0)$ of zero crossings by the non-stationary (but stationary in the sense of the covariance) Gaussian process $Y = \{Y_s,\ s \in I\!R\}$ defined by $Y_s := X_s - \psi_s$, i.e. $N_t^X(\psi) = N_t^Y(0)$.
As in [10], suppose that r satisfies on $[0, \delta]$, $\delta > 0$,

$$r(\tau) \;=\; 1 + \frac{r''(0)}{2}\tau^2 + \theta(\tau), \quad \text{with } \theta(\tau) > 0,\ \frac{\theta(\tau)}{\tau^2} \to 0,\ \frac{\theta'(\tau)}{\tau} \to 0,\ \theta'' \to 0,\ \text{as } \tau \to 0, \tag{1}$$

that the nonnegative function L defined by $L(\tau) := \dfrac{\theta''(\tau)}{\tau} = \dfrac{r''(\tau) - r''(0)}{\tau}$, $\tau > 0$, satisfies the Geman condition:

$$\exists \delta > 0, \quad L \in L^1([0, \delta]) \tag{2}$$

and assume that the modulus of continuity of $\dot\psi$ defined by $\gamma(\tau) := \sup\limits_{u\in[0,t]} \sup\limits_{|s|\le\tau} |\dot\psi(u+s) - \dot\psi(u)|$ is such that

$$\int_0^\delta \frac{\gamma(s)}{s}\, ds < \infty. \tag{3}$$

These conditions imply that $N_t^X(\psi)$ has a finite variance, as it has been proved by the authors in [10].
Let $(H_n)_{n\ge0}$ be the Hermite polynomials, defined by $H_n(x) = (-1)^n e^{x^2/2} \dfrac{d^n}{dx^n}\, e^{-x^2/2}$, which constitute a complete orthogonal system in the Hilbert space $L^2(I\!R, \varphi(u)du)$, φ denoting the standard normal density. Let $a_l(m)$ be the coefficients in this basis of the function $|\ .-m|$.

Proposition 1 *Under the conditions (1), (2) and (3), the number of crossings $N_t^X(\psi)$ of the function ψ by the process X has the following expansion in $L^2(\Omega)$:*

$$N_t^X(\psi) = \sqrt{-r''(0)}\, \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^{q} \frac{H_{q-l}(\psi_s)\varphi(\psi_s)}{(q-l)!}\, a_l\left(\frac{\dot\psi_s}{\sqrt{-r''(0)}}\right) H_{q-l}(X_s) H_l\left(\frac{\dot X_s}{\sqrt{-r''(0)}}\right) ds. \tag{4}$$

This type of results has been considered for the first time by Slud in [14]; he provided a MWI expansion of $N_t^X(\psi)$ by approximating the crossings of the process X by those of the discrete version of X.
Our approach is completely different and generalizes the one introduced in [8]; it consists in approaching the process, then the crossings, by using a process smoothed-by-convolution, for which the expansion can be readily obtained.
We believe that our method is maybe more intuitive, and can more easily be applied to other situations, as for instance to study the dynamic of specular points.

Remark. Proposition 1 recovers the result under the only hypothesis that the number of crossings belongs to $L^2(\Omega)$, thus completes the results of [8] for a fixed level $\psi_t \equiv x$ and for many functions ψ. Nevertheless, we do not know whether, under the Geman condition, $\int_0^\delta \frac{\gamma(s)}{s}\,ds = \infty$ implies that $N_t^X(\psi)$ does not belong to $L^2(\Omega)$.

Proof: Since the proof is based on the approach developed in [8] and on technical tools provided in [10], only the main ideas will be given. The proof can be sketched in four main steps.

• *Smooth approximation of X_t.*
Consider a twice differentiable even density function ϕ with support in $[-1,1]$ and define the continuous twice differentiable smoothed process as

$$X_t^\varepsilon = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \phi\left(\frac{t-u}{\varepsilon}\right) X_u\,du.$$

Let B be a complex Brownian motion such that $I\!\!E\left[B(d\lambda)\overline{B(d\lambda')}\right] = F(d\lambda)\mathbb{1}_{(\lambda=\lambda')}$.
We can write

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} B(d\lambda) \quad \text{and} \quad X_t^\varepsilon = \int_{-\infty}^{\infty} e^{it\lambda}\hat{\phi}(\varepsilon\lambda) B(d\lambda),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.
The correlation function of the process X_t^ε given by $r_\varepsilon(\tau) = \displaystyle\int_{-\infty}^{\infty} e^{i\tau\lambda}|\hat{\phi}(\varepsilon\lambda)|^2 F(d\lambda)$ satisfies

$$r_\varepsilon(0) \underset{\varepsilon\to 0}{\rightarrow} r(0) = \int_{-\infty}^{\infty} F(d\lambda) = 1, \tag{5}$$

and

$$r_\varepsilon(\tau) \;=\; r_\varepsilon(0) + \frac{r''_\varepsilon(0)}{2}\,\tau^2 + \theta_{X^\varepsilon}(\tau),$$

with θ_{X^ε} satisfying the same conditions as θ in (1).
Let us denote by $\sigma_\varepsilon := \sqrt{r_\varepsilon(0)}$ and $\eta_\varepsilon := \sqrt{-r''_\varepsilon(0)}$, and let us define the normalized process $\tilde{X}_t^\varepsilon = \dfrac{X_t^\varepsilon}{\sigma_\varepsilon}$ of variance 1 and correlation function ρ_ε, such that

$$\rho_\varepsilon(\tau) \;=\; 1 + \frac{\rho''_\varepsilon(0)}{2}\,\tau^2 + \theta_\varepsilon(\tau),$$

where $\rho''_\varepsilon(\tau) = \dfrac{r''_\varepsilon(\tau)}{r_\varepsilon(0)}$ and $\theta_\varepsilon(\tau) = \dfrac{\theta_{X^\varepsilon}(\tau)}{r_\varepsilon(0)}$.
Note that the smoothed process X_t^ε has the fourth derivative of its correlation function r_ε finite in 0, and so does $\tilde{X}_t^\varepsilon$, since these processes are twice differentiable:

$$r_\varepsilon^{(iv)}(0) < \infty \qquad \text{and} \qquad \rho_\varepsilon^{(iv)}(0) < \infty,$$

Consider also the smoothed function $\psi_t^\varepsilon = \dfrac{1}{\varepsilon}\displaystyle\int_{-\infty}^{\infty} \phi\left(\frac{t-u}{\varepsilon}\right)\psi_u\,du$, and let us write $\tilde{\psi}_t^\varepsilon = \frac{\psi_t^\varepsilon}{\sigma_\varepsilon}$.
The number of $\tilde{\psi}_t^\varepsilon$ - crossings by $\tilde{X}_t^\varepsilon$ is the number of ψ_t^ε - crossings by X_t^ε:

$$N_{\varepsilon,\,t}^{\psi} = card\{s \le t : \tilde{X}_s^\varepsilon = \tilde{\psi}_s^\varepsilon\} = card\{s \le t : X_s^\varepsilon = \psi_s^\varepsilon\}.$$

• *Hermite expansion for $N_{\varepsilon,\,t}^{\psi}$.*
As in [8], let us introduce

$$N_{t,\,h}^{X^\varepsilon}(\psi) := \frac{1}{h}\int_0^t \varphi\left(\frac{\tilde{X}_s^\varepsilon - \tilde{\psi}_s^\varepsilon}{h}\right)\frac{|\dot{X}_s^\varepsilon - \dot{\psi}_s^\varepsilon|}{\sigma_\varepsilon} ds = \frac{\eta_\varepsilon}{h\sigma_\varepsilon}\int_0^t \varphi\left(\frac{\tilde{X}_s^\varepsilon - \tilde{\psi}_s^\varepsilon}{h}\right)\left|\frac{\dot{X}_s^\varepsilon}{\eta_\varepsilon} - \frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon}\right| ds, \tag{6}$$

where φ denotes the standard Gaussian density and $h>0$.
By using the same approach as in [8], we can write in $L^2(\Omega)$,

$$N^{X^\varepsilon}_{t,\,h}(\psi) \;=\; \frac{\eta_\varepsilon}{\sigma_\varepsilon}\sum_{q=0}^{\infty}\int_0^t\left[\sum_{l=0}^{q} c_{q-l}(\tilde{\psi}^\varepsilon_s,h)a_l\left(\frac{\dot{\psi}^\varepsilon_s}{\eta_\varepsilon}\right)H_{q-l}(\tilde{X}^\varepsilon_s)H_l\left(\frac{\dot{X}^\varepsilon_s}{\eta_\varepsilon}\right)\right]ds, \tag{7}$$

where the Hermite coefficients $c_k(y,h)$ of the function $\dfrac{1}{h}\varphi\left(\dfrac{.\,-\,y}{h}\right)$ are given by

$$c_k(y,h)=\frac{1}{k!}\int_{-\infty}^{\infty}\varphi(v)H_k(y-hv)\varphi(y-hv)dv \;\underset{h\to 0}{\to}\; \frac{H_k(y)\varphi(y)}{k!}, \tag{8}$$

and where the Hermite coefficients $a_l(m)$ of the function $|\,.-m|$ are given by

$$\begin{aligned} a_0(m) &= I\!E|Z-m|, \quad \text{where } Z \text{ is a standard Gaussian r.v.,} \\ &= m\,[2\Phi(m)-1]+\sqrt{\frac{2}{\pi}}\;e^{-\frac{m^2}{2}}=\sqrt{\frac{2}{\pi}}\left[1+\int_0^m\int_0^u e^{-\frac{v^2}{2}}dvdu\right], \end{aligned} \tag{9}$$

$$\text{and}\quad a_l(m)=(-1)^{l+1}\sqrt{\frac{2}{\pi}}\frac{1}{l!}\int_0^1 e^{-\frac{m^2y^2}{2}}H_l(-my)y^{l-2}dy, \qquad l\geq 1. \tag{10}$$

So, as in [8], by taking $h\to 0$ in (7), we deduce the expansion of $N^\psi_{\varepsilon,\,t}$:

$$N^\psi_{\varepsilon,\,t}=\frac{\eta_\varepsilon}{\sigma_\varepsilon}\sum_{q=0}^{\infty}\int_0^t\sum_{l=0}^{q}\frac{H_{q-l}\left(\frac{\psi^\varepsilon_s}{\sigma_\varepsilon}\right)\varphi\left(\frac{\psi^\varepsilon_s}{\sigma_\varepsilon}\right)}{(q-l)!}\,a_l\left(\frac{\dot{\psi}^\varepsilon_s}{\eta_\varepsilon}\right)H_{q-l}\left(\frac{X^\varepsilon_s}{\sigma_\varepsilon}\right)H_l\left(\frac{\dot{X}^\varepsilon_s}{\eta_\varepsilon}\right)ds. \tag{11}$$

• *Convergence in $L^2(\Omega)$ of $N^\psi_{\varepsilon,\,t}$ to $N^X_t(\psi)$ as $\varepsilon\to 0$.*
We start proving that $I\!E[N^\psi_{\varepsilon,\,t}]^2 \underset{\varepsilon\to 0}{\to} I\!E[N^X_t(\psi)]^2$.
Since we have, by using (11) and the uniform convergence,

$$I\!E[N^\psi_{\varepsilon,\,t}]=\frac{\eta_\varepsilon}{\sigma_\varepsilon}\int_0^t\varphi\left(\frac{\psi^\varepsilon_s}{\sigma_\varepsilon}\right)a_0\left(\frac{\dot{\psi}^\varepsilon_s}{\eta_\varepsilon}\right)ds \;\underset{\varepsilon\to 0}{\to}\; \eta\int_0^t\varphi(\psi_s)a_0\left(\frac{\dot{\psi}_s}{\eta}\right)ds=I\!E[N^X_t(\psi)],$$

we only need to prove the convergence of the second factorial moment.
The second factorial moment of the number of $\tilde{\psi}^\varepsilon$-crossings by $\tilde{X}^\varepsilon$ can be deduced from the one of the number of zero-crossings by Y (see Cramér & Leadbetter, [7], p.209), namely

$$M_2^{\varepsilon,\,\psi}=\int_0^t\int_0^t dt_1dt_2\int_{I\!R^2}\left|\dot{x}_1-\dot{\tilde{\psi}}^\varepsilon_{t_1}\right|\left|\dot{x}_2-\dot{\tilde{\psi}}^\varepsilon_{t_2}\right|p^\varepsilon_{t_1,\,t_2}\left(\tilde{\psi}^\varepsilon_{t_1},\dot{x}_1,\tilde{\psi}^\varepsilon_{t_2},\dot{x}_2\right)d\dot{x}_1d\dot{x}_2\;, \tag{12}$$

where $p^\varepsilon_{t_1,\,t_2}(x_1,\dot{x}_1,x_2,\dot{x}_2)$ is the joint density of the vector $(\tilde{X}^\varepsilon_{t_1},\dot{\tilde{X}}^\varepsilon_{t_1},\tilde{X}^\varepsilon_{t_2},\dot{\tilde{X}}^\varepsilon_{t_2})$.
The formula holds whether $M_2^{\varepsilon,\,\psi}$ is finite or not.
It can also be expressed as

$$M_2^{\varepsilon,\,\psi}=2\int_0^t\int_{t_1}^t I_\varepsilon(t_1,t_2)dt_2dt_1=2\int_0^t\int_0^{t-t_1}I_\varepsilon(t_1,t_1+\tau)d\tau dt_1,$$

where

$$\begin{aligned} I_\varepsilon(t_1,t_2) &:= \int_{I\!R^2}|\dot{x}_1-\dot{\tilde{\psi}}^\varepsilon_{t_1}||\dot{x}_2-\dot{\tilde{\psi}}^\varepsilon_{t_2}|p^\varepsilon_{t_1,\,t_2}(\tilde{\psi}^\varepsilon_{t_1},\dot{x}_1,\tilde{\psi}^\varepsilon_{t_2},\dot{x}_2)d\dot{x}_1d\dot{x}_2 \\ &= p^\varepsilon_{t_1,\,t_2}(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_2})I\!E\left[|\dot{\tilde{X}}^\varepsilon_{t_1}-\dot{\tilde{\psi}}^\varepsilon_{t_1}||\,\dot{\tilde{X}}^\varepsilon_{t_2}-\dot{\tilde{\psi}}^\varepsilon_{t_2}|\,\middle|\,\tilde{X}^\varepsilon_{t_1}=\tilde{\psi}^\varepsilon_{t_1}\,,\tilde{X}^\varepsilon_{t_2}=\tilde{\psi}^\varepsilon_{t_2}\right], \end{aligned}$$

where $p^{\varepsilon}_{t_1,t_2}(x_1,x_2)$ is the joint density of the Gaussian vector $(\tilde{X}^{\varepsilon}_{t_1},\tilde{X}^{\varepsilon}_{t_2})$.
By using the following regression model:

$$(R) \quad \left\{ \begin{array}{l} \dot{\tilde{X}}^{\varepsilon}_{t_1} = \zeta_{\varepsilon} + \alpha_1 \tilde{X}^{\varepsilon}_{t_1} + \alpha_2 \tilde{X}^{\varepsilon}_{t_2} \\ \dot{\tilde{X}}^{\varepsilon}_{t_2} = \zeta^*_{\varepsilon} - \beta_1 \tilde{X}^{\varepsilon}_{t_1} - \beta_2 \tilde{X}^{\varepsilon}_{t_2} \end{array} \right.$$

where $\alpha_1(t_2-t_1)=\beta_2(t_2-t_1)=\dfrac{\rho'_{\varepsilon}(t_2-t_1)\rho_{\varepsilon}(t_2-t_1)}{1-\rho^2_{\varepsilon}(t_2-t_1)}$, $\alpha_2(t_2-t_1)=\beta_1(t_2-t_1)=-\dfrac{\rho'_{\varepsilon}(t_2-t_1)}{1-\rho^2_{\varepsilon}(t_2-t_1)}$,
and $(\zeta_{\varepsilon},\zeta^*_{\varepsilon})$ is jointly Gaussian such that $\sigma^2_{\varepsilon}(t_2-t_1)=Var(\zeta_{\varepsilon})=Var(\zeta^*_{\varepsilon})=-\rho''_{\varepsilon}(0)-\dfrac{(\rho'_{\varepsilon}(t_2-t_1))^2}{1-\rho^2_{\varepsilon}(t_2-t_1)}$
and $Cov(\zeta_{\varepsilon},\zeta^*_{\varepsilon})=-\rho''_{\varepsilon}(t_2-t_1)-\dfrac{(\rho'_{\varepsilon}(t_2-t_1))^2\rho_{\varepsilon}(t_2-t_1)}{1-\rho^2_{\varepsilon}(t_2-t_1)}$,
we obtain

$$I_{\varepsilon}(t_1,t_1+\tau)=p^{\varepsilon}_{\tau}(\tilde{\psi}^{\varepsilon}_{t_1},\tilde{\psi}^{\varepsilon}_{t_1+\tau})\mathbb{E}\left[|\zeta_{\varepsilon}+\alpha_1(\tau)\tilde{\psi}^{\varepsilon}_{t_1}+\alpha_2(\tau)\tilde{\psi}^{\varepsilon}_{t_1+\tau}-\dot{\tilde{\psi}}^{\varepsilon}_{t_1}||\zeta^*_{\varepsilon}-\alpha_2(\tau)\tilde{\psi}^{\varepsilon}_{t_1}-\alpha_1(\tau)\tilde{\psi}^{\varepsilon}_{t_1+\tau}-\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\tau}|\right].$$

For each $\tau>0$ and by using the uniform convergence, we have

$$\lim_{\varepsilon\to 0} I_{\varepsilon}(t_1,t_1+\tau)=I(t_1,t_1+\tau):=\int_{\mathbb{R}^2}|\dot{x}_1-\dot{\psi}_{t_1}||\dot{x}_2-\dot{\psi}_{t_1+\tau}|p_{t_1,t_1+\tau}(\psi_{t_1},\dot{x}_1,\psi_{t_1+\tau},\dot{x}_2)d\dot{x}_1d\dot{x}_2,$$

and thus $\lim\limits_{\varepsilon\to 0}\displaystyle\int_0^t\int_{\delta}^t I_{\varepsilon}(t_1,t_1+\tau)d\tau dt_1=\int_0^t\int_{\delta}^t I(t_1,t_1+\tau)d\tau dt_1.$
We now have to prove that

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_0^t\int_0^{\delta} I_{\varepsilon}(t_1,t_1+\tau)\,d\tau dt_1=0.$$

We can write

$$\int_0^t\int_0^{\delta} I_{\varepsilon}(t_1,t_1+\tau)\,d\tau dt_1\le\sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^t\int_0^{\delta}p^{\varepsilon}_{\tau}(\tilde{\psi}^{\varepsilon}_{t_1},\tilde{\psi}^{\varepsilon}_{t_1+\tau})\mathbb{E}[I_iJ_j]d\tau dt_1,$$

where

$$I_1=\left|\zeta_{\varepsilon}-\frac{\rho'_{\varepsilon}(\tau)}{1+\rho_{\varepsilon}(\tau)}\tilde{\psi}^{\varepsilon}_{t_1}\right| \quad , \quad I_2=\left|\left(1+\frac{\tau\rho'_{\varepsilon}(\tau)}{1-\rho^2_{\varepsilon}(\tau)}\right)\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\eta\tau}\right| \quad , \quad I_3=|\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\eta\tau}-\dot{\tilde{\psi}}^{\varepsilon}_{t_1}|$$

and

$$J_1=\left|\zeta^*_{\varepsilon}+\frac{\rho'_{\varepsilon}(\tau)}{1+\rho_{\varepsilon}(\tau)}\tilde{\psi}^{\varepsilon}_{t_1}\right|, \qquad J_2=\left|\rho_{\varepsilon}(\tau)\left(1+\tfrac{\tau\rho'_{\varepsilon}(\tau)}{1-\rho^2_{\varepsilon}(\tau)}\right)\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\nu\tau}\right|,$$
$$J_3=|(\rho_{\varepsilon}(\tau)-1)\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\nu\tau}|, \qquad J_4=|\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\nu\tau}-\dot{\tilde{\psi}}^{\varepsilon}_{t_1+\tau}|,$$

with $0\le\ \eta\ ,\ \nu\ \le 1$.
Let C be some positive constant which may vary from equation to equation.
First we have

$$\int_0^t\int_0^{\delta}p^{\varepsilon}_{\tau}(\psi^{\varepsilon}_{t_1},\psi^{\varepsilon}_{t_1+\tau})\mathbb{E}[I_1J_1]d\tau dt_1\le\mathbf{C}\int_0^{\delta}f_{\varepsilon}(\tau)\frac{1}{\tau}\left(\sigma^2_{\varepsilon}(\tau)+\left(\frac{\rho'_{\varepsilon}(\tau)}{1+\rho_{\varepsilon}(\tau)}\right)^2\right)d\tau, \tag{13}$$

where $f_{\varepsilon}(\tau):=\dfrac{\tau}{\sqrt{1-\rho_{\varepsilon}(\tau)}}$.
Note that

$$f_{\varepsilon}(\tau)=\left(\frac{-\rho''_{\varepsilon}(0)}{2}-\frac{\theta_{\varepsilon}(\tau)}{\tau^2}\right)^{-1/2}\ \underset{\varepsilon\to 0}{\to}\ \frac{\tau}{\sqrt{1-r(\tau)}}.$$

By using similar techniques as the ones introduced to prove the lemmas in [10], we can prove that f_ε is uniformly bounded when $\varepsilon \to 0$ in $[0,\delta]$, $\delta > 0$,

$$f_\varepsilon(\tau) \ \leq \ C, \tag{14}$$

that

$$\frac{\sigma_\varepsilon^2(\tau)}{\tau} \ \leq \ \mathbf{C}\frac{\tau\theta_\varepsilon'(\tau) - \theta_\varepsilon(\tau)}{\tau^3} \ \leq \ \mathbf{C}L(\tau), \tag{15}$$

from which we deduce under the Geman condition, by applying (14) and the Dominated Convergence Theorem, that

$$\lim_{\varepsilon\to 0}\int_0^\delta \frac{\sigma_\varepsilon^2(\tau)}{\sqrt{1-\rho_\varepsilon(\tau)}}\,d\tau \ = \ \int_0^\delta \frac{\sigma^2(\tau)}{\sqrt{1-r(\tau)}}\,d\tau, \quad \text{with} \quad \sigma^2(\tau) := -r''(0) - \frac{r'^{\,2}(\tau)}{1-r^2(\tau)}, \tag{16}$$

which tends to zero as $\delta \to 0$.
We can also prove that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^\delta \frac{1}{\sqrt{1-\rho_\varepsilon(\tau)}}\left(\frac{\rho_\varepsilon'(\tau)}{1+\rho_\varepsilon(\tau)}\right)^2 d\tau = 0, \tag{17}$$

by using the uniform bound (14) for $f_\varepsilon(\tau)$, and the fact that $\tau\left(\frac{\rho_\varepsilon'(\tau)}{\tau}\right)^2 \leq \mathbf{C}\tau(r''(0))^2$.
We can conclude by combining (13), (14), (16) and (17).
Now let us study $\displaystyle\int_0^t\int_0^\delta p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)\mathbb{E}[I_1J_2]d\tau dt_1$.
We have

$$\begin{aligned}
&\int_0^t\int_0^\delta p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)\mathbb{E}[I_1J_2]d\tau dt_1 \\
&\leq \mathbf{C}\left\{\int_0^\delta f_\varepsilon(\tau)\frac{|1+\rho_\varepsilon(\tau)-2|}{\tau}\,d\tau + \int_0^\delta (f_\varepsilon(\tau))^{3/2}\,\frac{|2(1-\rho_\varepsilon(\tau)) + \tau\rho_\varepsilon'(\tau)|}{\tau^3}\,d\tau\right\} \\
&\leq \mathbf{C}\left\{\int_0^\delta f_\varepsilon(\tau)d\tau \ + \ \int_0^\delta (f_\varepsilon(\tau))^{3/2}\left(\frac{\theta''(\tau)}{\tau} + \frac{\theta(\tau)}{\tau^3}\right)d\tau\right\}.
\end{aligned}$$

Again, the uniform bound (14) for $f_\varepsilon(\tau)$ and the Geman condition allow to conclude.
The term $\displaystyle\int_0^t\int_0^\delta p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)\mathbb{E}[I_2J_1]d\tau dt_1$ can be bounded in the same way.
Now, since $p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)\ J_3 \ \leq \ \mathbf{C}(\sqrt{1-\rho_\varepsilon(\tau)}\,)$, we obtain

$$\sum_{i=1}^3 \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^t\int_0^\delta p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)\mathbb{E}[I_iJ_3]d\tau dt_1 = 0.$$

Finally, the fact that $|\dot{\tilde\psi}_{t+\tau}^\varepsilon - \dot{\tilde\psi}_t^\varepsilon| \leq \mathbf{C}\gamma(\tau)$ and $p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)I_3 \leq \mathbf{C}f_\varepsilon(\tau)\frac{\gamma(\tau)}{\tau}$ implies

$$\sum_{i=1}^3 \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^t\int_0^\delta p_\tau^\varepsilon(\tilde\psi_{t_1}^\varepsilon, \tilde\psi_{t_1+\tau}^\varepsilon)\mathbb{E}[I_iJ_4]d\tau dt_1 = 0.$$

Thus we have proved that

$$\lim_{\varepsilon\to 0}\mathbb{E}[N_{\varepsilon,t}^\psi]^2 = \mathbb{E}[N_t^X(\psi)]^2. \tag{18}$$

Since it can be easily proved, in the same way as in [8] (p.245), that

$$\lim_{\varepsilon\to 0}\mathbb{E}[N_t^X(\psi)N_{\varepsilon,t}^\psi] = \mathbb{E}[N_t^X(\psi)]^2, \tag{19}$$

we deduce from (18) and (19) that $||N^{\psi}_{\varepsilon,t} - N^X_t(\psi)||_{L^2(\Omega)} \underset{\varepsilon\to 0}{\to} 0$.

• *Conclusion.*
Since, by Fatou lemma, for each Q positive,

$$\begin{aligned}\sum_{q=0}^{Q} \quad I\!\!E \quad &\left[\int_0^t \sum_{l=0}^{q} \frac{H_{q-l}(\psi_s)\varphi(\psi_s)}{(q-l)!}\, a_l\left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right) H_{q-l}(X_s) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) ds\right]^2 \\ &\leq \lim_{\varepsilon\to 0} I\!\!E\left[\sqrt{\frac{-r''_\varepsilon(0)}{r_\varepsilon(0)}}\, N^{\psi}_{\varepsilon,t}\right]^2 = I\!\!E[\sqrt{-r''(0)}\, N^X_t(\psi)]^2,\end{aligned}$$

then the formal expansion in the right hand side of (4) defines a random variable in $L^2(\Omega)$, denoted by $\mathcal{N}_t$.
Moreover

$$I\!\!E\left[\frac{\sqrt{r(0)}}{\eta} N^X_t(\psi) - \mathcal{N}_t\right]^2 \;\leq\; 2\left(I\!\!E\left[\frac{N^X_t(\psi)}{\sqrt{-r''(0)}} - \sqrt{\frac{r_\varepsilon(0)}{-r''_\varepsilon(0)}}\, N^{\psi}_{\varepsilon,t}\right]^2 \;+\; I\!\!E\left[\sqrt{\frac{r_\varepsilon(0)}{-r''_\varepsilon(0)}}\, N^{\psi}_{\varepsilon,t} - \mathcal{N}_t\right]^2\right).$$

The first term of the right hand side, tends to zero as shown in the previous step. For the second term, an argument of continuity for the projections into the chaos entails the result (see [8]). □

Note that, since $N^X_t(\psi) = N^{\tilde{Y}}_{\sqrt{\frac{-r''(0)}{r(0)}}\,t}(\psi)$, where $\tilde{Y}_s = \frac{1}{\sqrt{r(0)}}\left(X\left(s\sqrt{\frac{r(0)}{-r''(0)}}\right) - \psi\left(s\sqrt{\frac{r(0)}{-r''(0)}}\right)\right)$, the previous expansion can be easily generalized to a process X satisfying $I\!\!E[X_t^2] = r(0)$ (and $I\!\!E[\dot{X}_t^2] = -r''(0)$), as

$$N^X_t(\psi) = \sqrt{\frac{-r''(0)}{r(0)}} \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^{q} \frac{H_{q-l}\left(\frac{\psi_s}{\sqrt{r(0)}}\right)\varphi\left(\frac{\psi_s}{\sqrt{r(0)}}\right)}{(q-l)!}\, a_l\left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right) H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) ds.$$

3 Specular points

3.1 Characteristics of the number of specular points

In this section we shall apply the results of the previous section to describe the behavior of the specular points in a random curve.
As described by Longuet-Higgins ([11]), specular points are the moving images of a light source reflected at different points in a wave-like surface.
Let us consider a Gaussian field W defined on $I\!\!R^+ \times I\!\!R$, $I\!\!R^+$ for the temporal variable and $I\!\!R$ for the spatial one.
The first derivatives with respect to the spatial variable x and the temporal variable t will be denoted by W_x and W_t respectively; the second derivatives will be denoted by W_{xx}, W_{tt}, W_{tx} and W_{xt}.
The spectral representation of W is given by

$$W(t,x) = \int_\Lambda e^{i(\lambda x - \omega t)} \sqrt{f(\lambda)}\, dB(\lambda), \quad \text{with} \quad \Lambda := \{\lambda^2 = \omega^4\}, \tag{20}$$

where

$$B(\lambda) = \frac{1}{\sqrt{2}}(B_1(\lambda) + iB_2(\lambda)),$$

B_i being two standard Brownian motions, bilateral and independent. The correlation function r is given by

$$r(t,x) = \int_\Lambda e^{i(\lambda x - \omega t)} f(\lambda) d\lambda, \tag{21}$$

which can also be written as

$$r(t,x) = 4\int_0^\infty \cos(\omega^2 x - \omega t) f(\omega^2)\omega \, d\omega.$$

First let us consider the process at a fixed time, for instance $t = 0$.
In this case, $W(0,x)$ is a centered stationary Gaussian process with correlation function

$$r(0,x) := r(x) = 2\int_0^\infty \cos(\lambda x) f(\lambda)\, d\lambda.$$

Let us define a curve in the plan (x,z) by the equation $z = W(0,x)$.
Suppose that the coordinates of a point-source of light and an observer are $(0,h_1)$ and $(0,h_2)$ respectively, situated at heights h_1 and h_2 above the mean level. A specular point is characterized by the equations (see [11]):

$$W_x(0,x) = -\kappa x \qquad \text{with} \qquad \kappa = \frac{1}{2}\left(\frac{1}{h_1} + \frac{1}{h_2}\right),$$

which can be interpreted as *a crossing of the curve* $\psi(x) := -\kappa x$ *by the process* $W_x(0,x)$.
Let us assume that r is four times differentiable and that r'' satisfies the Geman condition:

$$\frac{r^{(iv)}(x) - r^{(iv)}(0)}{x} \in L^1([0,\delta]). \tag{22}$$

Then we can proved that the number $N_{[0,x]}$ of specular points in the interval $[0,x]$ verifies

Theorem 1 *Under the Geman condition (22),* $N_{[0,x]}$ *has the following expansion in* $L^2(\Omega)$*:*

$$N_{[0,x]} = \frac{\gamma}{\eta}\sum_{q=0}^{\infty}\sum_{l=0}^{q}\int_0^x \frac{H_{q-l}\left(\frac{-\kappa s}{\eta}\right)\varphi\left(\frac{-\kappa s}{\eta}\right)}{(q-l)!}\, a_l\left(\frac{-\kappa}{\gamma}\right) H_{q-l}\left(\frac{W_x(0,s)}{\eta}\right) H_l\left(\frac{W_{xx}(0,s)}{\gamma}\right) ds, \tag{23}$$

where $\eta = \sqrt{-r''(0)}$, $\gamma = \sqrt{r^{(iv)}(0)}$ *and* a_l *are defined in (10).*
Its expectation is given by

$$I\!E[N_{[0,x]}] \;=\; \sqrt{\frac{2}{\pi}}\frac{\kappa}{\eta}\left(\int_0^{\frac{\kappa}{\gamma}} e^{-\frac{u^2}{2}}\, du + \frac{\gamma}{\kappa}\, e^{-\frac{\kappa^2}{2\gamma^2}}\right) \int_0^x \varphi\left(\frac{\kappa s}{\eta}\right) ds, \tag{24}$$

and its variance by

$$Var(N_{[0,x]}) =$$
$$2\frac{\gamma}{\eta}\sum_{q=1}^{\infty}\int_0^x\int_0^{x-s} I\!E\left[F_q\left(s, \frac{W_x(0,s)}{\eta}, \frac{W_{xx}(0,s)}{\gamma}\right) F_q\left(s+\tau, \frac{W_x(0,s+\tau)}{\eta}, \frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right] d\tau ds,$$

where

$$F_q\left(s, \frac{W_x(0,s)}{\eta}, \frac{W_{xx}(0,s)}{\gamma}\right) = \sum_{l=0}^{q} \frac{H_{q-l}\left(\frac{-\kappa s}{\eta}\right)\varphi\left(\frac{-\kappa s}{\eta}\right)}{(q-l)!}\, a_l\left(\frac{-\kappa}{\gamma}\right) H_{q-l}\left(\frac{W_x(0,s)}{\eta}\right) H_l\left(\frac{W_{xx}(0,s)}{\gamma}\right).$$

Under the m-dependence, i.e. when assuming that $-r''(\tau) = 0$ *whenever* $\tau > m$*, then the asymptotic variance, as* $x \to \infty$*, is given by*

$$\begin{aligned} &Var(N_{[0,\infty]}) \\ = \;& 2\frac{\gamma}{\eta}\sum_{q=1}^{\infty}\int_0^\infty\int_0^m I\!E\left[F_q\left(s, \frac{W_x(0,s)}{\eta}, \frac{W_{xx}(0,s)}{\gamma}\right) F_q\left(s+\tau, \frac{W_x(0,s+\tau)}{\eta}, \frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right] d\tau ds. \end{aligned} \tag{25}$$

Note that the expectation (24) of $N_{[0,x]}$ was first obtained by Longuet-Higgins in [11] and [12], who considered a bilateral formula, i.e. $\mathbb{E}[N_{[-x,x]}]$, and showed that its limit as $x\to\infty$ is given by

$$\lim_{\frac{\gamma}{\kappa}\to\infty}\frac{\kappa}{\gamma}\,\mathbb{E}[N_{[0,+\infty]}]=\sqrt{\frac{1}{2\pi}}\quad;$$

it can be interpreted by saying that the number of specular points increases proportionally to the distance between the observer and the sea surface, since when $h_1=h_2$, $\frac{1}{\kappa}$ represents this latter distance.
It is also interesting to notice that the number of specular points is a finite random variable over all the line, on the contrary to the behavior of the crossings to a fixed level or of the number of maxima.

Proof.
Since the correlation function of $W_x(0,x)$ is $-r''(x)$, Proposition 1 provides the Hermite expansion of $N_{[0,x]}$ under the Geman condition (22), from which we deduce the expectation

$$\begin{aligned}\mathbb{E}[N_{[0,x]}] &= \frac{\gamma}{\eta}\int_0^x\varphi\left(\frac{\kappa s}{\eta}\right)a_0\left(\frac{-\kappa}{\gamma}\right)ds\\ &= \frac{\gamma}{\eta}\left(\int_0^x\varphi\left(\frac{\kappa s}{\eta}\right)ds\right)\left(\frac{\kappa}{\gamma}\left[2\Phi\left(\frac{-\kappa}{\gamma}\right)-1\right]+\sqrt{\frac{2}{\pi}}\,e^{-\frac{\kappa^2}{2\gamma^2}}\right)\;=\;(24),\end{aligned}$$

and also the variance.
Let us now study the limit of the variance of $N_{[0,\,x]}$ as $x\to\infty$, under the m-dependence.
$\star$ For each $q\geq 1$, we obtain

$$\begin{aligned}&\lim_{x\to\infty}\int_0^x\int_0^{x-s}\mathbb{E}\left[F_q\left(s,\frac{W_x(0,s)}{\eta},\frac{W_{xx}(0,s)}{\gamma}\right)F_q\left(s+\tau,\frac{W_x(0,s+\tau)}{\eta},\frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right]d\tau ds\\ &=\int_0^\infty\int_0^m\mathbb{E}\left[F_q\left(s,\frac{W_x(0,s)}{\eta},\frac{W_{xx}(0,s)}{\gamma}\right)F_q\left(s+\tau,\frac{W_x(0,s+\tau)}{\eta},\frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right]d\tau ds.\end{aligned}$$

The next inequality is a consequence of the diagram formula (see [5]):

$$\begin{aligned}&\left|\mathbb{E}\left[F_q\left(s,\frac{W_x(0,s)}{\eta},\frac{W_{xx}(0,s)}{\gamma}\right)F_q\left(s+\tau,\frac{W_x(0,s+\tau)}{\eta},\frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right]\right|\\ \leq\;&\mathbf{C}\left|\frac{\kappa s}{\eta}\right|^q\varphi\left(\frac{\kappa s}{\eta}\right)\left|\frac{\kappa(s+\tau)}{\eta}\right|^q\varphi\left(\frac{\kappa(s+\tau)}{\eta}\right)\sum_{a+b+c=q}|r''(\tau)|^a|r'''(\tau)|^b|r^{(iv)}(\tau)|^c,\end{aligned}$$

and the term equals zero when $\tau>m$. The Dominate Convergence Theorem allows then to compute the limit inside of the double integral.
$\star$ We shall study the asymptotic behavior as $x\to\infty$ of the second moment of $N_{[0,x]}$.
First let us consider the second factorial moment, and for ease of notation we shall suppose bellow that $-r''(0)=1$. We have

$$M_2^\psi\;=\;2\int_0^x\int_0^{x-s}p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)d\tau ds,$$

where

$$\begin{aligned}G(s,\tau)\;=\;\mathbb{E}\Bigg(&\left|\zeta-\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\kappa s-\frac{r'''(\tau)}{1-(r''(\tau))^2}\kappa(s+\tau)+\kappa\right|\\ &\times\left|\zeta^*+\frac{r'''(\tau)}{1-(r''(\tau))^2}\kappa s+\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\kappa(s+\tau)+\kappa\right|\Bigg),\end{aligned}$$

$\psi(x)=-\kappa x$, p_τ is the density of $(W_x(0,s),W_x(0,s+\tau))$ and (ζ,ζ^*) is defined in (R). Then

$$M_2^\psi\;=\;2\int_0^x\int_\delta^{x-s}p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau+2\int_0^x\int_0^\delta p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau.$$

By using the Cauchy-Schwarz inequality, we have

$$G(s,\tau) \le \mathbf{C}\left[\mathbb{E}\zeta^2 + \left(\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\right)^2 + \left(\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\right)^2 + \kappa^2\right](s^2+(s+\tau)^2).$$

Moreover, we can write

$$\begin{aligned} p_\tau(-\kappa s, -\kappa(s+\tau)) &= \frac{1}{2\pi\sqrt{1-(r''(\tau))^2}}\, e^{-\frac{\kappa^2}{2(1-r''(\tau)^2)}(s^2+2r''(\tau)s(s+\tau)+(s+\tau)^2)} \\ &= \frac{1}{2\pi\sqrt{1-(r''(\tau))^2}}\, e^{-\frac{\kappa^2}{1-r''(\tau)}\left(s+\frac{1}{2}\tau\right)^2} e^{-\frac{\kappa^2\tau^2}{4(1+r''(\tau))}}. \end{aligned} \tag{26}$$

If $\tau \in [\delta, x-s]$, it holds that $\quad p_\tau(-\kappa s, -\kappa(s+\tau)) \le \mathbf{C}\, e^{-\frac{\kappa^2}{1-r''(\tau)}(s+\frac{1}{2}\tau)^2}$.
Thus applying the Dominate Convergence Theorem gives

$$\lim_{x\to\infty}\int_0^x\int_\delta^{x-s} p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau = \int_0^\infty\int_\delta^m p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau.$$

To bound $G(s,\tau)$ in the interval $[0,\delta]$, we can proceed as in the previous section when bounding the terms $\mathbb{E}[I_iJ_i]$; all the terms can be bounded by functions belonging to $L^2[0,\delta]$, therefore the dominated convergence theorem yields

$$\lim_{x\to\infty}\int_0^x\int_0^{x-s} p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau = \int_0^\infty\int_0^m p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau < \infty.$$

On one hand, via Fatou lemma we have

$$\begin{aligned} &\infty > \lim_{x\to\infty}\mathbb{E}[N_{[0,x]}]^2 \ge \lim_{x\to\infty} Var\left(\frac{\eta}{\gamma}N_{[0,x]}\right) \\ = &\lim_{x\to\infty} 2\sum_{q=1}^\infty\int_0^x\int_0^{x-s}\mathbb{E}\left[F_q\left(s,\frac{W_x(0,s)}{\eta},\frac{W_{xx}(0,s)}{\gamma}\right)F_q\left(s+\tau,\frac{W_x(0,s+\tau)}{\eta},\frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right]dsd\tau \\ \ge &\ 2\sum_{q=1}^\infty\int_0^\infty\int_0^m\mathbb{E}\left[F_q\left(s,\frac{W_x(0,s)}{\eta},\frac{W_{xx}(0,s)}{\gamma}\right)F_q\left(s+\tau,\frac{W_x(0,s+\tau)}{\eta},\frac{W_{xx}(0,s+\tau)}{\gamma}\right)\right]dsd\tau. \end{aligned}$$

On the other hand, $N_{[0,x]} \uparrow N_{[0,\infty]}$ and so $\mathbb{E}[N^2_{[0,x]}] \;\to\; \mathbb{E}[N^2_{[0,\infty]}]$, as $x\to\infty$.
We can then conclude to (25). $\square$

3.2 Speed of specular points

Let us recall the co-area formula.
Let W be a $\mathcal{C}^3$ function of two variables and let us define the level curve by

$$\mathcal{C}(u) = \{x_1,x_2 \in Q(T,M) : W(x_1,x_2)=u\}, \quad \text{where} \quad Q(T,M)=[0,T]\times[0,M].$$

If v denotes a vector field, $\mathbf{n}$ denotes the vector normal to the level curve $\mathcal{C}(u)$ and $d\sigma$ is the length measure of $\mathcal{C}(u)$, since W is $\mathcal{C}^3$, then by the Green formula we obtain

$$\int_{-\infty}^\infty g(u)\int_{\mathcal{C}(u)} < v,\mathbf{n} > d\sigma du = \int_{Q(T,M)} g(W) < v,\nabla W > dx_1\,dx_2,$$

from which we can deduce immediately:
i) if $v = \dfrac{\nabla W}{||\nabla W||}$, then $\displaystyle\int_{-\infty}^\infty g(u)\mathcal{L}_Q(\mathcal{C}(u))du = \int_Q g(W)||\nabla W||dx_1\,dx_2,$

where $\mathcal{L}_Q(\mathcal{C}(u))$ denotes the length of the curve $\mathcal{C}(u)$;
ii) if α is such that $\nabla W := ||\nabla W||\, {}^t(\cos\alpha, \sin\alpha)$, if ζ denotes a continuous real function defined on $[0, 2\pi]$ and if

$$v := \zeta(\alpha)\frac{\nabla W}{||\nabla W||},$$

then we have

$$\int_{-\infty}^{\infty} g(u) \int_{\mathcal{C}(u)} \zeta(\alpha(s)) d\sigma(s) du = \int_Q g(W)\zeta(\alpha)||\nabla W|| dx_1\, dx_2.$$

Note this formula still holds when considering ζ as the indicator function of a measurable set, by using the monotone convergence theorem (see for instance [1]).
This co-area formula will help for the study of the speed of specular points.
Let us consider, as in [12], the simplified condition to have a specular point given by

$$W_x(t, x) = u.$$

Then a consequence of the implicit function theorem is:

$$W_{xx}dx + W_{xt}dt = 0, \qquad \text{i.e. } \frac{dx}{dt} = -\frac{W_{xt}}{W_{xx}}.$$

We are interested into the specular points which speed is bounded below by $-\,v_2$ and above by $-\,v_1$.
First let us compute its mean.
Let us consider the curve at level u: $\mathcal{C}(u) = \{(t, x) \in [0, T] \times [0, M] : W_x(t, x) = u\}$.
Applying the co-area formula provides

$$\int_{-\infty}^{\infty} g(u) \int_{\mathcal{C}(u)} < \mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\mathbf{n}\ ,\ \mathbf{n} > d\sigma = \int_Q g(W_x)\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\sqrt{W_{xt}^2 + W_{xx}^2}\ dxdt.$$

Note that $(W_x(t, x), W_{xt}(t, x), W_{xx}(t, x))$ is a Gaussian vector at fixed (t, x), with spectral representation given by

$$W(t, x) = \int_{\{\lambda^2=\omega^4\}} e^{i(\lambda x-\omega t)}\sqrt{f(\lambda)}dB(\lambda).$$

Let us define

$$M_{pq} = 2\int_0^{\infty} \lambda^p\omega^q f(\lambda)d\lambda = 2\int_0^{\infty} \lambda^{p+q/2}f(\lambda)d\lambda.$$

We easily obtain that

$$\mathbb{E}[W_x^2] = M_{20},\ \mathbb{E}[W_{xx}^2] = M_{40},\ \mathbb{E}[W_{xt}^2] = M_{22}$$
$$\mathbb{E}[W_xW_{xx}] = \mathbb{E}[W_xW_{xt}] = 0 \text{ and } \mathbb{E}[W_{xx}W_{xt}] = M_{31}.$$

Therefore, it comes

$$\int_{-\infty}^{\infty} g(u)\mathbb{E}\left[\int_{\mathcal{C}(u)} \mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\right] d\sigma du = TM\int_{-\infty}^{\infty} g(u)\frac{e^{-\frac{u^2}{2M_{20}}}}{\sqrt{2\pi M_{20}}}du\mathbb{E}\left[\sqrt{W_{xt}^2 + W_{xx}^2}\ \mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\right].$$

But by duality we have

$$\mathbb{E}\left[\int_{\mathcal{C}(u)} \mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma\right] = TM\frac{e^{-\frac{u^2}{2M_{20}}}}{\sqrt{2\pi M_{20}}}\mathbb{E}\left[\sqrt{W_{xt}^2 + W_{xx}^2}\ \mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\right]. \tag{27}$$

This quantity can be interpreted as the mean number of specular points which have a speed belonging to $[-v_2, -v_1]$.
Let us compute the expectation in (27). We have

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\sqrt{W_{xt}^2 + W_{xx}^2}\right] &= \frac{1}{2\pi(\lambda_1\lambda_2)^{1/2}}\int_0^{\infty}\int_{\beta+\arctan\, v_1}^{\beta+\arctan\, v_2} r^2 e^{-\frac{r^2}{2}(\frac{\cos^2\alpha}{\lambda_1}+\frac{\sin^2\alpha}{\lambda_2})}d\alpha dr\\ &= \frac{\lambda_1\lambda_2}{\sqrt{8\pi}}\int_{\beta+\arctan\, v_1}^{\beta+\arctan\, v_2}\frac{1}{(\lambda_1\sin^2\alpha + \lambda_2\cos^2\alpha)^{3/2}}d\alpha,\end{aligned}$$

which can also be written as

$$\mathbb{E}\left[\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)\sqrt{W_{xt}^2+W_{xx}^2}\right] = \frac{1}{\sqrt{8\pi}}\int_{\arctan a}^{\arctan b}(\lambda_1\cos^2\alpha+\lambda_2\sin^2\alpha)^{1/2}d\alpha$$

$$\text{where } a := \sqrt{\frac{\lambda_1}{\lambda_2}}\left(\frac{v_1-\tan\beta}{1+v_1\tan\beta}\right) \quad\text{and}\quad b := \sqrt{\frac{\lambda_1}{\lambda_2}}\left(\frac{v_2-\tan\beta}{1+v_2\tan\beta}\right), \tag{28}$$

and where $\lambda_1 \geq \lambda_2$ are the eigenvalues of the covariance matrix of (W_{xt}, W_{xx}), and β is the rotation angle which diagonalizes the covariance matrix.

We are now interested in getting the Hermite expansion of the curve length with speed included in $[-v_2, -v_1]$, denoted by $\mathcal{L}(\mathcal{C}_v(u))$.

Theorem 2 *The Hermite expansion of the curve length with speed included in* $[-v_2, -v_1]$ *is given by:*

$$\mathcal{L}(\mathcal{C}_v(u)) = \sum_{l=0}^{\infty}\sum_{0\leq n+m\leq l}\frac{H_{l-(n+m)}(u)\varphi(u)g_{n,m}}{[l-(n+m)]!}\int_0^T\int_0^M H_{l-(n+m)}(W_x(t,x))H_m\left(\frac{\eta_1(t,x)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(t,x)}{\sqrt{\lambda_2}}\right)dxdt,$$

where the Hermite coefficients $g_{n,m}$ *are defined in (30) below and where the random vector* $(\eta_1(t,x), \eta_2(t,x))$ *has a normal distribution* $\mathcal{N}\left(0, \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\right)$.

Proof. The proof of this result relies mainly on the application of the co-area formula and the method used in [9].
Let us consider the random vector: $(W_x(t,x), W_{xx}(t,x), W_{xt}(t,x))$, with covariance matrix $\Delta = (\Delta_{ij})_{1\leq i,j\leq 3}$ such that $\Delta_{11} = M_{20}$, $\Delta_{22} = M_{40}$, $\Delta_{33} = M_{22}$, $\Delta_{12} = \Delta_{13} = 0$, $\Delta_{23} = M_{31}$. Hence for fixed (t,x), the random variable $W_x(t,x)$ is independent of the vector $(W_{xx}(t,x), W_{xt}(t,x))$. The co-area formula allows writing

$$\int_{-\infty}^{\infty} f(u)\left[\int_{\mathcal{C}(u)}\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma\right]du$$

$$= \int_0^T\int_0^M f(W_x(t,x))\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)(W_{xx}^2(t,x)+W_{xt}^2(t,x))^{1/2}dxdt. \tag{29}$$

We can show that

$$\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-y}{h}\right)\left[\int_{\mathcal{C}(y)}\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma\right]dy \underset{h\to 0}{\to} \int_{\mathcal{C}(u)}\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma,$$

and the convergence is in $L^2(\Omega)$. By using the formula (29), we obtain

$$\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-y}{h}\right)\left[\int_{\mathcal{C}(y)}\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma\right]dy$$

$$= \frac{1}{h}\int_0^T\int_0^M\varphi\left(\frac{u-W_x(t,x)}{h}\right)\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)(W_{xx}^2(t,x)+W_{xt}^2(t,x))^{1/2}dxdt.$$

Let Σ be the covariance matrix of the random vector $(W_{xx}(t,x), W_{xt}(t,x))$ and P be the rotation matrix of angle β that diagonalizes Σ. Thus we can write

$$\begin{pmatrix}W_{xx}(t,x)\\ W_{xt}(t,x)\end{pmatrix} = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\eta_1(t,x)\\ \eta_2(t,x)\end{pmatrix}.$$

Let $\eta(t,x) := \arctan\left(\frac{\eta_2(t,x)}{\eta_1(t,x)}\right)$, so we can write

$$\begin{aligned}
&\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)(W_{xx}^2(t,x)+W_{xt}^2(t,x))^{1/2} = \mathbb{1}_{[v_1,v_2]}(\tan(\eta(t,x)+\beta))(\eta_1^2(t,x)+\eta_2^2(t,x))^{1/2} \\
&= \mathbb{1}_{[\arctan v_1-\beta,\arctan v_2-\beta]}(\eta(t,x))\ (\eta_1^2(t,x)+\eta_2^2(t,x))^{1/2} \\
&= \mathbb{1}_{\left[\frac{v_1-\tan\beta}{1+v_1\tan\beta},\frac{v_2-\tan\beta}{1+v_2\tan\beta}\right]}\left(\frac{\eta_2(t,x)}{\eta_1(t,x)}\right)\ \left(\eta_1^2(t,x)+\eta_2^2(t,x)\right)^{1/2} \\
&= \mathbb{1}_{[a,b]}\left(\frac{\sqrt{\lambda_1}\eta_2(t,x)}{\sqrt{\lambda_2}\eta_1(t,x)}\right)\left(\lambda_1\left(\frac{\eta_1(t,x)}{\sqrt{\lambda_1}}\right)^2+\lambda_2\left(\frac{\eta_2(t,x)}{\sqrt{\lambda_2}}\right)^2\right)^{1/2},
\end{aligned}$$

where a and b are given in (28).
In conclusion we have

$$\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)(W_{xx}^2(t,x)+W_{xt}^2(t,x))^{1/2} = \mathbb{1}_{[a,b]}\left(\frac{\sqrt{\lambda_1}\eta_2(t,x)}{\sqrt{\lambda_2}\eta_1(t,x)}\right)\left(\lambda_1\left(\frac{\eta_1(t,x)}{\sqrt{\lambda_1}}\right)^2+\lambda_2\left(\frac{\eta_2(t,x)}{\sqrt{\lambda_2}}\right)^2\right)^{1/2}.$$

Let us remark that, for fixed t and x, the random vector $\left(\frac{\eta_1(t,x)}{\sqrt{\lambda_1}},\frac{\eta_2(t,x)}{\sqrt{\lambda_2}}\right)$ has a normal distribution $\mathcal{N}(0,Id)$.
We look for the Hermite expansion of the function

$$G(Z_1,Z_2)=\mathbb{1}_{[a,b]}\left(\frac{Z_2}{Z_1}\right)\left(\lambda_1 Z_1^2+\lambda_2 Z_2^2\right)^{1/2} \text{ for } (Z_1,Z_2) \rightsquigarrow \mathcal{N}(0,Id).$$

So let us compute its Hermite coefficients. We have

$$\begin{aligned}
g_{n,m} &= \frac{1}{n!m!}\int_{\mathbb{R}^2} G(z_1,z_2)H_n(z_1)H_m(z_2)\varphi(z_1)\varphi(z_2)dz_1dz_2 \\
&= \frac{1}{2\pi n!m!}\int_0^\infty\int_0^{2\pi} G(\rho\cos\theta,\rho\sin\theta)H_n(\rho\cos\theta)H_m(\rho\sin\theta)e^{-\frac{\rho^2}{2}}\rho d\theta d\rho \\
&= \frac{1}{2\pi n!m!}\int_0^\infty\int_{\arctan a}^{\arctan b}(\lambda_1\cos^2\theta+\lambda_2\sin^2\theta)^{1/2}H_n(\rho\cos\theta)H_m(\rho\sin\theta)e^{-\frac{\rho^2}{2}}\rho^2 d\theta d\rho.
\end{aligned}$$

Since

$$\frac{H_{2n}(x)}{(2n)!}=\sum_{p=0}^{n}\frac{(-1)^{n-p}}{(2p)!(n-p)!2^{n-p}}\ x^{2p} \quad\text{ and }\quad \frac{H_{2n+1}(x)}{(2n+1)!}=\sum_{p=0}^{n}\frac{(-1)^{n-p}}{(2p+1)!(n-p)!2^{n-p}}\ x^{2p+1},$$

i.e.

$$\frac{H_n(x)}{n!}=\sum_{p=0}^{[n/2]}\frac{(-1)^{[n/2]-p}\ x^{2p}}{(2p)!([n/2]-p)!2^{[n/2]-p}}\left(\mathbb{1}_{(n\ even)}+\frac{x}{2p+1}\mathbb{1}_{(n\ odd)}\right),$$

we can write

$$\begin{aligned}
g_{n,m} = \frac{1}{2\pi}\sum_{p=0}^{[n/2]}\sum_{q=0}^{[m/2]}&\frac{(-1)^{[n/2]+[m/2]-(p+q)}}{(2p)!([n/2]-p)!(2q)!([m/2]-q)!2^{[n/2]+[m/2]-(p+q)}} \\
&\left(\mathbb{1}_{(n,m\ even)}I(2(p+q+1))J(2p,2q)+\frac{\mathbb{1}_{(n,m\ odd)}}{(2p+1)(2q+1)}I(2(p+q+2))J(2p+1,2q+1)\right. \\
&\left.+I(2(p+q+1)+1)\left(\frac{\mathbb{1}_{(n\ even,m\ odd)}}{(2q+1)}J(2p,2q+1)+\frac{\mathbb{1}_{(n\ odd,m\ even)}}{(2p+1)}J(2p+1,2q)\right)\right)
\end{aligned} \tag{30}$$

where

$$I(l):=\int_0^\infty e^{-\frac{\rho^2}{2}}\rho^l d\rho \qquad\text{and}\qquad J(\tilde{p},\tilde{q}):=\int_{\arctan a}^{\arctan b}(\lambda_1\cos^2\theta+\lambda_2\sin^2\theta)^{1/2}(\cos\theta)^{\tilde{p}}(\sin\theta)^{\tilde{q}}d\theta.$$

Note that $I(2l)=2^{l-1/2}\Gamma(l+1/2)$ and $I(2l-1)=2^l l!$ and, if $\lambda_1>\lambda_2$ and by putting $k^2:=1-\lambda_2/\lambda_1$, for $\tilde{p},\ \tilde{q}\ \neq 0$,

$$\begin{aligned}
J(\tilde{p},\tilde{q}) &= \sqrt{\lambda_1}\int_{\arctan a}^{\arctan b}\sqrt{1-k^2\sin^2\theta}(\cos\theta)^{\tilde{p}}(\sin\theta)^{\tilde{q}}d\theta \\
&= \frac{\sqrt{\lambda_1}}{2}\int_{a^2/(1+a^2)}^{b^2/(1+b^2)}(1-k^2x)^{1/2}(1-x)^{\frac{\tilde{p}-1}{2}}x^{\frac{\tilde{q}-1}{2}}dx \\
&= \frac{\sqrt{\lambda_1}}{2}\sum_{l=0}^{\infty}\left(\begin{array}{c}1/2\\ l\end{array}\right)(-1)^l k^{2l}\int_{a^2/(1+a^2)}^{b^2/(1+b^2)}(1-x)^{\frac{\tilde{p}-1}{2}}x^{\frac{2l+\tilde{q}-1}{2}}dx
\end{aligned}$$

$$=\frac{\sqrt{\lambda_1}}{2}\sum_{l=0}^{\infty}\left(\begin{array}{c}1/2\\ l\end{array}\right)(-1)^l k^{2l}\left[\mathcal{B}\left(\frac{b^2}{1+b^2},\frac{\tilde{p}+1}{2},\frac{2l+\tilde{q}+1}{2}\right)-\mathcal{B}\left(\frac{a^2}{1+a^2},\frac{\tilde{p}+1}{2},\frac{2l+\tilde{q}+1}{2}\right)\right],$$

where $\mathcal{B}$ denotes the incomplete beta function defined by $\mathcal{B}(\text{z,a,b}):=\int_0^z(1-x)^a x^b dx$.
So we obtain,

$$\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-v}{h}\right)\left[\int_{\mathcal{C}(v)}\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma\right]du$$

$$=\sum_{l=0}^{\infty}\sum_{0\leq n+m\leq l}\frac{c_{l-(n+m)}(u,h)g_{n,m}}{[l-(n+m)]!}\int_0^T\int_0^M H_{l-(n+m)}(W_x(t,x))H_m\left(\frac{\eta_1(t,x)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(t,x)}{\sqrt{\lambda_2}}\right)dxdt$$

with $g_{n,m}$ and c_k defined in (30) and (8) respectively.
Hence we can deduce, in the same way as in [9], the Hermite expansion of the curve length $\mathcal{L}\left(\mathcal{C}_v(u)\right)$ with speed included in $[-v_2,-v_1]$, as follows:

$$\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-y}{h}\right)\left[\int_{\mathcal{C}(y)}\mathbb{1}_{[v_1,v_2]}\left(\frac{W_{xt}}{W_{xx}}\right)d\sigma\right]dy \quad \overset{L^2}{\rightarrow} \quad \mathcal{L}\left(\mathcal{C}_v(u)\right), \quad \text{as } h\rightarrow 0,$$

where

$$\mathcal{L}\left(\mathcal{C}_v(u)\right)=\sum_{l=0}^{\infty}\sum_{0\leq n+m\leq l}\frac{H_{l-(n+m)}(u)\varphi(u)g_{n,m}}{[l-(n+m)]!}\int_0^T\int_0^M H_{l-(n+m)}(W_x(t,x))H_m\left(\frac{\eta_1(t,x)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(t,x)}{\sqrt{\lambda_2}}\right)dxdt.\square$$

We are of course interested in obtaining a central limit theorem for $\mathcal{L}\left(\mathcal{C}_v(u)\right)$; it will be the object of another work.

4 Specular points dynamics

In this section we are back to the process $W(t,x)$ whose representation spectral is given in (20). Longuet-Higgins has shown in [11] that the specular points evolving until a certain time up to the curvature of the curve, don't vanish. The number of images seen by the observer is not constant, specular points may appear or disappear, the images move. The creation or annihilation of specular points may be called a "twinkle": in those points the light reflection will be more intense and there will be a flash.
Defining the function

$$f(t,x)=W(t,x)+\frac{1}{2}\kappa x^2$$

the condition to have a twinckle will be translated as

$$\begin{cases} Y_1(t,x)=f_x(t,x)=0 & : \text{ to have a specular point} \\ Y_2(t,x)=f_{xx}(t,x)=0 & : \text{ to have a singularity in the curvature.} \end{cases}$$

Again we are interested in the number $\mathcal{TW}$ of flashes.
Let $\mathbf{Y}$ denotes the vectorial process $\mathbf{Y} = (Y_1(t,x), Y_2(t,x))$ such that:

$$\begin{cases} Y_1(t,x) = 0 \\ Y_2(t,x) = 0. \end{cases}$$

Let us suppose that the process W has fourth order continuous derivatives and let us M_{pq} denotes the mixed spectral moment:

$$M_{pq} = \int_{-\infty}^{\infty} \lambda^{p+q/2} f(\lambda)\, d\lambda.$$

Then we have

Proposition 2 *The mean number of twinkles at fixed time t and as $x \to \infty$, is given by*

$$\mathbb{E}[\mathcal{TW}] = \frac{t}{\sqrt{2\pi^3}} \frac{\sqrt{M_{60}\,(M_{40}M_{22} - M_{31}^2)}}{\kappa\; M_{40}}\; e^{-\frac{\kappa^2}{2M_{40}}} \left[e^{-\frac{a^2}{2}} + a \int_0^a e^{-\frac{v^2}{2}} dv \right], \tag{31}$$

where $a := \dfrac{M_{31}}{\sqrt{M_{40}\,(M_{40}M_{22} - M_{31}^2)}}$.

We get back the same formula as Longuet-Higgins ([11], p.853), except to a factor 2 that corresponds again to consider the integral in x over $(-\infty, \infty)$.

Proof.
We will need a multidimensional Rice formula to count the number of roots of a nonlinear system of equations having the same number of equations and variables. Such a formula, available in Cabaña ([6], p.80) and also in Azaïs & Wschebor ([2]), leads to

$$\begin{aligned}
&\mathbb{E}[\mathcal{TW}] = \\
&\int_0^t \int_0^x \tilde{p}_{(s,u))}(0,0)\; \mathbb{E}[|f_{xx}(s,u) f_{xxt}(s,u) - f_{xt}(s,u) f_{xxx}(s,u)| \mid f_x(s,u) = 0, f_{xx}(s,u) = 0] du\, ds \\
&= \int_0^t \int_0^x p(-\kappa u, -\kappa)\; \mathbb{E}[|W_{xt}(s,u) W_{xxx}(s,u)| \mid W_x(s,u) = -\kappa u, W_{xx}(s,u) = -\kappa] du\, ds,
\end{aligned} \tag{32}$$

where $\tilde{p}_{(s,u))}(u_1,u_2)$ is the joint density of the random vector (f_x, f_{xx})
and $p(u_1,u_2)$ is the joint density of (W_x, W_{xx}).
Note that

$$\mathbb{E}[W_x(s,u) W_{xt}(s,u)] = \mathbb{E}[W_x(s,u) W_{xx}(s,u)] = 0,\; \mathbb{E}[W_{xx}(s,u) W_{xt}(s,u)] = M_{31},\; \mathbb{E}[W_x W_{xxx}(s,u)] = -M_{40}.$$

By using the regression model:

$$\begin{cases} W_{xt}(s,u) = v_1 W_x(s,u) + \beta_1 W_{xx}(s,u) + e_1 \\ W_{xxx}(s,u) = v_2 W_x(s,u) + \beta_2 W_{xx}(s,u) + e_2, \end{cases}$$

where (e_1, e_2) is a Gaussian vector independent of $(W_x(s,u), W_{xx}(u,s))$, and $v_1 = 0$, $\beta_1 = \frac{-M_{31}}{M_{40}}$,
$v_2 = -\frac{M_{40}}{M_{20}}$ and $\beta_2 = 0$, the equation (32) becomes:

$$\begin{aligned}
\mathbb{E}[\mathcal{TW}] &= \int_0^t \int_0^x p(-\kappa u, -\kappa) \mathbb{E}\left[\left| \left(\frac{M_{31}}{M_{40}} \kappa + e_1 \right) \left(\frac{M_{40}}{M_{20}} \kappa\, u + e_2 \right) \right| \right] du\, ds \\
&= \frac{e^{-\frac{\kappa^2}{2M_{40}}}}{2\pi \sqrt{M_{20} M_{40}}} \int_0^t \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \mathbb{E}\left[\left| \left(\frac{M_{31}}{M_{40}} \kappa + e_1 \right) \left(\frac{M_{40}}{M_{20}} \kappa\, u + e_2 \right) \right| \right] du\, ds \\
&= t \frac{e^{-\frac{\kappa^2}{2M_{40}}}}{2\pi \sqrt{M_{20} M_{40}}} \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \mathbb{E}\left[\left| \left(\frac{M_{31}}{M_{40}} \kappa + e_1 \right) \left(\frac{M_{40}}{M_{20}} \kappa\, u + e_2 \right) \right| \right] du \\
&= t \frac{e^{-\frac{\kappa^2}{2M_{40}}}}{2\pi \sqrt{M_{20} M_{40}}} \mathbb{E}\left[\left| \frac{M_{31}}{M_{40}} \kappa + e_1 \right| \right] \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \mathbb{E}\left[\left| \frac{M_{40}}{M_{20}} \kappa\, u + e_2 \right| \right] du.
\end{aligned}$$

On one hand, since $\sigma_1^2 := \mathbb{E}[e_1^2] = \dfrac{M_{22}M_{40} - M_{31}^2}{M_{40}}$, and $\sigma_2^2 := \mathbb{E}[e_2^2] = \dfrac{M_{60}M_{20} - M_{40}^2}{M_{20}}$, then

$$\mathbb{E}\left|\frac{M_{31}}{M_{40}}\kappa + e_1\right| = \sigma_1\sqrt{\frac{2}{\pi}}\left[\frac{M_{31}}{M_{40}\sigma_1}\,\kappa\int_0^{\frac{M_{31}}{M_{40}\sigma_1}\kappa} e^{-\frac{v^2}{2}}\,dv + e^{-\frac{(\frac{M_{31}}{M_{40}\sigma_1})^2\kappa^2}{2}}\right].$$

On the other hand, we have

$$\begin{aligned}
\int_0^x e^{-\frac{\kappa^2u^2}{2M_{20}}}\,\mathbb{E}\left|\frac{M_{40}}{M_{20}}u\kappa + e_2\right| du \quad &= \quad \int_0^x e^{-\frac{\kappa^2u^2}{2M_{20}}}\sigma_2\sqrt{\frac{2}{\pi}}\left[\frac{M_{40}}{M_{20}\sigma_2}u\kappa\int_0^{\frac{M_{40}}{M_{20}\sigma_2}u\kappa} e^{-\frac{v^2}{2}}\,dv + e^{-\frac{\left(\frac{M_{40}}{M_{20}\sigma_2}\right)^2\kappa^2u^2}{2}}\right] du\\
&\underset{x\to\infty}{\longrightarrow} \quad \sigma_2\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{\kappa^2u^2}{2M_{20}}}\left[\frac{M_{40}}{M_{20}\sigma_2}u\kappa\int_0^{\frac{M_{40}}{M_{20}\sigma_2}u\kappa} e^{-\frac{v^2}{2}}\,dv + e^{-\frac{\left(\frac{M_{40}}{M_{20}\sigma_2}\right)^2\kappa^2u^2}{2}}\right] du\\
&= \quad \sigma_2\sqrt{\frac{2}{\pi}}\left[\frac{M_{40}^2}{\sigma_2^2M_{20}}\int_0^\infty e^{-\frac{\kappa^2u^2}{2M_{20}}\left(1+\frac{M_{40}^2}{\sigma_2^2M_{20}}\right)}du + \int_0^\infty e^{-\frac{\kappa^2u^2}{2M_{20}}\left(1+\frac{M_{40}^2}{\sigma_2^2M_{20}}\right)}du\right]\\
&= \quad \sigma_2\sqrt{\frac{2}{\pi}}\left[\left(\frac{M_{40}^2}{\sigma_2^2M_{20}}+1\right)\int_0^\infty e^{-\frac{\kappa^2u^2}{2M_{20}}\left(1+\frac{M_{40}^2}{\sigma_2^2M_{20}}\right)}du\right]\\
&= \quad \frac{1}{\kappa}(M_{40}^2+\sigma_2^2M_{20})^{1/2}.
\end{aligned}$$

Hence we obtain

$$\mathbb{E}[\mathcal{TW}] = \frac{t}{\kappa}\,\frac{e^{-\frac{\kappa^2}{2M_{40}}}}{\sqrt{2\pi^3}\sqrt{M_{20}M_{40}}}\,\sigma_1\left[\frac{M_{31}}{M_{40}\sigma_1}\,\kappa\int_0^{\frac{M_{31}}{M_{40}\sigma_1}\kappa} e^{-\frac{v^2}{2}}\,dv + e^{-\frac{\left(\frac{M_{31}}{M_{40}\sigma_1}\right)^2\kappa^2}{2}}\right](M_{40}^2+\sigma_2^2M_{20})^{1/2},$$

which gives (31), with $a := \frac{M_{31}}{M_{40}\sigma_1}$. $\square$.

The computation of the variance and to provide a CLT will be subject to further work.

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Marie Kratz
U.F.R. de Mathématiques et Informatique
Université René Descartes, ParisV
45 rue des Saints-Pères
75270 Paris cedex 06
FRANCE
`kratz@math-info.univ-paris5.fr`
`http://www.math-info.univ-paris5.fr/~ kratz`

José R. León
Escuela de Matemática, Facultad de Ciencias
Universidad Central de Venezuela
A.P. 47197 Los Chaguaramos
Caracas 1041-A
VENEZUELA
`jleon@euler.ciens.ucv.ve`
`http://www.euler.ciens.ucv.ve`