# Wiener integrals, Malliavin calculus and covariance measure structure

Ida Kruk [1] Francesco Russo [1] Ciprian A. Tudor [2]

[1] Université de Paris 13, Institut Galilée, Mathématiques,
99, avenue J.B. Clément, F-93430, Villetaneuse Cedex, France.

[2]SAMOS/MATISSE, Centre d'Economie de La Sorbonne,
Université de Panthéon-Sorbonne Paris 1,
90, rue de Tolbiac, 75634 Paris Cedex 13, France.

July 6, 2006

**Abstract**

We introduce the notion of *covariance measure structure* for square integrable stochastic processes. We define Wiener integral, we develop a suitable formalism for stochastic calculus of variations and we make Gaussian assumptions only when necessary. Our main examples are finite quadratric variation processes with stationary increments and the bifractional Brownian motion.

**Key words and phrases:** Square integrable processes, covariance measure structure, Malliavin calculus, Skorohod integral, bifractional Brownian motion.

**2000 Mathematics Subject Classification:** 60G12, 60G15, 60H05, 60H07.

## 1 Introduction

Different approaches have been used to extend the classical Itô's stochastic calculus. When the integrator stochastic process does not have the semimartingale property, then the powerful Itô's theory cannot be applied to integrate stochastically. Hence alternative ways have been then developed, essentially of two types:

- a trajectorial approach, that mainly includes the rough paths theory (see [33]) or the stochastic calculus via regularization (see [35]).
- a Malliavin calculus (or stochastic calculus of variations) approach.

Our main interest consists here in the second approach. Suppose that the integrator is a Gaussian process $X = (X_t)_{t\in[0,T]}$. The Malliavin derivation can be naturally developed on a Gaussian space. See, e.g. [45], [28] or [25]. A Skorohod (or divergence) integral can also be defined as the adjoint of the Malliavin derivative. The crucial ingredient is the canonical Hilbert space $\mathcal{H}$ (called also, improperly, by some authors reproducing kernel Hilbert space) of the Gaussian process $X$ which is defined as the closure of the linear space generated by the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R(t,s) \tag{1}$$

where $R$ denotes the covariance of $X$. Nevertheless, this calculus remains more or less abstract if the structure of the elements of the Hilbert space $\mathcal{H}$ is not known. When we say abstract, we refer to the fact that, for example, it is difficult to characterize the processes being integrable with respect to $X$, to estimate the $L^p$-norms of the Skorohod integrals or to push further this calculus to obtain an Itô type formula.

A particular case can be analyzed in a deeper way. We refer here to the situation when the covariance $R$ can be explicitly written as

$$R(t,s) = \int_0^{t\wedge s} K(t,u)K(s,u)du,$$

where $K(t,s)$, $0 < s < t < T$ is a deterministic kernel satisfying some regularity conditions. Enlarging, if need, our probability space, we can express the process $X$ as

$$X_t = \int_0^t K(t,s)dW_s \tag{2}$$

where $(W_t)_{t\in[0,T]}$ is a standard Wiener process and the above integral is understood in the Wiener sense. In this case, more concrete results can be proved (see [2]). The canonical space $\mathcal{H}$ can be written as

$$\mathcal{H} = (K^*)^{-1}\left(L^2([0,T])\right)$$

where the "transfer operator" $K^*$ is defined on the set of elementary functions as

$$K^*(\varphi)(s) = K(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s))\, K(dr,s)$$

and extended (if possible/when possible) to $\mathcal{H}$ (or a set of functions contained in $\mathcal{H}$). Consequently, a stochastic process $u$ will be Skorohod integrable with respect to $X$ if and only if $K^*(u)$ is Skorohod integrable with respect to $W$ and $\int u\delta X = \int (K^*u)\delta W$. Depending on the regularity of $K$ (in principal the Hölder continuity of $K$ and $\frac{\partial K}{\partial t}(t,s)$ are of interest) it becomes possible to have concrete results.

Of course, the most studied case is the case of the fractional Brownian motion (fBm), due to the multiple applications of this process in various area, like telecommunications, hydrology or economics. Recall that the fBm $(B_t^H)_{t\in[0,T]}$, with Hurst parameter $H\in(0,1)$ is defined as a centered Gaussian process starting from zero with covariance function

$$R(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t,s\in[0,T]. \tag{3}$$

The process $B^H$ admits the Wiener integral representation (2) and the kernel $K$ and the space $\mathcal{H}$ can be characterized by the mean of fractional integrals and derivatives. See [2], [3], [9], [32] among others. As a consequence, one can prove for a ny $H$ the following Itô's formula

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H)\delta B_s^H + H\int_0^t f''(B_s^H)s^{2H-1}ds.$$

One can also study the relation between "pathwise type" integrals and the divergence integral, the regularity of the Skorohod integral process or the Itô formula for indefinite integrals.

As we mentioned, if the deterministic kernel $K$ in the representation (2) is not explicitly known, then the Malliavin calculus with respect to the Gaussian process $X$ remains in an abstract form; and there are of course many situations when this kernel is not explicitly known. As main example, we have in mind the case of the *bifractional Brownian motion (bi-fBm).* This process, denoted by $B^{H,K}$, is defined as a centered Gaussian process starting from zero with covariance

$$R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right) \tag{4}$$

where $H\in(0,1)$ and $K\in(0,1]$. When $K=1$, then we have a standard fractional Brownian motion.

This process was introduced in [17] and a "pathwise type" approach to stochastic calculus was provided in [34]. An interesting property of $B^{H,K}$ consists in the expression of its quadratic variation (defined as usually as limit of Riemann sums, or in the sense of regularization, see [35]). The following properties hold true.

- If $2HK>1$, then the quadratic variation of $B^{H,K}$ is zero.
- If $2HK<1$ then the quadratic variation of $B^{H,K}$ does not exist.
- If $2HK=1$ then the quadratic variation of $B^{H,K}$ at time $t$ is equal to $2^{1-K}t$.

The last property is remarkable; indeed, for $HK = \frac{1}{2}$ we have a Gaussian process which has the same quadratic variation as the Brownian motion. Moreover, the processes is not a semimartingale (except for the case $K = 1$ and $H = \frac{1}{2}$), it is self-similar, has no stationary increments and it is a quasi-helix in the sense of J.P. Kahane [20], that is, for all $s \leq t$,

$$2^{-K}|t-s|^{2HK} \leq E\left|B_t^{H,K} - B_s^{H,K}\right|^2 \leq 2^{1-K}|t-s|^{2HK}. \tag{5}$$

We have no information on the form and/or the properties of the kernel of the bifractional Brownian motion. As a consequence, an intrinsic Malliavin calculus was not yet introduced for this process. On the other side, it is possible to construct a stochastic calculus of pathwise type, via regularization and one gets an Itô formula of the Stratonovich type (see [34])

$$f(B_t^{H,K}) = f(0) + \int_0^t f'(B_s^{H,K}) d^\circ B_s^{H,K}$$

for any parameters $H \in (0,1)$ and $K \in (0,1]$.

The purpose of this work is to develop a Malliavin calculus with respect to processes having a *covariance measure structure* in sense that the covariance is the distribution function of a (possibly signed) measure on $\mathcal{B}([0,T]^2)$. This approach is particularly suitable for processes whose representation form (2) is not explitely given.

We will see that under this assumption, we can define suitable spaces on which the construction of the Malliavin derivation/Skorohod integration is coherent.

In fact, our initial purpose is more ambitious; we start to construct a stochastic analysis for general (non-Gaussian) processes $X$ having a covariance measure $\mu$. We define Wiener integrals for a large enough class of deterministic functions and we define a Malliavin derivative and a Skorohod integral with respect to it; we can also prove certain relations and properties for these operators. However, if one wants to produce a consistent theory, then the Skorohod integral applied to deterministic integrands should coincide with the Wiener integral. This property is based on *integration by parts* on Gaussian spaces which is proved in Lemma 6.7. As it can be seen, that proof is completely based on the Gaussian character and it seems difficult to prove it for general processes. Consequently, in the sequel, we concentrate our study on the Gaussian case and we show various results as the continuity of the integral processes, the chaos expansion of local times, the relation between the "pathwise" and the Skorohod integrals and finally we derive an Itô formula. Our main examples include the Gaussian semimartingales, the fBm with $H \geq \frac{1}{2}$, the bi-fBm with $HK \geq \frac{1}{2}$ and processes with stationary increments. In the bi-fbm case, when $2HK = 1$, we find a very interesting fact, that is, the bi-fBm with $2HK = 1$ satisfies the same Itô formula as the standard Wiener process, that is

$$f(B_t^{H,K}) = f(0) + \int_0^t f'(B_s^{H,K}) \delta B_s^{H,K} + \frac{1}{2}\int_0^t f''(B_s^{H,K}) ds$$

where $\delta$ denotes the Skorohod integral.

We will also like to mention certain aspects that could be the object of a further study:

- the proof of the Tanaka formula involving weighted local times; for the fBm case, this has been proved in [7] but the proofs necessitates the expression of the kernel $K$.
- the two-parameter settings, as developed in e.g. [43].
- the proof of the Girsanov transform and the use of it to the study of stochastic equations driven by Gaussian noises, as e.g. in [30].

We organized our paper as follows. In Section 2 and 3 we explain the general context of our study: we define the notion of covariance measure structure and we give the basic properties of stochastic processes with this property. Section 4 contains several examples of processes having covariance measure $\mu$. Section 5 is consecrated to the construction of Wiener integrals for a large enough class of integrands with respect to (possibly non-Gaussian) process $X$ with $\mu$. In Section 6, for the same settings, we develop a Malliavin derivation and a Skorohod integration. Next, we work on a Gaussian space and our calculus assumes a more intrinsic form; we give concrete spaces of functions contained in the canonical Hilbert space of $X$ and this allows us to characterize the domain of the divergence integral, to have Meyer inequalities and other consequences. Finally, in Section 8 we present the relation "pathwise"-Skorohod integrals and we derive an Itô formula; some particular cases are discussed in details.

# 2 Preliminaries

In this paper, a rectangle will be a subset $I$ of $\mathbb{R}_+^2$ of the form

$$I = ]a_1, b_1] \times ]a_2, b_2]$$

and $T > 0$ will be fixed. Given $F : \mathbb{R}_+ \to \mathbb{R}$ we will denote

$$\Delta_I F = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2).$$

Such function will be said to vanish on the axes if $F(a_1, 0) = F(0, a_2) = 0$ for every $a_1, a_2 \in \mathbb{R}_+$.

Given a continuous function $F : [0,T] \to \mathbb{R}$ or a process $(X_t)_{t\in[0,T]}$, continuous in $L^2(\Omega)$, will be prolongated by convention to $\mathbb{R}$ by continuity.

**Definition 2.1** *$F : [0,T]^2 \to \mathbb{R}$ will be said to have a* **bounded planar variation** *if*

$$\sup_{\tau} \sum_{i,j=0}^{n} \left| \Delta_{]t_i,t_{i+1}]\times]t_j,t_{j+1}]} F \right| < \infty. \tag{6}$$

*where $\tau = \{0 = t_0 < \ldots < t_n = 1\}$ is a subdivision of $[0,T]$. A function $F$ will be said to be* **planarly increasing** *if for any rectangle $I \subset [0,T]^2$ we have $\Delta_I F \geq 0$.*

**Lemma 2.1** *Let $F : [0,T]^2 \to \mathbb{R}$ vanishing on the axes having a bounded planar variation. Then $F = F^+ - F^-$ where $F^+, F^-$ are planarly increasing and vanishing on the axes.*

**Proof:** It is similar to the result of the one-parameter result, which states that a bounded variation function can be decomposed into the difference of two increasing functions. The proof of this result is written for instance in [40] section 9-4. The proof translates into the planar case replacing $F(b) - F(a)$ with $\Delta_I F$. ■

**Lemma 2.2** *Let $F : [0,T]^2 \to \mathbb{R}_+$ be a continuous, planarly increasing function. Then there is a unique non-atomic, positive, finite measure $\mu$ on $\mathcal{B}([0,T]^2)$ such that for any $I \in \mathcal{B}([0,T]^2)$*

$$\mu(I) = \Delta_I F.$$

**Proof:** See Theorem 12.5 of [5]. ■

**Corollary 2.3** *Let $F : [0,T]^2 \to \mathbb{R}$ vanishing on the axes. Suppose that $F$ has bounded planar variation. Then, there is a signed, finite measure $\mu$ on $\mathcal{B}[0,T]^2$ such that for any rectangle $I$ of $[0,T]^2$*

$$\Delta_I F = \mu(I).$$

**Proof:** It is a consequence of Lemma 2.1 and 2.2. ■

We recall now the notion of finite quadratic planar variation introduced in [36].

**Definition 2.2** *A function $F : [0,T]^2 \to \mathbb{R}$ has* **finite quadratic planar variation** *if*

$$\frac{1}{\varepsilon^2} \int_{[0,T]^2} \left( \Delta_{]s_1,s_1+\varepsilon]\times]s_2,s_2+\varepsilon]} F \right)^2 ds_1 ds_2$$

*converges. That limit will be called the* **planar quadratic variation of** $F$.

We introduce now some notions related to stochastic processes. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $(Y_t)_{t\in[0,T]}$ with paths in $L^1_{loc}$ and $(X_t)_{t\in[0,T]}$ be a cadlag $L^2$-continuous process. Let $t \geq 0$. We denote by

$$\begin{aligned} I^-_\varepsilon(Y, dX, t) &= \frac{1}{\varepsilon} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds \\ I^+_\varepsilon(Y, dX, t) &= \frac{1}{\varepsilon} \int_0^t Y_s \frac{X_s - X_{s-\varepsilon}}{\varepsilon} ds \\ C_\varepsilon(X, Y, t) &= \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) ds. \end{aligned}$$

We set

$$\int_0^t Y d^- X \text{ (resp. } \int_0^t Y d^+ X)$$

the limit in probability of

$$I_\varepsilon^-(Y, dX, t) \text{ (resp. } I_\varepsilon^+(Y, dX, t)).$$

$\int_0^t Y d^- X$ (resp. $\int_0^t Y d^+ X$) is called (definite) **forward** (resp. **backward**) integral of Y with respect to X. We denote by $[X, Y]_t$ the limit in probability of $C_\varepsilon(X, Y, t)$. $[X, Y]_t$ is called **covariation** of $X$ and $Y$. If $X = Y$, $[X, X]$ is called **quadratic variation** of X, also denoted by $[X]$.

**Remark 2.4** *If $I$ is an interval with end-points $a < b$, then*

$$\int_0^T 1_I d^- X = \int_0^T 1_I d^+ X = X_b - X_a.$$

Let $(\mathcal{F}_t)_{t \in [0,T]}$ be a filtration satisfying the usual conditions. We recall, see [35], that if $X$ is an $(\mathcal{F}_t)$-semimartingale and $Y$ is a cadlag process (resp. an $(\mathcal{F}_t)$-semimartingale) then $\int_0^t Y d^- X$ (resp. $[Y, X]$) is the Itô integral (resp. the classical covariation).

# 3 Square integrable processes and covariance measure structure

In this section we will consider a cadlag zero-mean square integrable process $(X_t)_{t \in [0,T]}$ with covariance

$$R(s, t) = Cov(X_s, X_t).$$

For simplicity we suppose that $t \to X_t$ is continuous in $L^2(\Omega)$. $R$ defines naturally a finitely additive function $\mu_R$ (or simply $\mu$) on the algebra $\mathcal{R}$ of finite disjoint rectangles included in $[0, T]^2$ with values on $\mathbb{R}$. We set indeed

$$\mu(I) = \Delta_I R.$$

A typical example of square integrable processes are Gaussian processes.

**Definition 3.1** *A square integrable process will be said to have a* **covariance measure** *if $\mu$ extends to the Borel $\sigma$-algebra $\mathcal{B}([0, T]^2)$ to a signed $\sigma$-finite measure.*

We recall that $\sigma(I \text{ rectangle}, I \subset [0, T]^2) = \mathcal{B}([0, T]^2)$.

**Remark 3.1** *The process $(X_t)_{t \in [0,T]}$ has covariance measure if and only if $R$ has a bounded planar variation, see Corollary 2.3.*

**Definition 3.2** *Let us recall a classical notion introduced in [15] and [14]. A process* $(X_t)_{t\in[0,T]}$ *has* **finite energy** *(in the sense of discretization) if*

$$\sup_{\tau}\sum_{i=0}^{n-1}E(X_{t_{i+1}}-X_{t_i})^2<\infty.$$

*Note that if $X$ has a covariance measure then it has finite energy. Indeed for a given subdivision $t_0<t_1<\ldots<t_n$, we have*

$$\sum_{i=0}^{n-1}E(X_{t_{i+1}}-X_{t_i})^2=\sum_{i=0}^{n-1}\Delta_{]t_i,t_{i+1}]^2}R\le\sum_{i,j=0}^{n-1}\left|\Delta_{]t_i,t_{i+1}]\times]t_j,t_{j+1}]}R\right|.$$

**Remark 3.2** *Let $X$ be a process with covariance measure. Then $X$ has a supplementary property related to the energy. There is a function $\mathcal{E}:[0,T]\to\mathbb{R}_+$ such that, for each sequence of subdivisions $(\tau^N)=\{0=t_0<t_1<\ldots<t_n=T\}$, whose mesh converges to zero, the quantity*

$$\sum_{i=1}^{n}E\left(X_{t_{i+1}\wedge t}-X_{t_i\wedge t}\right)^2, \tag{7}$$

*converges uniformly in $t$, to $\mathcal{E}$.*

*Indeed*

$$\sum_{i=1}^{n}E\left(X_{t_{i+1}\wedge t}-X_{t_i\wedge t}\right)^2=\mu(D^N\cap[0,t]^2)$$

*where $D^N=\bigcup_{i=0}^{n-1}]t_i,t_{i+1}]^2$. We have $\bigcap_{N=0}^{\infty}D^N=\{(s,s)|s\in[0,T]\}$ . ¿From now on we will set*

$$D=\{(s,s)|s\in[0,T]\}\ \text{ and } D_t=D\cap[0,t]^2.$$

*Then, for every $t\in[0,T]$,*

$$\mathcal{E}(t)=\mu(D_t).$$

We will introduce the notion of energy in the sense of regularization (see [36]).

**Definition 3.3** *A process $(X_t)_{t\in[0,T]}$ is said to have* **finite energy** *if*

$$\lim_{\varepsilon\to0}E(C_\varepsilon(X,X,t))$$

*uniformly exists. This limit will be further denoted by $\mathcal{E}(X)(t)$.*

From now on, if we do not explicit contrary, we will essentially use the notion of energy in the sense of regularization.

**Lemma 3.3** *If $X$ has a covariance measure $\mu$, then it has finite energy. Moreover*

$$\mathcal{E}(X)(t) = \mu(D_t).$$

**Proof:** It holds that

$$\begin{aligned} E(C_\varepsilon(X,X,t)) &= \frac{1}{\varepsilon}\int_0^t dsE\left(X_{s+\varepsilon}-X_s\right)^2 = \frac{1}{\varepsilon}\int_0^t ds\Delta_{]s,s+\varepsilon]^2}R \\ &= \frac{1}{\varepsilon}\int_0^t ds \int_{]s,s+\varepsilon]^2} d\mu(y_1,y_2) = \frac{1}{\varepsilon}\int_{[0,t+\varepsilon]^2} d\mu(y_1,y_2) f_\varepsilon(y_1,y_2), \end{aligned}$$

where

$$f_\varepsilon(y_1,y_2) = \begin{cases} \frac{1}{\varepsilon}Leb(](y_1-\varepsilon)\vee(y_2-\varepsilon)\vee 0, y_1\wedge y_2]) & \text{if } y_1 \in ]y_2-\varepsilon, y_2+\varepsilon] \\ 0 & \text{otherwise.} \end{cases}$$

We observe that

$$|f_\varepsilon(y_1,y_2)| \leq \frac{2\varepsilon}{\varepsilon} = 2 \text{ and } f_\varepsilon(y_1,y_2) \stackrel{\varepsilon\to 0}{\longrightarrow} \begin{cases} 1 & : y_2 = y_1 \\ 0 & : y_2 \neq y_1. \end{cases}$$

So by Lebesgue dominated convergence theorem,

$$E(C_\varepsilon(X,X,t)) \to \mathcal{E}(X)(t),$$

with $\mathcal{E}(X)(t) = \mu(D_t)$. ■

We recall a result established in [36], see Proposition 3.9.

**Lemma 3.4** *Let $(X_t)_{t\in[0,T]}$ be a continuous, zero-mean Gaussian process with finite energy. Then $C_\varepsilon(X,X,t)$ converges in probability and it is deterministic for every $t \in [0,T]$ if and only if the planar quadratic variation of $R$ is zero. In that case $[X,X]$ exists and equals $\mathcal{E}(X)$.*

This allows to establish the following result.

**Proposition 3.5** *Let $(X_t)_{t\in[0,T]}$ be a zero-mean continuous Gaussian process, $X_0 = 0$, having a covariance measure $\mu$. Then*

$$[X,X]_t = \mu(D_t).$$

*In particular the quadratic variation is deterministic.*

**Proof:** First, if R has bounded planar variation, then it has zero planar quadratic variation. Indeed, by Corollary 2.3 $R$ has a covariance measure $\mu$ and so

$$\frac{1}{\varepsilon}\int_0^1 dt\left(\Delta_{]t,t+\varepsilon]^2}R\right)^2 \le \frac{1}{\varepsilon}\int_0^1 dt|\mu|\left(]t,t+\varepsilon]^2\right)\Gamma(\varepsilon), \tag{8}$$

where

$$\Gamma(\varepsilon) = \sup_{|s-t|<\varepsilon}\left|\Delta_{]s,t]^2}R\right|.$$

Since $R$ is uniformly continuous, $\Gamma(\varepsilon) \to 0$. Using the same argument as in the proof of Lemma 3.3 we conclude that (8) converges to zero.

Second, observe that Lemma 3.3 implies that $X$ has finite energy. Therefore the result follows from Lemma 3.4. ■

# 4 Some examples of processes with covariance measure

## 4.1 $X$ is a Gaussian martingale

It is well known, see [36], [39], that $[X]$ is deterministic. We denote $\psi(t) = [X]_t$. In this case

$$R(s_1, s_2) = \psi(s_1 \wedge s_2)$$

so that

$$\mu(B) = \int_B \delta(ds_2 - s_1)\psi(ds_1), \quad B \in \mathcal{B}([0,T]^2)$$

If $X$ is a classical Wiener process, then $\psi(x) = x$. The support of $\mu$ is the diagonal $D$ so $\mu$ and the Lebesgue measure are mutually singular.

## 4.2 $X$ is a Gaussian $(\mathcal{F}_t)$-semimartingale

We recall (see [39], [12]) that $X$ is a semimartingale if and only if it is a quasimartingale, i.e.

$$E\left(\sum_{j=0}^{n-1}\left|E\left(X_{t_{j+1}} - X_{t_j}|\mathcal{F}_{t_j}\right)\right|\right) \le K.$$

We remark that if $X$ is an $(\mathcal{F}_t)$-martingale, or a process such that $E\left(\|X\|_T\right) < \infty$, where $\|X\|_T$ is the total variation, then the above condition is easily verified. According to [39] $\mu$ extends to a measure.

### 4.3 $X$ is a fractional Brownian motion $B^H, H > \frac{1}{2}$

We recall that its covariance equals, for every $s_1, s_2 \in [0, T]$

$$R(s_1, s_2) = \frac{1}{2}\left(s_1^{2H} + s_2^{2H} - |s_2 - s_1|^{2H}\right).$$

In that case $\frac{\partial^2 R}{\partial s_1 \partial s_2} = 2H(2H-1)\left|s_2 - s_1\right|^{2H-2}$ in the sense of distributions. Since $R$ vanishes on the axes, we have

$$R(t_1, t_2) = \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \frac{\partial^2 R}{\partial s_1 \partial s_2}.$$

The function $R$ has bounded planar variation because $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ is non-negative. Therefore, for given $I = ]a_1, a_2] \times ]b_1, b_2]$, we have

$$|\Delta_I R| = \Delta_I R.$$

Hence for a subdivision $(t_i)_{i=0}^N$ of $[0, T]$

$$\sum_{i,j=0}^{N} \left|\Delta_{]t_i, t_{i+1}] \times ]t_j, t_{j+1}]} R\right| = \Delta_{]t_i, t_{i+1}] \times ]t_j, t_{j+1}]} R = R(T, T).$$

So the condition (6) is verified.

### 4.4 $X$ is a bifractional Brownian motion with $H \in (0,1), K \in (0,1]$ and $2HK \geq 1$

We refer to [17], [34] for the definition and the basic properties of this process. The covariance of the bi-fBm is given by (4). We can write its covariance as

$$R(s_1, s_2) = R_1(s_1, s_2) + R_2(s_1, s_2),$$

where

$$R_1(s_1, s_2) = \frac{1}{2^K}\left(s_1^{2H} + s_2^{2H}\right)^K - \left(s_1^{2HK} + s_2^{2HK}\right) \tag{9}$$

and

$$R_2(s_1, s_2) = -\frac{1}{2^K}|s_2 - s_1|^{2HK} + s_1^{2HK} + s_2^{2HK}. \tag{10}$$

We therefore have

$$\frac{\partial^2 R_1}{\partial s_1 \partial s_2} = \frac{4H^2 K(K-1)}{2^K}\left(s_1^{2H} + s_2^{2H}\right)^{K-2} s_1^{2H-1} s_2^{2H-1}.$$

Since $R_1$ is of class $C^2(]0, T]^2)$ and $\frac{\partial^2 R_1}{\partial s_1 \partial s_2}$ is always negative, $R_1$ is the distribution function of a negative absolutely continuous finite measure, having $\frac{\partial^2 R_1}{\partial s_1 \partial s_2}$ for density.

Concerning the term $R_2$ we suppose $2HK \geq 1$. $R_2$ is (up to a constant) also the covariance function of a fractional Brownian motion of index $HK$.

- If $2HK > 1$ then $R_2$ is the distribution function of an absolutely continuous positive measure with density $\frac{\partial^2 R_2}{\partial s_1 \partial s_2} = 2HK(2HK-1)\,|s_1 - s_2|^{2HK-2}$ which belongs of course to $L^1([0,1]^2)$.
- If $2HK = 1$, $R_2(s_1, s_2) = s_1 + s_2 - \frac{1}{2^K}|s_1 - s_2|$.

### 4.5 Processes with weakly stationary increments

A process $(X_t)_{t\in[0,T]}$ with covariance $R$ is said **with weakly stationary increments** if for any $s, t \in [0,T]$, $h \geq 0$, the covariance $R(t+h, s+h)$ does not depend on h.

**Remark 4.1** *1. If $(X_t)_{t\in[0,T]}$ is a Gaussian process then $(X_t)_{t\in[0,T]}$ is with weakly stationary increments if and only it has stationary increments, that is, for every subdivision $0 = t_0 < t_1 < \ldots < t_n$ and for every $h \geq 0$ the law of $(X_{t_1+h} - X_{t_0+h}, \ldots, X_{t_n+h} - X_{t_{n-1}+h})$ does not depend on h.*

*2. Malliavin-Skorohod calculus with respect to Gaussian processes with stationary increments was studied by [27] for a large class of functions $Q$ including logarithmic scales.*

We consider a zero-mean continuous in $L^2$ process $(X_t)_{t\in[0,T]}$ such that $X_0 = 0$ a.s. Let $d(s,t)$ be the associate canonical distance, i.e.

$$d^2(s,t) = E\left(X_t - X_s\right)^2, \quad s, t \in [0,T].$$

Since $(X_t)_{t\in[0,T]}$ has stationary increments one can write

$$d^2(s,t) = Q(t-s), \quad \text{where } Q(t) = d(0,t).$$

Therefore the covariance function $R$ can be expressed as

$$R(s,t) = \frac{1}{2}(Q(s) + Q(t) - Q(s-t)).$$

A typical example is provided when $X$ is a fractional Brownian motion $B^H$. In that case

$$Q(s) = s^{2H}.$$

**Remark 4.2** *Given a continuous increasing function $Q : \mathbb{R}_+ \to \mathbb{R}$ with $Q(0) = 0$, it is always possible to construct a centered Gaussian process with stationary increments, such that $VarX_t = Q(t)$ (see e.g. [26]).*

**Remark 4.3** *$X$ has finite energy if and only if $Q'(0+)$ exists. This follows immediately from the property*

$$E\left(C_\varepsilon(X, X, t)\right) = \frac{tQ(\varepsilon)}{\varepsilon}.$$

■

We can characterize conditions on $Q$ so that $X$ has a covariance measure.

**Proposition 4.4** *If $Q''$ is a Radon measure, then $X$ has a covariance measure.*

**Remark 4.5** *Previous assumption is equivalent to $Q'$ being of bounded variation. In that case $Q$ is absolutely continuous.*

**Proof** of the Proposition 4.4: Since

$$R(s_1, s_2) = \frac{1}{2} (Q(s_1) + Q(s_2) - Q(s_2 - s_1))$$

we have

$$\frac{\partial^2 R}{\partial s_1 \partial s_2} = -\frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} (Q(s_2 - s_1)) = \frac{1}{2} Q''(s_2 - s_1)$$

in the sense of distributions. This means in particular that for $\varphi, \psi \in \mathcal{D}(\mathbb{R})$ (the space of test functions with compact support)

$$\int_{\mathbb{R}^2} R(s_1, s_2) \varphi'(s_1) \psi'(s_2) ds_1 ds_2 = -\int_{\mathbb{R}} ds_2 \psi(s_2) \int_{\mathbb{R}} \varphi(s_1) \frac{Q(s_2 - ds_1)}{2}.$$

■

**Example 4.6** *We provide now an example of a process with stationary increments, investigated for financial applications purposes by [6]. It is called* **mixed fractional Brownian motion** *and it is defined as $X = W + B^H$, where $W$ is a Wiener process and $B^H$ is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, independent from $W$. [6] proves that $X$ is a semimartingale if and only if $H > \frac{3}{4}$.*

*$X$ is a Gaussian process with*

$$Q(t) = |t| + |t|^{2H}. \tag{11}$$

*Moreover*

$$Q''(dt) = 2\delta_0 + q'(t) dt,$$

*where $q(t) = |t|^{2H-1} 2H sign(t)$.*

The example above is still very particular.

Suppose that $Q''$ is a Radon measure. Then, the function $Q'$ can be decomposed in the following way

$$Q' = Q'_{sc} + Q'_d + Q'_{ac},$$

where $Q'_{sc}$ is continuous and singular, $Q'_{ac}$ is absolutely continuous and $Q''_d = \sum_n \gamma_n \delta_{x_n}$, with $(x_n)$ - sequence of nonnegative numbers and $\gamma_n \in \mathbb{R}$.

For instance if $Q$ is as in (11) then

$$Q'_{sc}(t) = 0, \quad Q'_{ac}(t) = 2H|t|^{2H-1} sign(t), \quad Q''_d(t) = 2\delta_0.$$

A more involved example is the following. Consider Gaussian zero-mean process $(X_t)_{t\in[0,T]}$ with stationary increments, $X_0 = 0$ such that

$$Q(t) = \begin{cases} |t| & : |t| \le \frac{1}{2} \\ 2^{2H-1}|t|^{2H} & : |t| > \frac{1}{2} \end{cases}$$

In this case it holds that

$$Q'(t) = \begin{cases} sign(t) & : |t| \le \frac{1}{2} \\ 2^{2H}H|t|^{2H-1}sign(t) & : |t| > \frac{1}{2} \end{cases}$$

and

$$Q''_d = 2\delta_0 + (2H-1)\delta_{\frac{1}{2}} + (2H-1)\delta_{-\frac{1}{2}},$$

$$Q''_{ac} = \begin{cases} 0 & : |t| < \frac{1}{2} \\ 2^{2H}H(2H-1)|t|^{2H-2} & : |t| > \frac{1}{2}. \end{cases}$$

**Proposition 4.7** *Let $(X_t)_{t\in[0,T]}$ be a Gaussian process with stationary increments such that $X_0 = 0$. Suppose that $Q''$ is a measure. Then*

$$[X]_t = \frac{Q''(\{0\})t}{2}.$$

**Proof:** It follows from the Proposition 3.5 and the fact that

$$\mu(ds_1, ds_2) = ds_1 Q''(ds_2 - s_1).$$

■

### 4.6 Non-Gaussian examples

A wide class of non-Gaussian processes having a covariance measure structure can be provided. We will illustrate how to produce such processes living in the second Wiener chaos. Let us define, for every $t \in [0,T]$,

$$Z_t = \int_{\mathbb{R}} \left( \int_0^t f(u, z_1) f(u, z_2) du \right) dB_{z_1} dB_{z_2},$$

where $(B_z)_{z\in\mathbb{R}}$ is a standard Brownian motion and $f : \mathbb{R}^2 \to \mathbb{R}$ is a Borel function such that

$$\int_0^T \int_0^T \left( \int_{\mathbb{R}} f(t,z) f(s,z) dz \right)^2 dsdt < \infty. \qquad (12)$$

Now, condition (12) assures that $\frac{\partial^2 R}{\partial s \partial t}$ belongs to $L^1([0,T]^2)$. Clearly, a large class of functions $f$ satisfies (12).

For example, the **Rosenblatt process** (see [42]) is obtained for $f(t,z) = (t-z)_+^{k-1}$ with $k \in (\frac{1}{4}, \frac{1}{2})$. In that case (12) is satisfied since $k > \frac{1}{4}$.

The covariance function of the process $Z$ is given by

$$\begin{aligned} R(t,s) &= \int_{\mathbb{R}^2} \left( \int_0^t f(u,z_1) f(u,z_2) du \right) \left( \int_0^s f(v,z_1) f(v,z_2) dv \right) dz_1 dz_2 \\ &= \int_0^t \int_0^s \left( \int_{\mathbb{R}^2} f(u,z_1) f(u,z_2) f(v,z_1) f(v,z_2) dz_1 dz_2 \right) dv du \end{aligned}$$

and thus

$$\begin{aligned} \frac{\partial^2 R}{\partial s \partial t} &= \int_{\mathbb{R}^2} f(t,z_1) f(t,z_2) f(s,z_1) f(s,z_2) dz_1 dz_2 \\ &= \left( \int_{\mathbb{R}} f(t,z) f(s,z) dz \right)^2 . \end{aligned}$$

In the case of the Rosenblatt process we get $\frac{\partial^2 R}{\partial s \partial t} = cst. |t-s|^{4K-2}$.

It is also possible to construct non-continuous processes that admit a covariance measure structure. Let us denote by $K$ the usual kernel of the fractional Brownian motion with Hurst parameter $H$ (actually, the kernel appearing in the Wiener integral representation (2) of the fBm) and consider $(\tilde{N}_t)_{t \in [0,T]}$ a compensated Poisson process (see e.g. [21]). Then $\tilde{N}$ is a martingale and we can define the integral

$$Z_t = \int_0^t K(t,s) d\tilde{N}_s.$$

The covariance of $Z$ can be written as

$$R(t,s) = \int_0^{t \wedge s} K(t,u) K(s,u) du = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

Then it is clear that for $H > \frac{1}{2}$ the above process $Z$ has covariance measure structure.

# 5 The Wiener integral

The Wiener integral, for integrators $X$ which are not the classical Brownian motion, was considered by several authors. Among the most recent references there are [32] for the case of fractional Brownian motion and [19] when $X$ is a second order process.

We will consider in this paragraph a zero-mean, square integrable, continuous in $L^2$, process $(X_t)_{t \in [0,T]}$ such that $X_0 = 0$. We denote by $R$ its covariance and we will suppose that $X$ has a covariance measure denoted by $\mu$ which is not atomic.

We construct here a Wiener integral with respect to such a process $X$. Our starting point is the following result (see for instance [36]): if $\varphi$ is a bounded variation continuous real function, it is well known that

$$\int_0^t \varphi d^- X = \varphi(t) X_t - \int_0^t X_s d\varphi_s, \ t \in [0,T].$$

Moreover, it holds that

$$\lim_{\varepsilon \to 0} I^-(\varepsilon, \varphi, dX, t) = \int_0^t \varphi d^- X \text{ in } L^2(\Omega). \tag{13}$$

We denote by $BV([0,T])$ the space of real functions with bounded variation, defined on $[0,T]$ and by $C^1([0,T])$ the set of functions on $[0,T]$ of class $C^1$. Clearly the above relation (13) holds for $\varphi \in C^1([0,T])$.

By $\mathcal{S}$ we denote the closed linear subspace of $L^2(\Omega)$ generated by $\int_0^t \varphi d^- X, \varphi \in BV([0,T])$. We define $\Phi : BV([0,T]) \to \mathcal{S}$ by

$$\Phi(\varphi) = \int_0^T \varphi d^- X.$$

We introduce the set $L_\mu$ as the vector space of Borel functions $\varphi : [0,T] \to \mathbb{R}$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := \int_{[0,T]^2} |\varphi(u)||\varphi(v)| d\,|\mu|\,(u,v) < \infty. \tag{14}$$

We will also use the alternative notation.

$$\|\varphi\|_{|\mathcal{H}|}^2 = \int_{[0,T]^2} |\varphi \otimes \varphi| d\,|\mu| < \infty. \tag{15}$$

**Remark 5.1** *A bounded function belongs to $L_\mu$, in particular if $I$ is a real interval, $1_I \in L_\mu$.*

For $\varphi,\ \phi \in L_\mu$ we set

$$\langle \varphi, \phi \rangle_{\mathcal{H}} = \int_{[0,T]^2} \varphi(u)\phi(v) d\mu(u,v). \tag{16}$$

**Lemma 5.2** *Let $\varphi,\ \phi \in BV([0,T])$. Then*

$$\langle \varphi, \phi \rangle_{\mathcal{H}} = E\left(\int_0^T \varphi d^- X \int_0^T \phi d^- X\right) \tag{17}$$

*and*

$$\|\varphi\|_{\mathcal{H}}^2 = E(\int_0^T \varphi d^- X)^2. \tag{18}$$

**Proof:** According to (13), when $\varepsilon \to 0$

$$E\left(\int_0^T \varphi(s)\frac{X_{s+\varepsilon}-X_s}{\varepsilon}ds \int_0^T \phi(u)\frac{X_{u+\varepsilon}-X_u}{\varepsilon}du\right). \tag{19}$$

converges to the right member of (17). We observe that (19) equals

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^T ds_1 \int_0^T ds_2 \varphi(s_1)\phi(s_2)E\left(\left(X_{s_1+\varepsilon}-X_{s_1}\right)\left(X_{s_2+\varepsilon}-X_{s_2}\right)\right)\\
&= \frac{1}{\varepsilon^2}\int_0^T ds_1 \int_0^T ds_2 \varphi(s_1)\phi(s_2)\Delta_{]s_1,s_s+\varepsilon]\times]s_2,s_2+\varepsilon]}R\\
&= \int_{[0,T]^2} d\mu(u_1,u_2)\frac{1}{\varepsilon^2}\int_{(u_1-\varepsilon)^+}^{u_1} ds_1\varphi(s_1)\int_{(u_2-\varepsilon)^+}^{u_2} ds_2\phi(s_2).
\end{aligned} \tag{20}$$

By Lebesgue dominated convergence theorem, when $\varepsilon \to 0$, the quantity (20) converges to

$$\int_{[0,T]^2} d\mu(u_1,u_2)\varphi(u_1)\phi(u_2)$$

and the lemma is therefore proved. ■

**Lemma 5.3** $(\varphi,\phi) \to \langle \varphi,\phi\rangle_{\mathcal{H}}$ *defines a semiscalar product on* $BV([0,T])$.

**Proof:** The bilinearity property is obvious. On the other hand, if $\varphi \in BV([0,T])$,

$$\langle \varphi,\varphi\rangle_{\mathcal{H}} = E\left(\int_0^T \varphi d^- X\right)^2 \geq 0. \tag{21}$$

■

We denote by $\|\cdot\|_{\mathcal{H}}$ the associated seminorm.

**Remark 5.4** *We use the terminology semiscalar product and seminorm since the property* $\langle \varphi,\varphi\rangle_{\mathcal{H}} \Rightarrow \varphi = 0$ *does not necessarily hold. Take for instance a process*

$$X_t = \begin{cases} 0, & t \leq t_0 \\ W_{t-t_0}, & t > t_0, \end{cases}$$

*where* $W$ *is a classical Wiener process.*

**Remark 5.5** *One of the difficulties in the sequel is caused by the fact that* $\|\cdot\|_{|\mathcal{H}|}$ *does not define a norm. In particular we do not have any triangle inequality.*

**Remark 5.6** *If $\mu$ is a positive measure, then $\| \cdot \|_{|\mathcal{H}|}$ constitutes a true seminorm. Indeed, if $f \in L_\mu$, we have*

$$\begin{aligned} \|f\|_{|\mathcal{H}|}^2 &= \int_{[0,T]^2} |f(u_1)||f(u_2)|d|\mu|(u_1, u_2) \\ &= \int_{[0,T]^2} |f|(u_1)|f|(u_2)d\mu(u_1, u_2) = \||f|\|_{\mathcal{H}}^2. \end{aligned}$$

*The triangle inequality follows easily.*

In particular if $X$ is a fractional Brownian motion $B$, $H \geq 1/2$, then $\|\cdot\|_{|\mathcal{H}|}$ constitutes a norm.

We introduce the marginal measure $\nu$ associated with $\mu$. We set

$$\nu(B) = |\mu|([0, T] \times B)$$

if $B \in \mathcal{B}([0, T])$.

**Lemma 5.7** *If $f \in L_\mu$, we have*

$$\|f\|_{\mathcal{H}} \leq \|f\|_{|\mathcal{H}|} \leq \|f\|_{L^2(\nu)},$$

*where $L^2(\nu)$ is the classical Hilbert space of square integrable functions on $[0, T]$ with respect to $\nu$.*

**Proof:** The first inequality is obvious. Concerning the second one, we operate via Cauchy-Schwartz inequality. Indeed,

$$\begin{aligned} \|f\|_{|\mathcal{H}|}^2 &= \int_{[0,T]^2} |f(u_1)f(u_2)|d|\mu|(u_1, u_2) \\ &\leq \left\{ \int_{[0,T]^2} f^2(u_1)d|\mu|(u_1, u_2) \int_{[0,T]^2} f^2(u_2)d|\mu|(u_1, u_2) \right\}^{\frac{1}{2}} = \int_0^T f^2(u)d\nu(u). \end{aligned}$$

■

Let $\mathcal{E}$ be the linear subspace of $L_\mu$ constituted by the linear combinations $\sum_i a_i 1_{I_i}$, where $I_i$ is a real interval.

**Lemma 5.8** *Let $\nu$ be a positive measure on $\mathcal{B}([0, T])$. Then $\mathcal{E}$ (resp. $C^\infty([0, T])$) is dense in $L^2(\nu)$.*

**Proof:**

i) We first prove that we can reduce to Borel bounded functions. Let $f \in L^2(\nu)$. We set $f_n = (f \wedge n) \vee (-n)$. We have $f_n \longrightarrow f$ pointwise (at each point). Consequently the quantity

$$\int_{[0,T]^2} |f_n - f|^2 d\nu \to_{n\to\infty} 0.$$

by the dominated convergence theorem.

ii) We can reduce to simple functions, i.e. linear combination of indicators of Borel sets. Indeed, any bounded Borel function $f$ is the limit of simple functions $f_n$, again pointwise. Moreover the sequence $(f_n)$ can be chosen to be bounded by $|f|$.

iii) At this point we can choose $f = 1_B$, where $B$ is a Borel subset of $[0,T]$. By the Radon property, for every $n$ there is an open subset $O$ of $[0,T]$ with $B \subset O$, such that $\nu(O \backslash B) < \frac{1}{n}$. This shows the existence of a sequence of $f_n = 1_{O_n}$, where $f_n \longrightarrow f$ in $L^2(\nu)$.

iv) Since every open set is a union of intervals, if $f = 1_O$, there is a sequence of step functions $f_n$ converging pointwise, monotonously to $f$.

v) The problem is now reduced to take $f = 1_I$, where $I$ is a bounded interval. It is clear that $f$ can be pointwise approximated by a sequence of $C^0$ functions $f_n$ such that $|f_n| \le 1$.

vi) Finally $C^0$ functions can be approximated by smooth functions via mollification; $f_n = \rho_n * f$ and $(\rho_n)$ is a sequence of mollifiers converging to the Dirac $\delta$-function.

The part concerning the density of elementary functions is contained in the previous proof. ■

We can now establish an important density proposition.

**Proposition 5.9** *The set $C^\infty([0,T])$ (resp. $\mathcal{E}$) is dense in $L_\mu$ with respect to $\|\cdot\|_{|\mathcal{H}|}$ and in particular to the seminorm $\|\cdot\|_{\mathcal{H}}$.*

**Remark 5.10** *As observed in Remark 5.5, in general $\|\cdot\|_{|\mathcal{H}|}$ does not constitute a norm. This is the reason, why we need to operate via Lemma 5.7.*

**Proof** of the Proposition 5.9: Let $f \in L_\mu$. We need to find a sequence $(f_n)$ in $C^\infty([0,T])$ (resp. $\mathcal{E}$) so that

$$\|f_n - f\|_{|\mathcal{H}|} \longrightarrow_{n\to\infty} 0.$$

The conclusion follows by Lemma 5.8 and 5.7. ■

**Corollary 5.11** *It holds that*

*i)* $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ *is a semiscalar product on* $L_\mu$.

*ii) The linear application*

$$\Phi : BV([0,T]) \longrightarrow L^2(\Omega)$$

*defined by*

$$\varphi \longrightarrow \int_0^T \varphi d^- X$$

*can be continuously extended to $L_\mu$ equipped with the $\|\cdot\|_{\mathcal{H}}$-norm. Moreover we will still have identity (18) for any $\varphi \in L_\mu$.*

**Proof:** The part i) is a direct consequence of the previous result. To check ii), it is only necessary to prove that $\Phi$ is continuous at zero. This follows from the property (18). ■

**Definition 5.1** *We will set $\int_0^T \varphi dX = \Phi(\varphi)$ and it will be called* **the Wiener integral** *of $\varphi$ with respect to $X$.*

**Remark 5.12** *Consider the relation $\sim$ on $L_\mu$ defined by*

$$\varphi \sim \phi \Leftrightarrow \|\varphi - \phi\|_{\mathcal{H}} = 0.$$

*Denoting $L^1_\mu$ as the quotient of $L_\mu$ through $\sim$ we obtain a vector space equipped with a true scalar product. However $L^1_\mu$ is not complete and so it is not a Hilbert space. For the simplicity of the notation, we will still denote by $L_\mu$ its quotient with respect to the relation $\sim$. Two functions $\phi, \varphi$ will be said to be equal in $\mathcal{H}$ if $\varphi \sim \phi$.*

*The fact that $L^1_\mu$ is a metric not complete space, does not disturb the linear extension. The important property is that $L^2(\Omega)$ is complete.*

**Lemma 5.13** *Let $h$ be cadlag. Then*
*i) $\int hd^- X = \int h_- dX$,*
*ii) $\int hd^+ X = \int hdX$,*
*where*

$$h_-(u) = \lim_{s \uparrow u} h(s).$$

**Proof:** We only prove point i), because the other one behaves similarly. Since $h$ is bounded, we recall by Remark 5.1 that $h \in L_\mu$. We have

$$\int_0^T h_u \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du = \int_0^T h^\varepsilon dX$$

with $h^\varepsilon_s = \frac{1}{\varepsilon} \int_{s-\varepsilon}^s h_u du$. Since

$$\|h^\varepsilon - h_-\|^2_{\mathcal{H}} = \int_0^T \int_0^T (h^\varepsilon(u_1) - h_-(u_1))(h^\varepsilon(u_2) - h_-(u_2))\, d\mu(u_1, u_2)$$

and $h^\varepsilon \to h_-$ pointwise, the conclusion follows by Lebesgue convergence theorem. ■

**Corollary 5.14** *If $h$ is cadlag, then*

$$\int_0^T hd^-X = \int_0^T hdX \ (= \int_0^T hd^+X).$$

**Proof:** Taking in account Lemma 5.13, it is enough to show that

$$\int_0^T (h - h_-)dX = 0.$$

This follows because

$$\begin{aligned}\|h - h_-\|_{\mathcal{H}}^2 &= \int\int_{[0,T]^2} (h - h_-)(u_1)(h - h_-)(u_2)d\mu(u_1, u_2) \\ &= \sum_{i,j} (h(a_i) - h(a_{i-}))(h(a_j) - h(a_{j-}))\mu(\{a_i, a_j\}) = 0\end{aligned}$$

and because $\mu$ is non-atomic. ■.

**Remark 5.15** *If $I$ is an interval with end-points $a < b$, then*

$$\int 1_I dX = X_b - X_a.$$

*This is a consequence of previous Corollary and Remark 2.4*

**Example 5.16 The bifractional Brownian motion case: significant subspaces of $L_\mu$.**

A significant subspace included in $L_\mu$ is the set $L^{\frac{1}{HK}}([0,T])$. For $K = 1$, we refer to [3]. Indeed, let us prove that

$$\|f\|_{|\mathcal{H}|}^2 \le C(H,K,T)\|f\|_{L^{\frac{1}{HK}}}^2. \tag{22}$$

It holds that

$$\begin{aligned}\|f\|_{|\mathcal{H}|}^2 &= \int_0^T \int_0^T |f(u_1)||f(u_2)| \frac{\partial^2 R}{\partial u_1 \partial u_2}(u_1, u_2) du_1 du_2 \\ &= C(H,K) \int_0^T \int_0^T |f(u_1)||f(u_2)| \left(u_1^{2H} + u_2^{2H}\right)^{K-2} u_1^{2H-1} u_2^{2H-1} du_1 du_2 \\ &\quad + C(H,K) \int_0^T \int_0^T |f(u_1)||f(u_2)||u_1 - u_2|^{2HK-2} du_1 du_2 \\ &:= A + B.\end{aligned}$$

The term $B$ has already been treated in the fBm case by using the Littlewood-Hardy inequality ([3]). Concerning the term $A$, by Hölder's inequality with exponent $\frac{1}{HK}$,

$$A \leq C(H,K) \left( \int_0^T \int_0^T |f(u_1)f(u_2)|^{\frac{1}{HK}} du_1 du_2 \right)^{HK}$$
$$\times \left( \int_0^T \int_0^T \left( \left(u_1^{2H} + u_2^{2H}\right)^{K-2} u_1^{2H-1} u_2^{2H-1} \right)^{\frac{1}{1-HK}} du_1 du_2 \right)^{1-HK}$$

and the only point to check is that the second factor in the above expression is finite. We can write

$$2 \int_0^T \int_0^{u_1} \left( \left(u_1^{2H} + u_2^{2H}\right)^{K-2} u_1^{2H-1} u_2^{2H-1} \right)^{\frac{1}{1-HK}} du_2 du_1$$
$$\leq 2C(H,K) \int_0^T \int_0^{u_1} \left(u_1^{2H} + u_2^{2H}\right)^{K-2} \left(u_1^{4H-2} + u_2^{4H-2}\right)^{\frac{1}{1-HK}} du_2 du_1$$
$$\leq C(H,K,T) \int_0^T \int_0^{u_1} \left(u_1^{2H} + u_2^{2H}\right)^{\frac{K-1}{1-HK}} du_2 du_1$$
$$= C(H,K,T) \int_0^T u_1^{\frac{2H(K-1)}{1-HK}+1} du_1 \left( \int_0^1 (1+z)^{\frac{2H(K-1)}{1-HK}} dz \right),$$

where at the last line we used the change of variables $u_1 = zu_2$. This last quantity is finite since $\frac{2H(K-1)}{1-HK} + 1 > -1$.

**Remark 5.17** *The inclusion*

$$L^2([0,T]) \subset L_\mu$$

*follows easily for $HK > \frac{1}{2}$ since (we treat only the term $A$, for $B$ see [3])*

$$2 \int_0^T \int_0^{u_1} \left( \left(u_1^{2H} + u_2^{2H}\right)^{K-2} u_1^{2H-1} u_2^{2H-1} \right)^{\frac{1}{1-HK}} du_2 du_1$$
$$\leq C(H,K) \left( \int_0^T |f(u)| u^{2HK+2H-1} du \right)^2$$
$$\leq C(H,K) \int_0^T |f(u)|^2 du \int_0^T u^{2HK-2} du \leq C(H,K) \|f\|^2_{L^2([0,T])}.$$

Let us summarize a few points of our construction. The space $L_\mu$ given by (14) is, due to Remark 5.12, a space with scalar product and it is in general incomplete. The norm of this space is given by the inner product (16). We also define (15) which is not a norm in general but it becomes a norm when $\mu$ is a positive measure.

We denote by $\mathcal{H}$ the abstract completion of $L_\mu$.

**Remark 5.18** *Remark 5.1 says that* $1_{[0,a]}$ *belongs to* $L_\mu$ *for any* $a \in [0,T]$*. Therefore* $\mathcal{H}$ *may be seen as the closure of* $1_{[0,t]}, t \in [0,T]$ *with respect to the scalar product*

$$< 1_{[0,s]}, 1_{[0,t]} >_{\mathcal{H}} = R(s,t).$$

$\mathcal{H}$ is now a Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$; it coincides with (17) when restricted to $L_\mu$. $\mathcal{H}$ is isomorphic to the self-reproducing kernel space. Generally that space is the space of $v : [0,T] \to \mathbb{R}$, $v(t) = E(X_t Z)$ with $Z \in L^2(\Omega)$. Therefore, if $Z = \int g dX$, $g \in L_\mu$, we have

$$v(t) = \int_{[0,t]} \int_{[0,1]} g(s_2) R(ds_1, ds_2).$$

**Proposition 5.19** *Suppose that* $X$ *is Gaussian. For* $f \in L_\mu$*, we have*

$$E\left(\int_0^T f dX\right)^2 = E\left(\int_0^T f^2(s) d[X]_s + 2\int_{[0,T]^2} 1_{(s_1 > s_2)} f(s_1) f(s_2) d\mu(s_1, s_2)\right).$$

**Proof:** It is a consequence of the Proposition 3.5 and Corollary 5.11. ■

# 6 Wiener analysis for non-Gaussian processes having a covariance measure structure

The aim of this section is to construct some framework of Malliavin calculus for stochastic integration related to continuous processes $X$, which are $L^2$-continuous, with a covariance measure defined in Section 2 and $X_0 = 0$. We denote by $C_0([0,T])$ the set of continuous functions on $[0,T]$ vanishing at zero. In this section we will also suppose that the law of process $X$ on $C_0([0,T])$ has full support, i. e. the probability that $X$ belongs to any subset of $C_0([0,T])$ is strictly positive.

We will start with a general framework. We will define the Malliavin derivative with some related properties in this general, not necessarily Gaussian, framework. A Skorohod integral with respect to $X$ can be defined as the adjoint of the derivative operator in some suitable spaces. Nevertheless, Gaussian properties are needed to go into a more practical and less abstract situation: for instance if one wants to exhibit concrete examples of processes belonging to the domain of the Skorohod integral and estimates for the $L^p$ norm of the integral. A key point, where the Gaussian nature of the process intervenes is Lemma 6.7. We refer also to the comments following that lemma.

We denote by $Cyl$ the set of smooth and cylindrical random variables of the form

$$F = f\left(\int \phi_1 dX, \ldots, \int \phi_n dX\right), \tag{23}$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $\phi_i \in L_\mu$. Here $\int \phi_i dX$ represents the Wiener integral introduced before Remark 5.18.

We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the canonical filtration associated with $X$ fulfilling the usual conditions. The underlying probability space is $(\Omega, \mathcal{F}_T, P)$, where $P$ is some suitable probability. For our consideration, it is not restrictive to suppose that $\Omega = C_0([0,T])$, so that $X_t(\omega) = \omega(t)$ is the canonical process. We suppose moreover that the probability measure $P$ has $\Omega$ as support. According to Section II.3 of [23], $\mathcal{F}C_b^\infty$ is dense in $L^2(\Omega)$, where

$$\mathcal{F}C_b^\infty = \{f(l_1, \dots, l_m),\ m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1, \dots, l_m \in \Omega^*\}.$$

On the other side, using similar arguments as in [22] one can prove that for every $l \in \Omega^*$ there is a sequence of random variables $Z_n \in \mathcal{S}, Z_n \to l$ in $L^2(\Omega)$. Thus $Cyl$ is dense in $L^2(\Omega)$.

The derivative operator $D$ applied to $F$ of the form (23) gives

$$DF = \sum_{i=1}^n \partial_i f\left(\int \phi_1 dX, \dots, \int \phi_n dX\right)\phi_i.$$

In particular $DF$ belongs a.s. to $L_\mu$ and moreover $E\|DF\|^2_{|\mathcal{H}|} < \infty$.

Recall that the classical Malliavin operator $D$ is an unbounded linear operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ where $\mathcal{H}$ is the abstract space defined in Section 5.

We define first the set $\left|\mathbb{D}^{1,2}\right|$, constituted by $F \in L^2(\Omega)$ such that there is a sequence $(F_n)$ of the form (23) and there exists $Z : \Omega \to L_\mu$ verifying two conditions:

**i)** $F_n \longrightarrow F$ in $L^2(\Omega)$;

**ii)** $E\|DF_n - Z\|^2_{|\mathcal{H}|} := E\int_0^T\int_0^T |D_u F_n - Z| \otimes |D_v F_n - Z|\, d|\mu|(u,v) \overset{n\to\infty}{\longrightarrow} 0$.

The set $\mathbb{D}^{1,2}$ will be the vector subspace of $L^2(\Omega)$ constituted by functions $F$ such that there is a sequence $(F_n)$ of the form (23)

**i)** $F_n \longrightarrow F$ in $L^2(\Omega)$;

**ii)** $E\,\|DF_n - DF_m\|^2_{\mathcal{H}} \overset{n,m\to\infty}{\longrightarrow} 0$.

We will denote by $Z = DF$ the $\mathcal{H}$-valued r.v. such that $\|Z - DF_n\|_{\mathcal{H}} \overset{L^2(\Omega)}{\longrightarrow} 0$. If $Z \in L_\mu$ a.s. then

$$\|DF\|^2_{\mathcal{H}} = \int_{[0,T]^2} D_{s_1}F D_{s_2}F d\mu(s_1, s_2).$$

Note that $\left|\mathbb{D}^{1,2}\right| \subset \mathbb{D}^{1,2}$ and $\mathbb{D}^{1,2}$ is a Hilbert space if equipped with the scalar product

$$\langle F, G\rangle_{1,2} = E(FG) + E\,\langle DF, DG\rangle_{\mathcal{H}}. \tag{24}$$

In general $\left|\mathbb{D}^{1,2}\right|$ is not a linear space equipped with scalar product since (15) is not necessarily a norm.

**Remark 6.1** *Cyl is a vector algebra. Moreover, if $F$, $G \in Cyl$*

$$D(F \cdot G) = GDF + FDG. \tag{25}$$

We prove some immediate properties of the Malliavin derivative.

**Lemma 6.2** *Let $F \in Cyl$, $G \in \left|\mathbb{D}^{1,2}\right|$. Then $F \cdot G \in \left|\mathbb{D}^{1,2}\right|$ and (25) still holds.*

**Proof:** Let $(G_n)$ be a sequence in $Cyl$ such that

$$\begin{aligned} &E(G_n - G)^2 \longrightarrow_{n\to+\infty} 0, \\ &E\left\{\int_{[0,T]^2} |DG_n - DG| \otimes |DG_n - DG| \, d|\mu|\right\} \longrightarrow_{n\to+\infty} 0. \end{aligned} \tag{26}$$

Since $F \in L^\infty(\Omega)$, $FG_n \longrightarrow FG$ in $L^2(\Omega)$. Remark 6.1 implies that

$$D(FG_n) = G_n DF + FDG_n.$$

So

$$\int_{[0,T]^2} d|\mu| \, |G_n DF - GDF| \otimes |G_n DF - GDF| = (G_n - G)^2 \int_{[0,T]^2} |DF| \, |DF| \, d|\mu| \tag{27}$$

If $F$ is of type (23) then

$$DF = \sum_{i=1}^{n} Z_i \phi_i,$$

where $\phi_i \in L_\mu$, $Z_i \in L^\infty(\Omega)$. Therefore the expectation of (27) is bounded by

$$cst. \sum_{i,j=1}^{n} \int_{[0,T]^2} |\phi_i| \otimes |\phi_j| \, d|\mu| \, E\left((G_n - G)^2 Z_i Z_j\right). \tag{28}$$

When $n$ converges to infinity, (28) converges to zero since $G_n \to G$ in $L^2(\Omega)$. On the other hand

$$\begin{aligned} &\int_{[0,T]^2} d|\mu| \left(|F(DG_n - DG)| \otimes |F(DG_n - DG)|\right) \\ &= |F|^2 \int_{[0,T]^2} d|\mu| \, |DG_n - DG| \otimes |DG_n - DG|. \end{aligned}$$

Since $F \in L^\infty(\Omega)$, previous term converges to zero because of (26). By additivity the result follows. ■

We denote by $L^2(\Omega; L_\mu)$ set of stochastic processes $(u_t)_{t\in[0,T]}$ verifying

$$E\left(\|u\|^2_{|\mathcal{H}|}\right) < \infty.$$

We can now define the divergence operator (or the Skorohod integral) which is an unbounded map defined from $Dom(\delta) \subset L^2(\Omega; L_\mu)$ to $L^2(\Omega)$. We say that $u \in L^2(\Omega; L_\mu)$ belongs to $Dom(\delta)$ if there is a zero-mean square integrable random variable $Z \in L^2(\Omega)$ such that

$$E(FZ) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right) \tag{29}$$

for every $F \in Cyl$. In other words

$$E(FZ) = E\left(\int_{[0,T]^2} D_{s_2} F u(s_2) \mu(ds_1, ds_2)\right) \text{ for every } F \in Cyl. \tag{30}$$

Using Riesz theorem we can see that $u \in Dom(\delta)$ if and only if the map

$$F \mapsto E\left(\langle DF, u\rangle_{\mathcal{H}}\right)$$

is continuous linear form with respect to $\|\cdot\|_{L^2(\Omega)}$. Since $Cyl$ is dense in $L^2(\Omega)$, $Z$ is uniquely characterized. We will set

$$Z = \int_0^T u\delta X.$$

$Z$ will be called *the Skorohod integral of $u$ towards $X$.*

**Remark 6.3** *If (29) holds, then it will be valid by density for every $F \in \mathbb{D}^{1,2}$.*

An important preliminary result in the Malliavin calculus is the following.

**Proposition 6.4** *Let $u \in Dom(\delta)$, $F \in \left|\mathbb{D}^{1,2}\right|$. Suppose $F \cdot \int_0^T u_s \delta X_s \in L^2(\Omega)$. Then $Fu \in Dom(\delta)$ and*

$$\int_0^T F \cdot u_s \delta X_s = F \int_0^T u_s \delta X_s - \langle DF, u\rangle_{\mathcal{H}}.$$

**Proof:** We proceed using the duality relation (29). Let $F_0 \in Cyl$. We need to show

$$E\left(F_0\left\{F\int_0^T u_s\delta X_s - \langle DF, u\rangle_{\mathcal{H}}\right\}\right) = E\left(\langle DF_0, Fu\rangle_{\mathcal{H}}\right). \tag{31}$$

Lemma 6.2 implies that $F_0 F \in \left|\mathbb{D}^{1,2}\right|$. The left member of (31) gives

$$\begin{aligned} &E\left(F_0 F \int_0^T u_s \delta X_s\right) - E\left(F_0 \langle DF, u\rangle_{\mathcal{H}}\right) \\ &= E\left(\langle D(F_0F), u\rangle_{\mathcal{H}}\right) - E\left(F_0\langle DF, u\rangle_{\mathcal{H}}\right) = E(< D(F_0F) - F_0 DF, u >_{\mathcal{H}}). \end{aligned} \tag{32}$$

This gives the right member of (31) because of the Lemma 6.2. Remark 6.3 allows to conclude. ∎

We state a useful Fubini type theorem which allows to interchange Skorohod and measure theory integrals. When $X$ is a Brownian motion and the measure theory integral is Lebesgue integral, then the result is stated in [28].

**Proposition 6.5** *Let $(G, \mathcal{G}, \lambda)$ be a $\sigma$-finite measured space. Let $u : G \times [0,T] \times \Omega \longrightarrow \mathbb{R}$ be a measurable random field with the following properties*

**i)** *For every $x \in G$, $u(x,\cdot) \in Dom(\delta)$,*

**ii)**

$$E \int_G d\lambda(x_1) \int_G d\lambda(x_2) \int_{[0,T]^2} d|\mu| \, |u| \, (x_1,\cdot) \otimes |u| \, (x_2,\cdot) < \infty,$$

**iii)** *There is a measurable version in $\Omega \times G$ of the random field $\left(\int_0^T u(x,t)\delta X_t\right)_{x\in G}$,*

**iv)** *It holds*

$$\int_G d\lambda(x) E \left( \int_0^T u\,(x,t)\,\delta X_t \right)^2 < \infty.$$

*Then $\int_G d\lambda(x) u(x,\cdot) \in Dom(\delta)$ and*

$$\int_0^T \left( \int_G d\lambda(x) u(x,\cdot) \right) \delta X_t = \int_G d\lambda(x) \left( \int_0^T u(x,t)\delta X_t \right).$$

**Proof:** We need to prove two properties:

**a)**

$$\int_G d\lambda(x) \, |u| \, (x,\cdot) \in L^2(\Omega; L_\mu)$$

**b)** For every $F \in Cyl$ we have

$$E \left( F \int_G d\lambda(x) \int_0^T u(x,t)\delta X_t \right) = E \left( \left\langle DF, \int_G d\lambda(x) u(x,\cdot) \right\rangle_{\mathcal{H}} \right).$$

It is clear that without restriction to the generality we can suppose $\lambda$ to be a finite measure. Concerning a) we write

$$\begin{aligned} E \left( \left| \int_G d\lambda(x) \, |u| \, (x,\cdot) \right|^2_{|\mathcal{H}|} \right) &= \int_{[0,T]^2} d\,|\mu|\,(s_1,s_2) \int_G d\lambda(x_1) \, |u| \, (x_1,s_1) \int_G d\lambda(x_2) \, |u| \, (x_2,s_2) \\ &= \int_{G\times G} d\lambda(x_1) d\lambda(x_2) \int_{[0,T]} d\,|\mu|\,(s_1,s_2) \, |u| \, (x_1,s_1) \, |u| \, (x_2,s_2). \end{aligned} \tag{33}$$

Taking the expectation of (33), the result a) follows from ii). For the part b) let us consider $F \in Cyl$. Classical Fubini theorem implies

$$
\begin{aligned}
& E\left(F \int_G d\lambda(x) \left(\int_0^T u(x,t)\delta X_t\right)\right) \\
&= \int_G d\lambda(x) E\left(F \int_0^T u(x,t)\delta X_t\right) = \int_G d\lambda(x) E\left(\langle DF, u(x,\cdot)\rangle_{\mathcal{H}}\right) \\
&= E\left\{\int_G d\lambda(x) \int_{[0,T]^2} D_{s_1} F u(x,s_1) D_{s_2} F u(x,s_2) d\mu(s_1,s_2)\right\} \\
&= E\left(\int_{[0,T]^2} d\mu(s_1,s_2) D_{s_1} F D_{s_2} F \int_G d\lambda(x) u(x,s_1) \int_G d\lambda(x) u(x,s_2)\right) \\
&= \left\langle DF, \int_G d\lambda(x) u(x,\cdot)\right\rangle_{\mathcal{H}}.
\end{aligned}
$$

At this point the proof of the proposition is concluded. ■

We denote by $L_{\mu,2}$ the set of $\phi : [0,T]^2 \to \mathbb{R}$ such that

- $\phi(t_1,\cdot) \in L_\mu, \ \forall t_1 \in [0,T]$,
- $t_1 \to \|\phi(t_1,\cdot)\|_{|\mathcal{H}|} \in L_\mu$.

For $\phi \in L_{\mu,2}$ we set

$$\|\phi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} = \int_{[0,T]^2} \|\phi(t_1,\cdot)\|_{|\mathcal{H}|} \|\phi(t_2,\cdot)\|_{|\mathcal{H}|} d|\mu|(t_1,t_2).$$

Similarly to $|\mathbb{D}^{1,2}|$ we will define $|\mathbb{D}^{1,2}(L_\mu)|$ and even $|\mathbb{D}^{1,p}(L_\mu)|$, $p \geq 2$.

We first define $Cyl(L_\mu)$ as the set of smooth cylindrical random elements of the form

$$u_t = \sum_{\ell=1}^n \psi_\ell(t) G_\ell, \quad t \in [0,T], \psi_\ell \in L_\mu, G_\ell \in Cyl. \tag{34}$$

.

On $L_{\mu,2}$ we also define the following inner semiproduct:

$$< u_1, u_2 >_{\mathcal{H}\otimes\mathcal{H}} = \int_{[0,T]^2} < u_1(t_1,\cdot), u_2(t_2,\cdot) >_{\mathcal{H}} d\mu(t_1,t_2).$$

This inner product naturally induces a seminorm $\|u\|_{\mathcal{H}\otimes\mathcal{H}}$ and we have of course

$$\|u\|_{|\mathcal{H}|\otimes|\mathcal{H}|} \geq \|u\|_{\mathcal{H}\otimes\mathcal{H}}.$$

We denote by $|\mathbb{D}^{1,p}(L_\mu)|$ the vector space of random elements $u : \Omega \to L_\mu$ such that there is a sequence $u_n \in Cyl(L_\mu)$ and

**i)** $\|u-u_n\|^2_{|\mathcal{H}|} \longrightarrow_{n\to\infty} 0$ in $L^p(\Omega)$,

**ii)** there is $Z:\Omega\longrightarrow L_{\mu,2}$ with $\|Du_n - Z\|_{|\mathcal{H}|\otimes|\mathcal{H}|} \longrightarrow 0$ in $L^p(\Omega)$.

We convene here that

$$Du_n : (t_1,t_2) \longrightarrow D_{t_1}u_n(t_2).$$

Note that until now we did not need the Gaussian assumption on $X$. But this is essential in following result. It says that when the integrand is deterministic, the Skorohod integral coincide with the Wiener integral.

**Proposition 6.6** *Suppose $X$ to be a Gaussian process. Let $h\in L_\mu$. Then*

$$\int_0^T h\delta X_s = \int_0^T h dX_s.$$

**Proof:** Let $F\in Cyl$. The conclusion follows from the following Lemma 6.7 and density arguments. ■

**Lemma 6.7** *Let $F\in Cyl$. Then*

$$E\left(\langle DF,h\rangle_{\mathcal{H}}\right) = E\left(F\int_0^T hdX\right). \tag{35}$$

**Proof:** We use the method given in [28], Lemma 1.1. After normalization it is possible to suppose that $\|h\|_{\mathcal{H}}=1$. There is $n\ge 1$ such that $F=\tilde f\left(\int hdX, \int\tilde\phi_1 dX,\ldots,\int\tilde\phi_n dX\right)$, $h,\tilde\phi_1,\ldots,\tilde\phi_1\in L_\mu$, $\tilde f\in C_b^\infty(\mathbb{R}^n)$. We set $\phi_0=h$ and we proceed by Gram-Schmidt othogonalization. The first step is given by

$$Y_1 = \int hdX - \left\langle h,\tilde\phi_1\right\rangle_{\mathcal{H}}\int hdX = \int\phi_1 dX,$$

where $\phi_1 = \frac{h-\langle h,\tilde\phi_1\rangle h}{\|h-\langle h,\tilde\phi_1\rangle h\|}$ and so on. Therefore it is possible to find a sequence $\phi_0,\ldots,\phi_n\in L_\mu$ orthonormal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, such that

$$F = f\left(\int\phi_0 dX,\ldots,\int\phi_n dX\right),\ f\in C_b^\infty(\mathbb{R}^{n+1}).$$

Let $\rho$ be the density of the standard normal distribution in $\mathbb{R}^{n+1}$, i.e.

$\rho(x) = (2\pi)^{-\frac{n+1}{2}} \exp\left(-\frac{1}{2}\sum_{i=0}^n x_i^2\right)$. Then we have

$$\begin{aligned} E\left(\langle DF, h\rangle_{\mathcal{H}}\right) &= E\left(\sum_{i=0}^n \partial_i f\left(\int \phi_0 dX, \ldots, \int \phi_n dX\right)\right)\langle \phi_i, h\rangle_{\mathcal{H}} \\ &= E\left(\partial_0 f\left(\int \phi_0 dX, \ldots, \int \phi_n dX\right)\right) = \int_{\mathbb{R}^{n+1}} \partial_0 f(y)\rho(y)dy \\ &= \int_{\mathbb{R}} f(y)\rho(y)y_0 dy_0 = E\left(F \int h dX\right) \end{aligned}$$

which completes the proof of the lemma. ■

**Remark 6.8** *It must be pointed out that the Gaussian property of $X$ appears to be crucial in the proof of Lemma 6.7. Actually we used the fact that uncorrelated Gaussian random variables are independent and also the special form of the derivative of the Gaussian kernel. As far as we know, there are two possible proofs of this integration by parts on the Wiener spaces, both using the Gaussian structure: one (that we used) presented in Nualart [28] and other given in Bass [4] using a Bismut's idea and based on the Fréchet form of the Malliavin derivative.*

# 7 The case of Gaussian processes with a covariance measure structure

Let $X = (X_t)_{t\in[0,T]}$ be a zero-mean Gaussian process such that $X_0 = 0$ a.s. that is continuous. A classical result of [13] (see Th. 1.3.2. and Th. 1.3.3) says that

$$\sup_{t\in[0,T]} |X| \in L^2. \tag{36}$$

This implies in particular that $X$ is $L^2$-continuous. We suppose also as in previous section that the law of $X$ in $C_0([0,T])$ has full support.

We suppose moreover that *it has covariance $R$ with covariance measure $\mu$*. Since $X$ is Gaussian, according to the Section 5, the canonical Hilbert space $\mathcal{H}$ of $X$ (called reproducing kernel Hilbert space by some authors) provides an abstract Wiener space and an abstract structure of Malliavin calculus was developed, see for instance [38, 29, 45].

Recently, several papers were written in relation to fractional Brownian motion and Volterra processes of the type $X_t = \int_0^t G(t,s)dW_s$, where $G$ is a deterministic kernel, see for instance [2, 8]. In this work we remain close to the intrinsic approach based on the covariance as in [38, 29, 45]. However their approach is based on a version of self-reproducing kernel space $\mathcal{H}$ which is abstract. Our construction focuses on the linear subspace $L_\mu$ of $\mathcal{H}$ which is constituted by functions.

### 7.1 Properties of Malliavin derivative and divergence operator

We introduce some elements of the Malliavin calculus with respect to $X$. Remark 5.18 says that the abstract Hilbert space $\mathcal{H}$ introduced in Section 5 is the topological linear space generated by the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

In general, the elements of $\mathcal{H}$ may not be functions but distributions. This is actually the case of the fractional Brownian motion with $H > \frac{1}{2}$, see Pipiras and Taqqu [32]. Therefore it is more convenient to work with one subspace of $\mathcal{H}$ that contains only functions, for instance $L_\mu$.

We establish here some peculiar and useful properties of Skorohod integral.

**Proposition 7.1** *Let $u \in Cyl(L_\mu)$. Then $u \in Dom(\delta)$ and $\int_0^T u\delta X \in L^p(\Omega)$ for every $1 \le p < \infty$.*

**Proof:** Let $u = G\psi$, $\psi \in L_\mu$, $G \in Cyl$. Proposition 6.6 says that $\psi \in Dom(\delta)$. Applying Proposition 6.4 with $F = G$ and $u = \psi$, we get that $\psi G$ belongs to $Dom(\delta)$ and

$$\int_0^T u\delta X = G \int_0^T \psi \delta X_s - \int_{[0,T]^2} \psi(t_1) D_{t_2} G d\mu(t_1,t_2).$$

If $G = g(Y_1, \ldots, Y_n)$, where $Y_i = \int \phi_i dX,\ 1 \le i \le n$, then

$$\int_0^T u\delta X = -\sum_{j=1}^n < \phi_j, \psi >_{\mathcal{H}} \partial_j g(Y_1,\ldots,Y_n) + g(Y_1,\ldots,Y_n) \int_0^T \psi dX. \tag{37}$$

The right member belongs obviously to each $L^p$ since $Y_j$ is a Gaussian random variable and $g, \partial_j g$ are bounded. The final result for $u \in Cyl(L_\mu)$ follows by linearity. ■

**Remark 7.2** *(37) provides an explicit expression of $\int_0^T u\delta X$.*

We discuss now the commutativity of the derivative and Skorohod integral. First we observe that if $F \in Cyl$, $(D_t F) \in Dom(\delta)$. Moreover, if $u \in Cyl(L_\mu)$, $(D_{t_1} u(t_2))$ belongs to $|\mathbb{D}^{1,2}(L_{\mu,2})|$. Similarly to (1.46) Ch. 1 of [[28]], we have the following property.

**Proposition 7.3** *Let $u \in Cyl(L_\mu)$. Then*

$$\int_0^T u\delta X \in |\mathbb{D}^{1,2}|$$

*and we have for every $t$*

$$D_t \left( \int_0^T u\delta X \right) = u_t + \int_0^T (D_t u_s) \delta X_s. \tag{38}$$

**Proof:** It is enough to write the proof for $u = \psi G$ where $G \in Cyl$ of the type

$$G = g(Y_1, \dots, Y_n),$$

$Y_i = \int \phi_i dX$, $1 \le i \le n$. According to (37) in the proof of Proposition 7.1, the left member of (38) gives

$$\begin{aligned} -\sum_{j=1}^n \quad & < \phi_j, \psi >_{\mathcal{H}} D_t(\partial_j g(Y_1, \dots, Y_n)) + D_t G \int_0^T \psi dX + G\psi(t) \\ & = -\sum_{j=1}^n < \phi_j, \psi >_{\mathcal{H}} \sum_{l=1}^n \partial^2_{il} g(Y_1, \dots, Y_n)\phi_l(t) \\ & \quad + \sum_{j=1}^n \partial_j g(Y_1, \dots, Y_n) \int_0^T \psi dX \phi_j(t) + g(Y_1, \dots, Y_n)\psi(t). \end{aligned} \tag{39}$$

On the other hand

$$D_t u(s) = \psi(s) \sum_{j=1}^n \partial_j g(Y_1, \dots, Y_n)\phi_j(t).$$

Applying again (37), through linearity, we obtain for $t \in [0, T]$,

$$\begin{aligned} \int_0^T D_t u \delta X &= \sum_{j=1}^n \phi_j(t) \int_0^T \psi \partial_j g(Y_1, \dots, Y_n)\delta X \\ &= \sum_{j=1}^n \phi_j(t) \left[ -\sum_{l=1}^n < \phi_l, \psi >_{\mathcal{H}} \partial^2_{lj} g(Y_1, \dots, Y_n) + \partial_j g(Y_1, \dots, Y_n) \int_0^T \psi dX \right]. \end{aligned}$$

Coming back to (39) we get

$$D_t \left( \int_0^T \psi G \delta X \right) = \int_0^T D_t(\psi G)\delta X + \psi(t) G.$$

■

We can now evaluate the $L^2(\Omega)$ norm of the Skorohod integral.

**Proposition 7.4** *Let $u \in |\mathbb{D}^{1,2}(L_\mu)|$. Then $u \in Dom(\delta)$, $\int_0^T u\delta X \in L^2(\Omega)$ and*

$$E\left( \int_0^T u \delta X \right)^2 = E(\|u\|^2_{\mathcal{H}}) + E\left( \int_{[0,T]^2} d\mu(t_1, t_2) \int_{[0,T]^2} d\mu(s_1, s_2) D_{s_1} u_{t_1} D_{t_2} u_{s_2} \right). \tag{40}$$

*Moreover*

$$E\left( \int_0^T u \delta X \right)^2 \le E\left( \|u\|^2_{|\mathcal{H}|} + \int_{[0,T]^2} d|\mu|(t_1, t_2) \, \|D.u_{t_1}\|^2_{|\mathcal{H}|} \right). \tag{41}$$

**Remark 7.5** *Let $u, v \in |\mathbb{D}^{1,2}(L_\mu)|$. Polarization identity implies*

$$E\left(\int_0^T u\delta X \int_0^T v\delta X\right) = E\left(< u, v >_{\mathcal{H}}\right)$$
$$+E\left(\int_{[0,T]^2} d\mu(t_1,t_2)\int_{[0,T]^2} d\mu(s_1,s_2) D_{s_1}u_{t_1}D_{t_2}v_{s_2}\right). \quad (42)$$

**Proof** (of Proposition 7.4): Let $u \in Cyl(L_\mu)$. By the Proposition 7.3, since $\int_0^T u\delta X \in \mathbb{D}^{1,2}$ we get

$$\begin{aligned}
E\left(\int_0^T u\delta X\right)^2 &= E\left(\left\langle u, D\int_0^T u\delta X\right\rangle_{\mathcal{H}}\right)\\
&= E\left(\int_{[0,T]^2} u_{t_1}D_{t_2}\left(\int_0^T u\delta X\right)d\mu(t_1,t_2)\right)\\
&= E\left(\int_{[0,T]^2} u_{t_1}\left(u_{t_2}+\int_0^T D_{t_2}u_s\delta X_s\right)d\mu(t_1,t_2)\right)\\
&= E\left(\|u\|^2_{\mathcal{H}}\right)+\int_{[0,T]^2} d\mu(t_1,t_2)E\left(u_{t_1}\int_0^T D_{t_2}u_s\delta X_s\right).
\end{aligned}$$

Using again the duality relation, we get

$$E\left(\|u\|^2_{\mathcal{H}}+\int_0^T d\mu(t_1,t_2) < D_{\cdot}u_{t_1}, D_{t_2}u_{\cdot} >_{\mathcal{H}}\right),$$

which constitutes formula (40).

Moreover, using Cauchy-Schwarz, we obtain

$$E\left(\int_0^T u\delta X\right)^2 \le E\left(\|u\|^2_{\mathcal{H}}\right)+E\left(\int_{[0,T]^2} d|\mu|(t_1,t_2)\,\|D_{t_1}u_{\cdot}\|_{\mathcal{H}}\,\|D_{\cdot}u_{t_2}\|_{\mathcal{H}}\right). \quad (43)$$

Since

$$\int_{[0,T]^2} d|\mu|(t_1,t_2)\,\|Du_{t_2}\|^2_{\mathcal{H}} = \int_{[0,T]^2} d|\mu|(t_1,t_2)\,\|D_{t_1}u\|^2_{\mathcal{H}}$$

(43) is equal or smaller than

$$E\left(\int_{[0,T]^2} d|\mu|(t_1,t_2)\,\|Du_{t_2}\|^2_{\mathcal{H}}\right)$$

and this shows (41).

Using the fact that $Cyl(L_\mu)$ is dense in $|\mathbb{D}^{1,2}(L_\mu)|$ we obtain the result. ∎

### 7.2 Continuity of the Skorohod integral process

It is possible to connect previous objects with the classical Wiener analysis on an abstract Wiener space, related to Hilbert spaces $\mathcal{H}$, see [[29]],[[38]].

In the classical theory the Malliavin gradient (derivative) $\nabla$ and the divergence operator $\delta$ are well defined with its domain. For instance $\delta : \mathbb{D}^{1,2}(\mathcal{H}) \to L^2(\Omega)$ is continuous and $\mathbb{D}^{1,2}(\mathcal{H})$ is contained in the classical domain. However as we said the realizations of $u \in \mathbb{D}^{1,2}(\mathcal{H})$ may not be functions.

If $u \in |\mathbb{D}^{1,2}(L_\mu)|$, it belongs to $\mathbb{D}^{1,2}(\mathcal{H})$ and its norm is given by

$$\|u\|_{1,2}^2 = E\left(\|u\|_{\mathcal{H}}^2 + \int_{[0,T]^2} d\mu(s_1,s_2)\, \|D.u_{s_1}\|_{\mathcal{H}}\, \|D.u_{s_2}\|_{\mathcal{H}}\right).$$

Classically $\nabla u$ is an element of $L^2(\Omega, \mathcal{H} \otimes \mathcal{H})$, where $\mathcal{H} \otimes \mathcal{H}$ is the Hilbert space of bilinear continuous functionals $\mathcal{H} \otimes \mathcal{H} \to \mathbb{R}$ equipped with the Hilbert-Schmidt norm.

Given $u \in |\mathbb{D}^{1,2}(L_\mu)| \subset \mathbb{D}^{1,2}(\mathcal{H})$, we have $Du \in L^2(\Omega; L_{\mu,2})$. The associated gradient $\nabla u$ is given by

$$(h,k) \mapsto \int_{[0,T]^2} D_{s_1} u_{t_1} h(s_2) k(t_2) d\mu(s_1,s_2) d\mu(t_1,t_2),$$

where $h, k \in L_\mu$. Its Hilbert-Schmidt norm coincides with

$$\int_{[0,T]^2} < Du_{s_1}, Du_{s_2} >_{\mathcal{H}} d\mu(s_1,s_2).$$

**Remark 7.6** *If $u \in |\mathbb{D}^{1,2}(L_\mu)|$*

$$\int_0^T u_s dX_s = \delta(u).$$

**Remark 7.7** *The standard Sobolev-Wiener space $\mathbb{D}^{1,p}(\mathcal{H})$, $p \geq 2$ is included in the classical domain of $\delta$ and the Meyer's inequality holds:*

$$E|\delta(u)|^p \leq C(p) E\left(\|u\|_{\mathcal{H}}^p + \|\nabla u\|_{\mathcal{H}\otimes\mathcal{H}}^p\right). \tag{44}$$

*This implies that if $u \in |\mathbb{D}^{1,2}(L_\mu)|$*

$$E\left|\int_0^T u\delta X\right|^p \leq C(p) E\left(\|u\|_{\mathcal{H}}^p + \left\{\int_{[0,T]^2} < Du_{s_1}, Du_{s_2} >_{\mathcal{H}} d\mu(s_1,s_2)\right\}^{\frac{p}{2}}\right). \tag{45}$$

*Consequently this gives*

$$E\left(\left|\int_0^T u\delta X\right|^p\right) \leq C(p) E\left\{\int_{[0,T]^2} |< Du_{s_1}, Du_{s_2} >_{\mathcal{H}}|\, d|\mu|(s_1,s_2)\right\}^{\frac{p}{2}}. \tag{46}$$

*The last inequality can be shown in a similar way as in the case of Brownian motion. One applies Proposition 3.2.1 p. 158 in [28] and then one argument in the proof of Proposition 3.2.2 again in [28].*

The Meyer inequalities are very useful in order to prove the continuity of the trajectories for Skorohod integral processes. We illustrate this in the next proposition.

**Proposition 7.8** *Assume that the covariance measure of the process $X$ satisfies*

$$[\mu((s,t]\times(s,t])]^{p-1} \le |t-s|^{1+\beta}, \quad \beta > 0 \tag{47}$$

*for some $p>1$ and consider a process $u \in |\mathbb{D}^{1,2p}(L_\mu)|$ such that*

$$\int_0^T\int_0^T\left(\int_0^T\int_0^T |D_a u_r||D_b u_\theta| d|\mu|(a,b)\right)^p d|\mu|(r,\theta) < \infty. \tag{48}$$

*Then the Skorohod integral process $\left(Z_t = \int_0^t u_s \delta X_s\right)_{t\in[0,T]}$ admits a continuous version.*

**Proof:** We can assume that the process $u$ is centered because the process $\int_0^t E(u_s)\delta X_s$ always admits a continuous version under our hypothesis. By (46), (47) and (48 we have

$$\begin{aligned}
E|Z_t - Z_s|^{2p} &\le c(p)E\left(\int_0^T\int_0^T\int_0^T\int_0^T \left|D_a u_r 1_{(s,t]}(r)\right|\left|D_b u_\theta 1_{(s,t]}(\theta)\right| d|\mu|(a,b)d|\mu|(r,\theta)\right)^p \\
&\le c(p)\left[\mu((s,t]\times(s,t])\right]^{p-1}\int_0^T\int_0^T\left(\int_0^T\int_0^T |D_a u_r||D_b u_\theta| d|\mu|(a,b)\right)^p d|\mu|(r,\theta) \\
&\le c(p,T)|t-s|^{1+\beta}.
\end{aligned}$$

■

**Remark 7.9** *In the fBm case we have that*

$$\mu((s,t]\times(s,t]) = |t-s|^{2H}$$

*and (47) holds with $pH>1$. In the bifractional case, it follows from [17] that*

$$|\mu((s,t]\times(s,t])| \le 2^{1-K}|t-s|^{2HK}$$

*and therefore (47) holds if $pHK>1$.*

**Remark 7.10** *The case of the bifractional Brownian motion can be also treated using the fact that $L^{\frac{1}{HK}}$ is included in $L_\mu$. When $K=1$, we refer to [3], Section 5, for estimates of the stochastic integral and conditions for the continuity of the integral process. It is clear that all these results can be extended to $K\in(0,1]$. If $pHK>1$ and $u$ verifies*

$$E\|u\|^p_{L^{\frac{1}{HK-\varepsilon}}([0,T])} + E\|Du\|^p_{L^{\frac{1}{HK-\varepsilon}}([0,T]^2)} < \infty. \tag{49}$$

*with* $0 < \varepsilon < HK - \frac{1}{p}$*, we have*

$$E\left(\sup_{t\in[0,T]}\left|\int_0^t u_s\delta X_s\right|^p\right)$$
$$\leq c(p,H,K)\left[\left(\int_0^T |Eu_s|^{\frac{1}{HK-\varepsilon}}\right)^{p(HK-\varepsilon)} + E\left(\int_0^T\left(\int_0^T |D_s u_r|^{\frac{1}{HK}}dr\right)^{\frac{HK}{HK-\varepsilon}}ds\right)^{p(HK-\varepsilon)}\right].$$

*It can be also proved that if* $u$ *satisfies (49) with* $\varepsilon = 0$ *and* $pHK > 1$ *then the indefinite Skorohod integral has* $HK$ *-Hölder continuous paths.*

### 7.3 On local times

We will make in this paragraph a few observations on the chaotic expansion of the local time of a Gaussian process $X$ having a covariance measure structure. Our analysis is basic and we will only aim to anticipate a possible further study. We illustrate the fact that the covariance measure appears to play an important role for the existence and the regularity of the local time.

Let us use the standard way to introduce the local time $L(t,x)$ of the process $X$; that is, for every $t \in [0,T]$ and $x \in \mathbb{R}$, $L(t,x)$ is defined as the density of the occupation measure

$$\mu_A = \int_0^t 1_A(X_s)ds, \qquad A \in \mathcal{B}(\mathbb{R}).$$

It can be formally written as

$$L(t,x) = \int_0^t \delta_x(X_s)ds,$$

where $\delta$ denotes the Dirac delta function and $\delta_x(X_s)$ represents a distribution in the Watanabe sense.

SInce $X$ is a Gaussian process, it is possible to construct related multiple Wiener-Itô integrals. We refer to [28] or [24] for the elements of this construction.

There is a standard method to compute the Wiener-Itô chaos expansion of $L(t,x)$. It consists to approach the Dirac function by Gaussian kernels $p_\varepsilon$ of variance $\varepsilon$ and to take the limit in $L^2(\Omega)$ as $\varepsilon \to 0$. We get (see e.g. [11])

$$L(t,x) = \sum_{n=0}^{\infty}\int_0^t \frac{p_{R(s,s)}(x)}{R(s,s)^{\frac{n}{2}}} H_n\left(\frac{x}{\sqrt{R(s,s)}}\right) I_n\left(1_{[0,s]}^{\otimes n}\right)ds \tag{50}$$

for all $t \in [0,T]$, $x \in \mathbb{R}$ where $I_n$ denotes the multiple integral of order $n$ with respect to $X$ and $H_m$ represents the Hermite polynomial of degree $m$. One can compute the second moment of $L(t,x)$ by using the isometry of multiple stochastic integrals

$$EI_n\left(1_{[0,s]}^{\otimes n}\right)I_m\left(1_{[0,t]}^{\otimes m}\right)=\begin{cases} R(s,t)^m & \text{if } m=n \\ 0 & \text{if } m\neq n\end{cases}$$

Using standard bounds as in [11], it follows that the $L^2$ norm of (50) is finite if

$$\sum_{n\geq 1} n^{-\frac{1}{2}}\int_0^t\int_0^t \frac{|\mu([0,u]\times[0,v])|^n}{(|\mu([0,u]\times[0,u])|\,|\mu([0,v]\times[0,v])|)^{\frac{n+1}{2}}}dvdu<\infty.$$

It can be seen that the existence of the local time $L(t,x)$ as random variable in $L^2(\Omega)$ is closely related to the properties of the covariance measure $\mu$. A possible condition to ensure the existence of $L$ could be

$$\int_0^t\int_0^t \frac{|\mu([0,u]\times[0,v])|^n}{(|\mu([0,u]\times[0,u])|\,|\mu([0,v]\times[0,v])|)^{\frac{n+1}{2}}}dvdu<n^{-\beta}$$

with $\beta>\frac{1}{2}$. Of course, this remains rather abstract and it is interesting to be checked in concrete cases. We refer to [31] for the Brownian case, to [11] for the fractional Bownian case and to [34] for the bifractional case.

We also mention that the properties of the covariance measure of Gaussian processes are actually crucial to study sample path regularity of local times like level sets, Hausdorff dimension etc. in the context of the existence of *local non-determinism.* We refer e.g. to Xiao [44] for a complete study of path properties of Gaussian random fields.

# 8 Itô formula in the Gaussian case

The next step will consist in expressing the relation between Skorohod integral and integrals obtained via regularization. The first result is illustrative. It does not enter into specificity of the examples.

**Theorem 8.1** *Let $(Y_t)_{t\in[0,T]}$ be a cadlag process. We take into account the following hypotheses*

**a)** $\sup_{t\leq T}|Y_t|$ *is square integrable.*

**b)** $Y\in|\mathbb{D}^{1,2}(L_\mu)|$. *Moreover $DY$ verifies*

$$|D_{t_1}Y_{t_2}|\leq Z_2,\quad \forall(t_1,t_2)\in[0,T]^2\quad |\mu|\ a.e. \tag{51}$$

*where $Z_2$ is a square integrable random variable.*

**c)** *For $|\mu|$ almost all $(t_1, t_2) \in [0,T]^2$*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_2-\varepsilon}^{t_2} D_{t_1} Y_s ds \tag{52}$$

*exists a.s. This quantity will be denoted $(D_{t_1} Y_{t_2 -})$. Moreover for each $s$, the set of $t$ such that $D_t Y_{s-} = D_t Y_s$ is null with respect to the marginal measure $\nu$.*

**c')** *For $|\mu|$ almost all $(t_1, t_2)$,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_2}^{t_2+\varepsilon} D_{t_1} Y_s ds \tag{53}$$

*exists a.s. It will be denoted by $(D_{t_1} Y_{t_2+})$. Moreover for each $s$, the set of $t$ such that $D_t Y_{s+} = D_t Y_s$ is null with respect to measure $\nu$.*

*If a), b), c) (resp. a), b), c')) are verified then $Y \in Dom(\delta)$ and the forward integral $\int_0^T Y d^- X$ (resp. the backward integral $\int_0^T Y d^+ X$) exists and*

$$\int_0^t Y d^- X = \int_0^T Y \delta X + \int_{[0,T]^2} D_{t_1} Y_{t_2 -} d\mu(t_1, t_2)$$

*(resp.*

$$\int_0^t Y d^+ X = \int_0^T Y \delta X + \int_{[0,T]^2} D_{t_1} Y_{t_2 +} d\mu(t_1, t_2).)$$

**Remark 8.2** *i) Condition (51) implies trivially*

$$E\left( \int_{[0,T]^2} |D_{t_1} Y_{s_1}| \int_{[0,T]^2} |D_{t_2} Y_{s_2}| \right) d|\mu|(s_1, s_2) d|\mu|(t_1, t_2) < \infty. \tag{54}$$

*ii) By Proposition 7.4 we know that $Y \in Dom(\delta)$.*

**Remark 8.3** *In the case of Malliavin calculus for classical Brownian motion, see Section 4, one has*

$$d\mu(s_1, s_2) = \delta(ds_2 - s_1) ds_1.$$

*So (54) becomes*

$$E\left( \iint_{[0,T]^2} |D_t Y_s|^2 \, ds dt \right) < \infty. \tag{55}$$

**Remark 8.4** *Condition (51) may be replaced by the existence a.s. of the trace $TrDu$, where*

$$TrDu = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \langle DY_s, 1_{]s, s+\varepsilon]} \rangle_{\mathcal{H}} ds. \tag{56}$$

*This is a direct consequence of Fubini theorem. A similar condition related to symmetric integral appears in [1].*

**Lemma 8.5** *Let $(Y_t)_{t\in[0,T]}$ be a process fulfilling points a), b), c) of Theorem 8.1. We set*

$$Y_t^\varepsilon = \frac{1}{\varepsilon}\int_{(t-\varepsilon)^+}^t Y_s ds. \tag{57}$$

*Then $Y^\varepsilon \in Dom(\delta)$ and for every $t$*

$$\int_0^t Y^\varepsilon \delta X \overset{\varepsilon\to 0}{\longrightarrow} \int_0^t Y\delta X \ \textit{in } L^2(\Omega). \tag{58}$$

**Proof:** First one can prove that if $Y \in Cyl(L_\mu)$, $Y^\varepsilon \in Cyl(L_\mu)$ and

$$D_t Y_s^\varepsilon = \frac{1}{\varepsilon}\int_{s-\varepsilon}^s D_t Y_s dr. \tag{59}$$

Then we can establish that $Y^\varepsilon$ is a suitable limit of elements in $Cyl(L_\mu)$ so that $Y^\varepsilon \in |\mathbb{D}^{1,2}(L_\mu)|$. We omit details at this level. Relation (59) extends then to every $Y$ fulfilling the assumptions of the theorem. According to Proposition 7.4, $Y^\varepsilon \in Dom(\delta)$. Relation (41) in Proposition 7.4 gives

$$E\left(\int_0^T (Y - Y^\varepsilon)\delta X\right)^2 \le E(\|Y - Y^\varepsilon\|_{\mathcal{H}}^2) + E\left(\int_{[0,T]^2} d|\mu|(t_1,t_2)\|D_\cdot(Y_{t_1} - Y_{t_1}^\varepsilon)\|_{\mathcal{H}}^2\right). \tag{60}$$

We have to show that both expectations converge to zero. The first expectation gives

$$E\left(\int_{[0,T]^2} d\mu(t_1,t_2)(Y_{t_1} - Y_{t_1}^\varepsilon)(Y_{t_2} - Y_{t_2}^\varepsilon)\right). \tag{61}$$

Using assumption a) of the theorem, Lebesgue dominated convergence theorem implies that (61) converges to

$$E\left(\int_{[0,T]^2} d\mu(t_1,t_2)(Y_{t_1} - Y_{t_1-})(Y_{t_2} - Y_{t_2-})\right).$$

For each $\omega$ a.s. the discontinuities of $Y(\omega)$ are countable. The fact that $|\mu|$ is non-atomic implies that previous expectation is zero.

We discuss now the second expectation. It gives

$$\int_{[0,T]^2} d|\mu|(t_1,t_2)\int_{[0,T]^2} E\left(D_{s_1}(Y_{t_1} - Y_{t_1}^\varepsilon)D_{s_2}(Y_{t_2} - Y_{t_2}^\varepsilon)\right) d|\mu|(s_1,s_2). \tag{62}$$

Taking in account assumptions b), c) of the theorem, previous term converges to

$$E\left(\int_{[0,T]^2} d|\mu|(t_1,t_2)\int_{[0,T]^2} d|\mu|(s_1,s_2)(D_{s_1}Y_{t_1} - D_{s_1}Y_{t_1-})(D_{s_2}(Y_{t_2} - D_{s_2}Y_{t_2-})\right).$$

Using Cauchy-Schwarz this is bounded by

$$E\left(\int_0^T d\nu(t)\int_0^T d\nu(s)(D_sY_t - D_sY_{t-})^2\right).$$

This quantity is zero because of c). ■

**Remark 8.6** *If point c') is verified (instead of c) it is possible to state a similar version of the lemma with $Y_t^\varepsilon = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} Y_s ds$.*

It is interesting to observe that convergence (58) holds weakly in $L^2(\Omega)$ even without assumption c). This constitutes the following proposition.

**Proposition 8.7** *Let $(Y_t)_{t\in[0,T]}$ be a process fulfilling points a), b) of Theorem 8.1. We set $Y^\varepsilon$ as in (57). Then for every $t$,*

$$\int_0^t Y^\varepsilon \delta X \stackrel{\varepsilon\to 0}{\longrightarrow} \int_0^t Y\delta X \tag{63}$$

*weakly in $L^2(\Omega)$.*

**Proof:** One can prove directly that $Y^\varepsilon$ belongs to $Dom(\delta)$ because of Fubini type Proposition 6.5. Indeed, we set

$$G = [0,T],\ \nu(ds) = ds,\ u(s,t) = Y_s 1_{]s,s+\varepsilon]}(t)$$

and we verify the assumptions of the Proposition. Using Proposition 7.4 and points a), b) it is clear that $E(\int_0^T Y^\varepsilon \delta X)^2$ is bounded. Then it is possible to show that the left term in (58) admits a subsequence $(\int_0^T Y^{\varepsilon_n}\delta W)$ converging weakly to some square integrable random variable $Z$.

Let $F \in Cyl$. By duality of Skorohod integral

$$\begin{aligned} E\left(F\int_0^T Y^{\varepsilon_n}\delta X\right) &= E\left(\langle DF, Y^{\varepsilon_n}\rangle_{\mathcal{H}}\right) \\ &= E\left(\int_{[0,T]^2} D_{s_1}FY_{s_2}^{\varepsilon_n}\mu(ds_1,ds_2)\right) \\ & E\left(\int_{[0,T]^2} D_{s_1}FY_{s_2-}\mu(ds_1,ds_2)\right). \end{aligned}$$

Now since $X$ is $L^2$ continuous, it is not difficult to see that the

$$|\mu|\left(\{s_1\}\times[0,T]\right) = |\mu|\left([0,T]\times\{s_2\}\right) = 0. \tag{64}$$

Using Banach-Steinhaus theorem and the density of $Cyl$ in $L^2(\Omega)$, the convergence (58) is established. For $\omega$ a.s the set $N(\omega)$ of discontinuity of $Y(\omega)$ is countable. Consequently $|\mu|\left([0,T]\times N(\omega)\right)=0$ and so

$$\begin{aligned} E\left(\int_{[0,T]^2} D_{s_1}FY_{s_2-}\mu(ds_1,ds_2)\right) &= E\left(\int_{[0,T]^2} D_{s_1}FY_{s_2}\mu(ds_1,ds_2)\right) \\ &= E\left(< DF, Y >_{\mathcal{H}}\right) = E\left(F\int_0^T Y\delta X\right). \end{aligned}$$

■

**Proof of the Theorem 8.1:** We only operate for the forward integral. The backward case can be treated similarly.

Proposition 6.4 implies that

$$\begin{aligned} I^-(\varepsilon, Y, dX, T) &= \frac{1}{\varepsilon}\int_0^T dsY_s\int_0^T 1_{]s,s+\varepsilon]}(t)\delta X_t \\ &= \frac{1}{\varepsilon}\int_0^T ds\int_0^T Y_s 1_{]s,s+\varepsilon]}(t)\delta X_t + \frac{1}{\varepsilon}\int_0^T ds\left(\int_{[0,T]^2} d\mu(t_1,t_2)D_{t_1}Y_s 1_{]s,s+\varepsilon]}(t_2)\right) \\ &= I_1(T,\varepsilon) + I_2(T,\varepsilon) \end{aligned}$$

. Proposition 6.5 says that

$$I_1(T,\varepsilon) = \int_0^T Y_t^\varepsilon \delta X_t.$$

According to Lemma 8.5, $I_1(T,\varepsilon)$ converges in $L^2(\Omega)$ to $\int_0^T Y\delta X$.

We observe now that $I_2(T,\varepsilon)$ gives

$$\int_{[0,T]^2} d\mu(t_1,t_2)\frac{1}{\varepsilon}\int_{t_2-\varepsilon}^{t_2} dsD_{t_1}Y_s. \tag{65}$$

Assumptions b) and c) together with Lebesgue dominated convergence theorem show that (65) converges in $L^2(\Omega)$ to

$$\int_{[0,T]^2} d\mu(t_1,t_2)D_{t_1}Y_{t_2}.$$

■

In particular, we retrieve the result in Remark 5.13.

**Corollary 8.8** *Let $h$ be a cadlag function $h:[0,T]\to\mathbb{R}$. Then*

$$\int_0^T hdX = \int_0^T hd^-X = \int_0^T hd^+X.$$

**Proof:** This is obvious because the Malliavin derivative of $h$ vanishes. ■.

**Corollary 8.9** *Let $(Y_t)_{t\in[0,T]}$ to be a process fulfilling assumptions a), b) c) c') of Theorem 8.1. Then the symmetric integral of $Y$ with respect to $X$ is defined and*

$$\int_0^T Y d^o X = \int_0^T Y \delta X + \frac{1}{2}\int_{[0,T]^2} \left(D_{t_1} Y_{t_2+} + D_{t_1} Y_{t_2-}\right) d\mu(t_1, t_2)$$

*and*

$$[X, Y]_T = \int_{[0,T]^2} \left(D_{t_1} Y_{t_2+} + D_{t_1} Y_{t_2-}\right) d\mu(t_1, t_2).$$

**Example 8.10 The case of a Gaussian martingale $X$.**

We recall by Section 4 that $[X] = \psi$, where $\psi$ is a deterministic increasing function vanishing at zero. Under assumption a), b), c) of Theorem 8.1

$$\int_0^T \psi d^- X = \int_0^T Y \delta X + \int_0^T d\psi(t_1) D_{t_1} Y_{t_1-}.$$

Let $Y$ be $\mathcal{F}$-progressively measurable cadlag, such that $\int_0^T Y_s^2 d[X]_s < \infty$ a.s. In [37] it is also shown that $\int_0^T Y d^- X$ equals the Itô integral $\int_0^T Y dX$.

It is possible to see that the Itô integral $Z = \int_0^T Y dX$ verifies the duality relation (25) and so $Y \in Dom(\delta)$. Moreover

$$\int_0^T Y d^- X = \int_0^T Y \delta X.$$

We discuss now Itô formula.

**Proposition 8.11** *Let $f \in C^2(\mathbb{R})$ such that $f''$ is bounded. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\delta X_s + \int_{\Delta_T} f''(X_{s_2}) d\mu(s_1, s_2) + \frac{1}{2}\int_0^T f''(X_s) d\mathcal{E}(s),$$

*where*

$$\mathcal{E}(t) = \mu(D_t), \;\; D_t = \{(s,s) | s \le t\}, \;\; \Delta_T = \{(s_1, s_2) | s_2 > s_1\}.$$

**Proof:** Itô formula for finite quadratic variation processes was established for instance by [36]. It says

$$f(X_t) = f(X_0) + \int_0^t f(X) d^- X + \frac{1}{2}\int_0^t f''(X) d[X].$$

Now we need to apply Theorem 8.1. For this we need to verify its hypotheses. The assumption a) is verified because

$$\sup_{t\leq T}|f(X_t)| \leq \sup|f'|\sup_{t\leq T}|X_t|.$$

Since $X$ is a Gaussian process, (36) recalls that $\sup_{t\leq T}|X_t| \in L^2(\Omega)$. On the other hand, setting $Y_t = f'(X_t)$,

$$D_{t_1}Y_{t_2} = f''(X_{t_2})1_{]0,t_2]}(t_1)$$

and so b) is also verified.

$$\lim_{\varepsilon\to 0}\int_{t_2-\varepsilon}^{t_2} D_{t_1}Y_s ds = \begin{cases} 0 & t_1 > t_2 \\ f''(X_{t_2}) & t_1 \leq t_2 \end{cases}$$

and c) is verified. Therefore

$$\int_0^T f'(X)\delta X = \int_0^T f'(X)d^-X + \int_{\Delta_T} f''(X_{t_2})d\mu(t_1,t_2).$$

Moreover, by Lemma 3.3 and 3.4 we have

$$\frac{1}{2}\int_0^T f''(X_s)d[X]_s = \frac{1}{2}\int_0^T f''(X_s)d\mathcal{E}(s).$$

■

We would like to examine some particular cases. For this we decompose $\mu$ into $\mu_d + \mu_{od}$ where for every $A \in B([0,T]^2)$

$$\mu_d(A) = \mu(A\cap D_T), \quad \mu_{od}(A) = \mu(A\backslash D_T).$$

Hence $\mu_d$ is concentrated on the diagonal, $\mu_{od}$ outside the diagonal.

We recall that

$$\mathcal{E}(t) = \mu(D_t)$$

where $\mathcal{E} = \mathcal{E}(X)$ is the energy function defined in Section 4. Consider the repartition functions $R_d$, $R_{od}$ of $\mu_d$, $\mu_{od}$. We have

$$R_d(s_1,s_2) = \mu_d\left(]0,s_1]\times]0,s_2]\right) = \mathcal{E}(s_1\wedge s_2)$$

and

$$R_{od}(s_1,s_2) = \mu_{od}\left(]0,s_1]\times]0,s_2]\right).$$

**Remark 8.12** *i) Setting $\psi = \mathcal{E}$, there is a Gaussian martingale with covariance $R_d$. It is enough to take $M_t = W_{\psi(t)}$, where $(W_t)$ is a classical Brownian motion.*
*ii) $R_{od}(s_1,s_2) = Cov(X_{s_1},X_{s_2}) - Cov(M_{s_1},M_{s_2})$.*

**Proposition 8.13** *Suppose that*

**i)** $\mathcal{E}$ *is absolutely continuous, i.e. there is a locally integrable function* $\mathcal{E}'$ *such that*

$$\mathcal{E}(t) = \int_0^t \mathcal{E}'(s)ds :$$

**ii)** $\mu_{od}$ *is absolutely continuous with respect to Lebesgue measure. In particular one has*

$$\mu_{od}(]0, s_1]\times]0, s_2]) = \int_{]0,s_1]\times]0,s_2]} \frac{\partial^2 R}{\partial s_1 \partial s_2} ds_1 ds_2.$$

*In fact it is clear that*

$$\frac{\partial^2 R}{\partial s_1 \partial s_2} = \frac{\partial^2 R_{od}}{\partial s_1 \partial s_2} \text{ on } [0, T]^2 \backslash D_T.$$

*Then the conclusion of the Theorem 8.1 holds replacing assumptions*

- *Y is cadlag,*
- *c) (resp. c'),*

*with*

- *For t a.e. Lebesgue*

$$D_t Y_{t-} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t D_t Y_s ds \text{ exists a.s.} \tag{66}$$

*(resp.*

$$D_t Y_{t+} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} D_t Y_s ds \text{ exists a.s.)} \tag{67}$$

*Moreover the conclusion of the theorem can be stated as*

$$\int_0^T Y d^- X = \int_0^T Y \delta X + \int_0^T D_t Y_{t-} \mathcal{E}'(t) dt + \int_{[0,T]^2} D_{t_1} Y_{t_2} \frac{\partial^2 R}{\partial s_1 \partial s_2}(t_1, t_2) dt_1, dt_2. \tag{68}$$

*(resp.*

$$\int_0^T Y d^+ X = \int_0^T Y \delta X + \int_0^T D_t Y_{t+} \mathcal{E}'(t) dt + \int_{[0,T]^2} D_{t_1} Y_{t_2} \frac{\partial^2 R}{\partial s_1 \partial s_2}(t_1, t_2) dt_1, dt_2.) \tag{69}$$

**Remark 8.14** *If c) and c') of Theorem 8.1, with (66) and (67) are verified then*

$$\int_0^T Y d^o X = \int_0^T Y \delta X + \int_0^T (D_t Y_{t+} + D_t Y_{t-}) \mathcal{E}'(t) dt + \int_{[0,T]^2} D_{t_1} Y_{t_2} \frac{\partial^2 R}{\partial s_1 \partial s_2}(t_1, t_2) dt_1 dt_2$$

*and*

$$[X, Y]_t = \int_0^T (D_t Y_{t+} + D_t Y_{t-}) \mathcal{E}'(t) dt.$$

Before proceeding to the proof, we recall a basic result which can be found in [41].

**Lemma 8.15** *Let $g \in L^p(\mathbb{R})$, $1 \le p < \infty$. We set*

$$g_\varepsilon(x) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^x g(y) dy \text{ or } g_\varepsilon(x) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} g(y) dy.$$

*Then $g_\varepsilon \to g$ a.e. and in $L^p$.*

**Proof** of Proposition 8.13: The proof follows the same line as the proof of Theorem 8.1. a) First we need to adapt Lemma 8.5 to show that $\lim_{\varepsilon \to 0} \int_0^T Y_s^\varepsilon \delta X_s = \int_0^T Y_s \delta X_s$ in $L^2(\Omega)$ where $Y^\varepsilon$ still denotes the same approximation process. Again we need to show that the right hand side of (60) converges to zero when $\varepsilon \to 0$. Its first term gives $(I_1 + I_2)(\varepsilon)$, where

$$\begin{aligned} I_1(\varepsilon) &= E\left(\int_{[0,T]^2} ds_1 ds_2 \frac{\partial^2 R}{\partial s_1 \partial s_2} (Y_{s_1}^\varepsilon - Y_{s_1})(Y_{s_2}^\varepsilon - Y_{s_2})\right), \\ I_2(\varepsilon) &= E\left(\int_0^T ds \mathcal{E}'(s)(Y_s^\varepsilon - Y_s)^2)\right). \end{aligned}$$

Lemma 8.15 implies that $Y^\varepsilon \longrightarrow Y$ a.e. $dP \otimes Leb$. Lebesgue dominated convergence theorem and Assumption a) imply that $I_1(\varepsilon) \longrightarrow 0$ and $I_2(\varepsilon) \longrightarrow 0$, when $\varepsilon$ converges to zero. It remains to control the second term in (60) which is given by (62). This second term gives $K_1(\varepsilon) + K_2(\varepsilon)$ with

$$\begin{aligned} K_1(\varepsilon) &= E\left(\int_{[0,T]^2} ds_1 ds_2 \left|\frac{\partial^2 R}{\partial s_1 \partial s_2}\right| \int_{[0,T]^2} dt_1 dt_2 \left|\frac{\partial^2 R}{\partial t_1 \partial t_2}\right| |D_{s_1} Y_{t_1}^\varepsilon - D_{s_1} Y_{s_1}||D_{s_2} Y_{s_2}^\varepsilon - D_{s_2} Y_{s_2}|\right), \\ K_2(\varepsilon) &= E\left(\int_0^T ds |\mathcal{E}'(s)| \int_0^T dt (D_s Y_t^\varepsilon - D_s Y_t)^2)\right). \end{aligned}$$

Point b) and (66) allow to show that $K_1(\varepsilon) + K_2(\varepsilon) \longrightarrow 0$.

b) The other point concerns the convergence of $I_2(T, \varepsilon)$ appearing in the proof of Theorem 8.1. To prove the convergence of (65) we separate again $\mu = \mu_d + \mu_{od}$ and we use (66) on the diagonal. Finally Lemma 8.15, 51 and Lebesgue dominated convergence theorem show that for $t_1,\ t_2,\ t_1 \neq t_2$ a.e.

$$\frac{1}{\varepsilon}\int_{t_2}^{t_2+\varepsilon} ds D_{t_1} Y_s \stackrel{\varepsilon\to 0}{\longrightarrow} D_{t_1} Y_{t_2} \quad \text{a.e.} \quad dP \otimes dt_1 dt_2.$$

■

We conclude with Itô formula which under assumptions i), ii) of Proposition 8.13 can be stated as follows.

**Corollary 8.16** *Let $f \in C^2$ with $f'$ bounded. Suppose $R$ of class $C^1(\Delta)$.*

$$\begin{aligned} f(X_T) &= f(X_0) + \int_0^T f'(X)\delta X \\ &\quad + \int_0^T f''(X_s)\left\{\frac{\partial R_{od}}{\partial s_2}(s,s) - \frac{\partial R_{od}}{\partial s_2}(0,s)\right\} ds + \frac{1}{2}\int_0^T f''(X_s)\mathcal{E}'(s)ds. \end{aligned}$$

**Proof:** It follows from Proposition 8.11 and the fact that

$$\int_{\Delta_T} d\mu(s_1,s_2) f''(X_{s_2}) = \int_0^T ds_2 f''(X_{s_2}) \int_0^{s_2} ds_1 \frac{\partial^2 R}{\partial s_1 \partial s_2}(s_1,s_2).$$

■

In previous Corollary it is important to write $R_{od}$ and not $R$ since $R_{od}$ only has a density and it can be integrated.

**Example 8.17** *Let us apply the obtained results to some particular examples.*

**a)** *Case of a Gaussian martingale with absolutely continuous quadratic variation $\psi$.*

$$\begin{aligned} &R(t_1,t_2) = \psi(t_1 \wedge t_2) \\ &\mathcal{E}'(t) = \dot{\psi}(t), \quad R = R_d, \\ &\frac{\partial^2 R}{\partial t_1 \partial t_2} = 0 \text{ a.e. Lebesgue.} \end{aligned}$$

Let $Y$ be as in Proposition 8.13. Then

$$\int_0^T Y d^- X = \int_0^T Y\delta X + \int_0^T D_t Y_{t-}\dot{\psi}(t)dt.$$

**b)** *The case of fractional Brownian motion $H > 1/2$.*
We have

$$R = R_{od}$$
$$\frac{\partial^2 R}{\partial t_1 \partial t_2} = 2H(2H-1)|t_2 - t_1|^{2H-2}. \tag{70}$$

One obtains the classical results for fractional Brownian motion as in [3], for instance

$$\int_0^T Y d^- X = \int_0^T Y \delta X + H(2H-1) \int_{[0,T]^2} D_{t_1} Y_{t_2} |t_2 - t_1|^{2H-2} dt_1 dt_2.$$

Itô formula becomes the now classical one

$$f(X_t) = f(X_0) + \int_0^t f'(X) \delta X + H \int_0^T f''(X_s) s^{2H-1} ds. \tag{71}$$

**c)** *The case of bifractional Brownian motion.*
Due to the properties of the quadratic variation of $X$, the division into two cases should be done.

c1) If $2HK > 1$, then the problem can be treated similarly with the above fBm case. Applying Corollary 8.16 with $R_{od} = R$, $\mathcal{E} = 0$, we will obtain

$$\begin{aligned} f(X_t) &= f(0) + \int_0^t f'(X_s) \delta X_s + \int_0^t f''(X_s) \left( \partial_2 R(s,s) - \partial_2 R(0,s) \right) ds \\ &= f(0) + \int_0^t f'(X_s) \delta X_s + HK \int_0^t f''(X_s) s^{2HK-1} ds. \end{aligned}$$

c2) If $2HK = 1$, we come back to the notations of Section 4. It holds that

$$R_d = R_2, R_{od} = R_1, \mathcal{E}(t) = [X,X]_t = 2^{1-K} t,$$
$$R_1(t_1, t_2) = \int_{[0,t_1] \times [0,t_2]} \frac{\partial^2 R}{\partial s_1 \partial s_2} ds_1 ds_2.$$

since $R_1$ vanishes on the real axes. Therefore under the assumptions on $Y$ of Proposition 8.13 we have

$$\int_0^T Y d^- X = \int_0^T Y \delta X + 2^{-K+1} \int_0^T D_t Y_{t-} dt + \int_{[0,T]^2} D_{t_1} Y_{t_2} \frac{\partial^2 R}{\partial t_1 \partial t_2}(t_1, t_2) dt_1 dt_2.$$

In fact
$$\frac{\partial R_1}{\partial s_2}(s_1, s_2) = 2^{-K} \left\{ (s_2^{2H} + s_1^{2H})^{K-1} s_2^{2H-1} - 1 \right\}.$$

So

$$\frac{\partial R_1}{\partial s_2}(s_2, s_2) = \frac{1}{2} - 2^{-k}, \text{ and } \frac{\partial R_1}{\partial s_2}(0, s_2) = 0.$$

In conclusion Itô formula stated by Corollary 8.16 gives

$$\begin{aligned} f(X_t) &= f(0) + \int_0^t f'(X_s)\delta X_s + \int_0^t f''(X_s)\left(\frac{1}{2} - 2^{-K}\right) ds + \int_0^t f''(X_s)2^{-K} ds \\ &= f(0) + \int_0^t f'(X_s)\delta X_s + \frac{1}{2}\int_0^t f''(X_s) ds. \end{aligned}$$

We are glad to conclude our paper by this very appealing formula.

# References


[1] E. Alos, J.A. Leon and D. Nualart (2001): *Stratonovich calculus for fractional Brownian motion with Hurst parameter less than* $\frac{1}{2}$. Taiwanese Journal of Math., 4, pag. 609-632.

[2] E. Alos, O. Mazet and D. Nualart (2001): *Stochastic calculus with respect to Gaussian processes.* Annals of Probability, 29, pag. 766-801.

[3] E. Alos and D. Nualart (2002): *Stochastic integration with respect to the fractional Brownian motion.* Stochastics and Stochastics Reports, 75, pag. 129-152.

[4] R. Bass (1997): *Elliptic operators and diffusions.* Springer-Verlag.

[5] P. Billingsley (1985): *Probability and Measure*, Second Edition, John Wiley and Sons, Inc.

[6] P. Cheridito (2001): *Mixed fractional Brownian motion.* Bernoulli, no. 6, 913–934.

[7] L. Coutin, D. Nualart and C.A. Tudor (2001): *The Tanaka formula for the fractional Brownia motion.* Stochastic Proc. Applic., 94(2), pag. 301-315.

[8] L. Decreusefond (2005): *Stochastic integration with respect to Volterra processes.* Ann. Inst. H. Poincaré Probab. Statist. 41, no. 2, pag. 123–149.

[9] L. Decreusefond and A.S. Ustunel (1998): *Stochastic analysis of the fractional Brownian motion.* Potential Analysis, 10, pag. 177-214.

[10] T. Duncan, Y. Z. Hu and B. Pasik-Duncan (2000): *Stochastic Calculus for Fractional Brownian Motion I. Theory.* SIAM J. Control Optim., 38 (2), pag. 582-612.

[11] M. Eddahbi, R. Lacayo, J.L. Solé, C. A. Tudor and J. Vives (2005): *Regularity of the local time for the d–dimensional fractional Brownian motion with N–parameters.* Stoc. Analysis and Applications, 23(2), pag. 383-400.

[12] M. Emery (1982): *Covariance des semimartingales gaussiennes*, C.R. Acad. Sc. Paris. t. 295, 703-705.

[13] X. Fernique (1975): *Régularité des trajectoires des fonctions aléatoires gaussiennes.* Ecole d' été de Probabilités de Saint-Flour, IV-1974, pp. 1–96. Lecture Notes in Math., Vol. 480, Springer-Verlag.

[14] H. Föllmer (1981): *Calcul d'Itô sans probabilités.* Séminaire de Probabilités XV, Lecture Notes in Mathematics 850, pag. 143-150.

[15] S. Graversen and M. Rao (1985): *Quadratic variation and energy.* Nagoya J. Math., 100, pag. 163-180.

[16] J. Horowitz (1977): *Une remarque sur les bimesures*, Seminaire de Probibilités XI, Lecture Notes in Math. 581, pag. 59-64.

[17] C. Houdré and J. Villa (2003): *An example of infinite dimensional quasi-helix.* Contemporary Mathematics, Amer. Math. Soc., 336, pag. 195-201.

[18] C.N. Jain and D. Monrad (1982): *Gaussian Quasimartingales*, Z. Warscheinlichkeitstheorie verw. Gebiete 59, 139-159.

[19] M. Jolis (2005): *On the Wiener integral with respect to the fractional Brownian motion on an interval.* Preprint Universitat autonoma Barcelona.

[20] J. P. Kahane (1981): *Hélices et quasi-hélices.* Adv. Math., 7B, pag. 417-433.

[21] I. Karatzas and S. Shreve (1991): *Brownian Motion and Stochastic Calculus. Second Edition.* Springer-Verlag.

[22] H. Kuo (1975): *Gaussian Measures in Banach spaces.* Lecture Notes in Math. 436, Springer-Verlag.

[23] Z. Ma and M. Röckner (1992): *Introduction to the theory of (Non-Symetric) Dirichlet forms*, Springer-Verlag.

[24] P. Major (1981): *Multiple Wiener-Ito integrals.* Springer-Verlag.

[25] P. Malliavin (1997): *Stochastic analysis.* Springer Verlag.

[26] M.B. Marcus (1973): *Continuity of Gaussian processes and random Fourier series.* Annals of Probability, 1, pag. 968-981.

[27] O. Mocioalca and F. Viens (2005): *Skorohod integration and stochastic calculus beyond the fractional Brownian scale.* J. Funct. Anal., no. 2, pag. 385-434.

[28] D. Nualart (1995): *The Malliavin calculus and related topics.* Springer-Verlag.

[29] D. Nualart (1998): *Analysis on Wiener spaces ans anticipating stochastic calculus.* St. Flour Summer School, Lecture Notes in Mathematics, Springer-Verlag.

[30] D. Nualart and Y. Ouknine (2002): *Regularization of differential equations by fractional noise.* Stoc. Proc. Appl., 102, pag. 103-116.

[31] D. Nualart and J. Vives (1992): *Smoothness of Brownian local times and related functionals.* Potential Analysis, 1(3), pag. 257-263.

[32] V. Pipiras and M. Taqqu (2000): *Integration questions related to fractional Brownian motion.* Probability theory and related fields, 118, pag. 251-291.

[33] Z. Qian and T. Lyons (2002): *System control and rough paths.* Clarendon Press, Oxford.

[34] F. Russo and C.A. Tudor (2006): *On the bifractional Brownian motion.* Stochastic Processes and their applications, 116, pag. 830-856.

[35] F. Russo and P. Vallois (1993): *Forward backward and symmetric stochastic integration.* Prob. Theory Rel. Fields, 97, pag. 403-421.

[36] F. Russo and P. Vallois (2000): *Stochastic calculus with respect to a finite quadratic variation process.* Stochastics and Stochastics Reports, 70, pag. 1-40.

[37] F. Russo and P. Vallois (2005): *Elements of stochastic integration vis regularization.* Preprint LAGA-Paris 13 2004-28. Revised version 2005. To appear in Séminaire de Probabilités.

[38] I. Shigekawa (2004): *Stochastic analysis*, Memoirs of the AMS, Vol. 224.

[39] C. Stricker (1983): *Semimartingales gaussiennes - application au problème de l'innovation*, Z. Warscheinlichkeitstheorie verw. Gebiete 64, pag. 303-313.

[40] A.E. Taylor (1965): *General Theory of Functions and Integration*, Blaisdell Publishing Company.

[41] E. Stein (1970): *Singular integrals and differentiability properties of functions*, Princeton University Press, No 30.

[42] M. Taqqu (1975): *Weak convergence to fractional Brownian motion and to the Rosenblatt process.* Z. Wahrscheinlichkeitstheorie verw. Geb. 31, pag. 287-302.

[43] C.A. Tudor and F. Viens (2003): *Itô formula and local times for the fractional Brownian sheet.* Electronic Journal of Prob., 8, paper 14, pag. 1-31.

[44] Y. Xiao (2005): *Strong local nondeterminism and the sample path properties of Gaussian random fields.* Preprint.

[45] S. Watanabe (1994): *Lecture on stochastic differential equations and Malliavin calculus*, Tata institute of fundamental research, Springer-Verlag.