Asymptotic Properties of the Detrended Fluctuation Analysis of Long Range Dependence Processes

Jean-Marc Bardet and Imen Kammoun

Abstract— **In the past few years, a certain number of authors have proposed analysis methods of the time series built from a long range dependence noise. One of these methods is the Detrended Fluctuation Analysis (DFA), frequently used in the case of physiological data processing. The aim of this method is to highlight the long-range dependence of a time series with trend. In this study asymptotic properties of DFA of the fractional Gaussian noise are provided. Those results are also extended to a general class of stationary long-range dependent processes. As a consequence, the convergence of the semi-parametric estimator of the Hurst parameter is established. However, several simple exemples also show that this method is not at all robust in case of trend.**

Index Terms— **Detrended ¤uctuation analysis, fractional Gaussian noise, stationary process, self-similar process, Hurst parameter, trend, long-range dependence processes.**

I. Introduction

In the past few years, numerous methods of analysis of a trended long range process have been proposed. One of these methods is the Detrended Fluctuation Analysis (DFA), frequently used in the case of physiological data processing in particular the heartbeat signals recorded in healthy or sick subjects (see for instance [10], [13], [17], [18] and [19]). Indeed, it can be interesting to £nd some constants among the ¤uctuations of physiological data. The parameter of long-range dependence (so called Hurst parameter) of the original signal, or the self-similarity parameter of the aggregated signal could be a new way of interpretation and explanation for a physiological behavior.

The DFA method is a version for time series with trend of the method of aggregated variance used for long-memory stationary process (see for instance [13]). It consists on 1. aggregated the process by windows with £xed length, 2. detrended the process from a linear regression in each windows, 3. computed the standard error of the residual errors (the DFA function) for all data, 4. estimated the coef£cient of the power law from a log-log regression of the DFA function on the length of the chosen window. After the £rst stage, the process is supposed to behave like a self-similar process with stationary increments added with a trend. The second stage is supposed to remove the trend. Finally, the third and fourth stages are the same than those of the aggregated method (for zero-mean stationary process).

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The processing of experimental data, and in particular physiological data, exhibits a major problem that is the non-stationarity of the signal . Hu *et al.* (2001) have studied different types of non-stationarities associated with examples of trends (linear, sinusoidal and power-law trends) and deduced their effect on an added noise and the kind of competition who exists between this two signals. They have also explained (see Chen *et al.*, 2002) the effects of three other types of non-stationarities, which are often encountered in real data. The DFA method was applied to signals with segments removed, with random spikes or with different local behavior. The results were compared with the case of stationary correlated signals.

In Taqqu *et al.* (1999), the case of the fractional Gaussian noise (FGN) is studied. A theoretical proof to the power law followed by the expectation of the DFA function of this process is established. This is an important £rst step for proving the convergence of the estimator of the Hurst parameter. The study we propose here is a kind of achievement of this work. Indeed the convergence rate of the Hurst parameter estimator is obtained, in a semi-parametric frame.

The paper is organized as follows. In Section II, the DFA method is presented and two general properties are proved. The Section III is devoted to provide asymptotic properties (beforehand illustrated by simulations) of the DFA function in case of the FGN. Section IV contains an extension of these results for a general class of stationary long-range dependence processes. Finally, in Section V, the method is proved not to be robust in different particular cases of trended processes, while the proofs of the different results are in the Appendix I.

II. Definitions and first properties of the DFA method

The Detrended Fluctuation Analysis (DFA)

The DFA method was introduced in [18]. The aim of this method is to highlight the self-similarity of a time series with trend. Let $(Y(1),\dots,Y(N))$ be a sample of a time series $(Y(n))_{n\in }$.

1) The £rst step of the DFA method is a "discrete integration" of the sample, *i.e.* a calculation of $(X(1),\dots,X(N))$ where

$$X(k)=\sum_{i=1}^{k}Y(i)\quad\text{for } k\in\{1,\dots,N\}. \tag{1}$$

2) The second step is a division of $\{1, \ldots, N\}$ in $[N/n]$ windows of length n (for $x \in \mathbb{R}$, $[x]$ is the integer part of x). In each window, the least squares regression line is computed, which represents the linear trend of the process in the window. Then, we denote by $\widehat{X}_n(k)$ for $k = 1, \ldots, N$ the process formed by this piecewise linear interpolation. Then the DFA function is the standard deviation of the residuals obtained from the difference between $X(k)$ and $\widehat{X}_n(k)$, therefore,
$$F(n) = \sqrt{\frac{1}{n \cdot [N/n]} \sum_{k=1}^{n\cdot[N/n]} \left(X(k) - \widehat{X}_n(k)\right)^2}$$
3) The third step consists on a repetition of the second step with different values $(n_1, \ldots, n_m)$ of the window's length. Then the graph of the $\log F(n_i)$ by $\log n_i$ is drawn. The slope of the least squares regression line of this graph provides an estimation of the self-similarity parameter of the process $(X(k))_{k\in}$ or the Hurst parameter of the $(Y(n))_{n\in}$ process (see above for the explanations).

From the construction of the DFA method, it is interesting to de£ne the restriction of the DFA function in a window. Thus, for $n \in \{1, \ldots, N\}$, one de£nes the partial DFA function computed in the j-th window, *i.e.*
$$F_j^2(n) = \frac{1}{n} \sum_{i=n(j-1)+1}^{nj} (X(i) - \widehat{X}_n(i))^2 \qquad (2)$$
for $j \in \{1, \ldots, [N/n]\}$. Then, it is obvious that
$$F^2(n) = \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} F_j^2(n). \qquad (3)$$
Remark: In the Hu *et al.* and Kantelhardt *et al.* papers (for details [10], [12] and [13]), the de£nition of the time series $(X(n))_{n\in}$ computed from $(Y(n))_{n\in}$ is different of (1), *i.e.*
$$\tilde{X}(k) = \sum_{i=1}^{k} (Y(i) - \overline{Y}_N) \ , \quad \text{for} \quad k \in \{1, \ldots, N\}$$
with $\overline{Y}_N = \frac{1}{N} \sum_{j=1}^{N} Y(j)$.

It is obvious to see that in both the de£nitions, $(X(k) - \widehat{X}_n(k))$ is the same and therefore the value of $F(n)$ is the same.

Lemma 2.1: With the previous notations, let $\tilde{F}(n)$ be the DFA function built from $(\tilde{X}(k))$, *i.e.*
$$\tilde{F}(n) = \sqrt{\frac{1}{n \cdot [N/n]} \sum_{k=1}^{n\cdot[N/n]} \left(\tilde{X}(k) - \widehat{\tilde{X}}_n(k)\right)^2}.$$
Then for $n \in \{1, \ldots, N\}$, $F(n) = \tilde{F}(n)$.

Proof: Consider the j-th window, $j \in \{1, \ldots, [N/n]\}$ and de£ne the vectors $X^{(j)} = (X(1 + n(j-1)), \ldots, X(nj))'$ and $\tilde{X}^{(j)} = (\tilde{X}(1 + n(j-1)), \ldots, \tilde{X}(nj))' = X^{(j)} - (1 + n(j-1), \ldots, nj)' \cdot \overline{Y}_N$. In this j-th window, de£ne E_j the vector subspace of $\mathbb{R}^n$ generate by the two vectors of $\mathbb{R}^n$, $(1, \ldots, 1)'$ and $\big((j-1)n+1, (j-1)n+2, \ldots, nj\big)'$. It is well known that if P_A is the matrix of the orthogonal projection on a vector subspace A of $\mathbb{R}^n$, then
$$F_j^2(n) = \frac{1}{n} \left(P_{E_j^\perp} \cdot X^{(j)}\right)' \cdot P_{E_j^\perp} \cdot X^{(j)}$$
$$\text{and} \quad \tilde{F}_j^2(n) = \frac{1}{n} \left(P_{E_j^\perp} \cdot \tilde{X}^{(j)}\right)' \cdot P_{E_j^\perp} \cdot \tilde{X}^{(j)},$$
where $E_j^\perp$ is the orthogonal vector subspace of E_j.
But $(1 + n(j-1), \ldots, nj)' \cdot \overline{Y}_N \in E_j$, and therefore
$$\begin{aligned} P_{E_j^\perp} \cdot \tilde{X}^{(j)} &= P_{E_j^\perp} \cdot X^{(j)} - P_{E_j^\perp} \cdot (1 + n(j-1), \ldots, nj)' \, \overline{Y}_N \\ &= P_{E_j^\perp} \cdot X^{(j)}, \end{aligned}$$
and thus, $F_j^2(n) = \tilde{F}_j^2(n)$, that implies $F(n) = \tilde{F}(n)$. □

In order to simplify the following proofs, the case of the DFA method applied to a stationary process $\{Y(t),\, t \geq 0\}$ can be considered. The following lemma shows that the law of $F_j^2(n)$ does not depend on j

Lemma 2.2: Let $\{Y(t),\, t \geq 0\}$ a stationary process. Then, with $X(k) = \sum_{i=1}^{k} Y(i)$ for $k \in \{1, \ldots, N\}$, for any $n \in \{1, \ldots, N\}$, the times series $(F_j^2(n))_{1 \leq j \leq [N/n]}$ is a stationary process.

Proof: Set $j \in \{1, \ldots, [N/n]\}$ and de£ne the vector $X^{(j)} = (X(1 + n(j-1)), \ldots, X(nj))'$. Then,
$$\begin{aligned} X^{(j)} - X(n(j-1)+1) \cdot (1, \ldots, 1)' \\ \overset{\mathcal{L}}{=} X^{(1)} - X(1) \cdot (1, \ldots, 1)'. \end{aligned} \qquad (4)$$
Indeed
$$X^{(j)} - X(n(j-1)+1) \cdot (1, \ldots, 1)' = \Big(0, Y(2 + n(j-1)), \ldots, \sum_{k=2}^{n-1} Y(k + n(j-1)), \sum_{k=2}^{n} Y(k + n(j-1))\Big)$$
and
$$X^{(1)} - X(1) \cdot (1, \ldots, 1)' = \Big(0, Y(2), \ldots, \sum_{k=2}^{n-1} Y(k), \sum_{k=2}^{n} Y(k)\Big)$$
We have $(Y(2), \ldots, Y(n)) \overset{\mathcal{L}}{=} (Y(2 + (j-1)n), \ldots, Y(jn))$ because $\{Y(t),\, t \geq 0\}$ is a stationary process. Then with $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ a Borelian function de£ned by $g(y_2, \ldots, y_n) = (y_2, \ldots, \sum_{k=2}^{n-1} y_k, \sum_{k=2}^{n} y_k)$, it is clear that $g(Y(2), \ldots, Y(n)) \overset{\mathcal{L}}{=} g(Y(2 + (j-1)n), \ldots, Y(jn))$ and therefore (4) is true.

Now, in each window j, and with the same de£nition of the vector subspace E_j than in the proof of Lemma 2.1,
$$\begin{aligned} F_j^2(n) &= \frac{1}{n} \left(P_{E_j^\perp} \cdot X^{(j)}\right)' \cdot P_{E_j^\perp} \cdot X^{(j)} \\ &= \frac{1}{n} \left(X^{(j)} - X(n(j-1)+1) \cdot (1, \ldots, 1)'\right)' \cdot P_{E_j^\perp} \cdot \\ &\qquad \left(X^{(j)} - X(n(j-1)+1) \cdot (1, \ldots, 1)'\right), \end{aligned}$$

with $P_{E_j^\perp} \cdot (1,\dots,1)' = (0,\dots,0)'$. But $E_1 = E_j$ and thus $E_j^\perp = E_1^\perp$. Therefore, with (4), we obtain $F_j^2(n) \stackrel{\mathcal{L}}{=} F_1^2(n)$ for all $j \in \{1,\dots,[N/n]\}$.

Moreover, for all $m \in \mathbb{N}^*$, $(j_1,\dots,j_m) \in \{1,\dots,[N/n]\}^m$ and $t \in \mathbb{N}^*$, the same reasoning can be resumed for the case of vectors $(F_{j_1}^2(n),\dots,F_{j_m}^2(n))$ and $(F_{j_1+t}^2(n),\dots,F_{j_m+t}^2(n))$. Indeed,

$$\Big(X^{(j_1)'} - X(n(j_1-1)+1)\cdot(1,\dots,1),\dots, X^{(j_m)'} - X(n(j_m-1)+1)\cdot(1,\dots,1)\Big)' \stackrel{\mathcal{L}}{=} \Big(X^{(j_1+t)'} - X(n((j_1+t)-1)+1)\cdot(1,\dots,1),\dots, X^{(j_m+t)'} - X(n((j_m+t)-1)+1)\cdot(1,\dots,1)\Big)'$$

and $P_{E_{j_1}} = \cdots = P_{E_{j_m}} = P_{E_{j_1+t}} = \cdots = P_{E_{j_m+t}}$. This achieves the proof. □

Finally, in order to consider trended processes, the following property for two independent processes could be considered.

Lemma 2.3: Let $Y = \{Y(k),\, k \in \mathbb{N}\}$ and $Y' = \{Y'(k),\, k \in \mathbb{N}\}$ be two independent processes, with $\mathbb{E}(Y(k)) = 0$ for all $k \in \mathbb{N}$, and denote respectively F_Y^2, $F_{Y'}^2$ and $F_{Y+Y'}^2$ the DFA functions associated to Y, Y' and $Y+Y'$. Then, for $n \in \{1,\cdots,N\}$,

$$\mathbb{E}(F_{Y+Y'}^2(n)) = \mathbb{E}(F_Y^2(n)) + \mathbb{E}(F_{Y'}^2(n)).$$

Proof: With X and X' the aggregated processes associated to Y and Y', it is obvious that

$$\begin{aligned}
&\mathbb{E}(F_{Y+Y'}^2(n)) \\
&= \frac{1}{n\cdot[N/n]} \sum_{k=1}^{n\cdot[N/n]} \mathbb{E}\Big(\big(X(k)+X'(k)-\widehat{X}_n(k)-\widehat{X}'_n(k)\big)^2\Big) \\
&= \mathbb{E}(F_Y^2(n)) + \mathbb{E}(F_{Y'}^2(n)) + \frac{2}{n\cdot[N/n]}\cdot \sum_{k=1}^{n\cdot[N/n]} \mathbb{E}\Big(\big(X(k)-\widehat{X}_n(k)\big)\big(X'(k)-\widehat{X}'_n(k)\big)\Big).
\end{aligned}$$

From the independence of X and X' and thanks to the assumption $\mathbb{E}(Y(k)) = 0$ for all $k \in \mathbb{N}$ which implies $\mathbb{E}(X(k)) = 0$ and $\mathbb{E}(\widehat{X}(k)) = 0$ for all $k \in \mathbb{N}$, $\mathbb{E}\Big(\big(X(k) - \widehat{X}_n(k)\big)\big(X'(k)-\widehat{X}'_n(k)\big)\Big) = 0$. □

III. ASYMPTOTIC PROPERTIES OF THE DFA FUNCTION FOR A FGN

In this section, we study the asymptotic (both the sample size N and the length of window n increase to ∞) behavior of the DFA when $(Y(n))_{n\in}$ is a stationary Gaussian process called fractional Gaussian noise (FGN), *i.e.* $(X_1,\dots,X_N)$ is a Gaussian process having stationary increments and called a fractional Brownian motion (FBM). First, one reminds some de£nitions and properties of both this processes.

De£nition and £rst properties of the FBM and the FGN

Let $\{X^H(t),\, t \geq 0\}$ be a fractional Brownian motion (FBM) with parameters $H \in]0,1[$ and $\sigma^2 > 0$, *i.e.* a real zero mean Gaussian process satisfying,

1) $X^H(0) = 0$ a.s.
2) $E[(X^H(t) - X^H(s))^2] = \sigma^2|t-s|^{2H} \quad \forall (t,s) \in \mathbb{R}_+^2$.

Here there are some properties of a FBM $\{X^H(t),\, t \geq 0\}$ (see more details in Samorodnitsky and Taqqu, 1994)

- The process $\{X^H(t),\, t \geq 0\}$ has stationary increments. As a consequence, if we denote $\{Y^H(t),\, t \geq 0\}$ the process de£ned by $Y^H(t) = X^H(t+1) - X^H(t)$ for $t \geq 0$, then $\{Y^H(t),\, t \geq 0\}$ is a zero mean stationary Gaussian process so-called a fractional Gaussian noise (FGN).
- $\{X^H(t),\, t \geq 0\}$ is a self-similar process satisfying $\forall c > 0$, $X^H(ct) \stackrel{\mathcal{L}}{=} c^H X^H(t)$ and H is also called the exponent of self-similarity.
- The covariance function of the fractional Brownian motion $\{X^H(t),\, t \geq 0\}$ is

$$\mathrm{Cov}(X^H(t), X^H(s)) = \frac{\sigma^2}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}) \quad \forall (s,t) \in \mathbb{R}_+^2. \tag{5}$$

Some numerical results of the DFA of a FGN

The following Figures 1 and 2 show an example of the DFA method applied to a FGN with different values of H ($H = 0.6$ in the £rst £gure and $H = 0.2, 0.4, 0.5, 0.7, 0.8$ in the second one, with $N = 10000$ for both ones). Such a sample path is generated with a circulant matrix algorithm (see for instance Bardet *et al.*, 2002). Let us remark that if $(Y(n))_{n\in}$ is a sample path of a discretized FGN, then $(X(1),\dots,X(N))$ is a sample path of the associated discretized FBM.

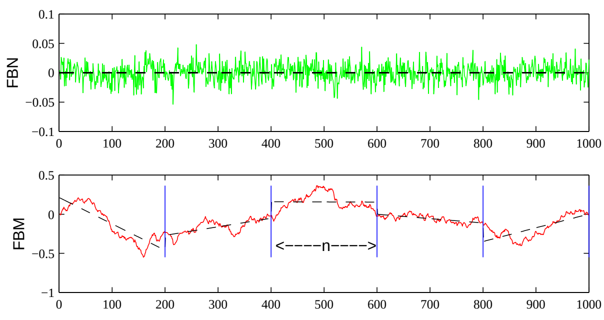

Fig. 1. Two £rst step of the DFA method applied to a path of a discretized FGN (with $H = 0.6$ and $N = 10000$)

In the right of Figure 2 appear the different estimations of H computed from the DFA method. Those values have to be compared with theoretical ones. The results seem to be quite good and it seems that, under certain conditions, the asymptotic behavior of the DFA function $F(n)$ can be written like

$$F(n) \simeq c(\sigma, H)\cdot n^H, \tag{6}$$

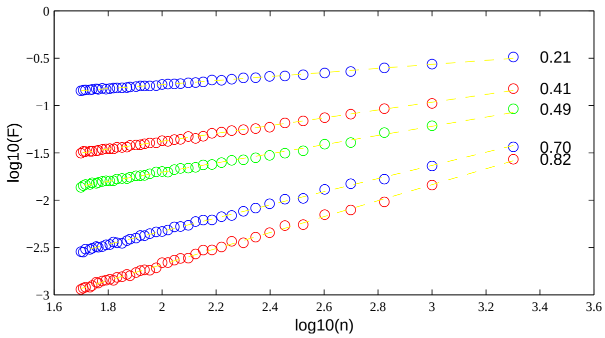

Fig. 2. Results of the DFA method applied to a path of a discretized FGN for different values of $H = 0.2, 0.4, 0.5, 0.7, 0.8$) (with also $N = 10000$)

where c is a positive function depending only on σ and H (see its expression above). The approximation (6) explains that the slope of the least square regression line of $(\log F(n_i))$ by $\log(n_i)$ for different values of n_i provides an estimation of H. One provides now a mathematical proof of this result.

Let $\{X^H(t), t \geq 0\}$ be a FBM, built as a cumulated sum of a FGN $\{Y^H(t), t \geq 0\}$. We first give some asymptotic properties of $F_1^2(n)$.

Property 3.1: Let $\{X^H(t), t \geq 0\}$ be a FBM with parameters $0 < H < 1$ and $\sigma^2 > 0$. Then, for n and j large enough,

$$\begin{array}{lll} 1.\ \mathbb{E}(F_1^2(n)) & = & \sigma^2 f(H)\cdot n^{2H}\Big(1+O(\frac{1}{n})\Big), \\ 2.\ \mathrm{Var}\big(F_1^2(n)\big) & = & \sigma^4 g(H)\cdot n^{4H}\Big(1+O(\frac{1}{n})\Big), \\ 3.\ \mathrm{Cov}(F_1^2(n),F_j^2(n)) & = & \sigma^4 h(H)\cdot n^{4H}\cdot j^{2H-3}\cdot \\ & & \Big(1+O(\frac{1}{n})+O(\frac{1}{j})\Big), \end{array}$$

with $f(H) = \dfrac{(1-H)}{(2H+1)(H+1)(H+2)}$, g depending only on H, see (19), and $h(H) = \dfrac{H^2(H-1)(2H-1)^2}{48(H+1)(2H+1)(2H+3)}$.

The proofs of these results (and of the other ones) are provided in the Appendix I.

In order to obtain a central limit theorem for the logarithm of the DFA function, one considers a normalized DFA functions

$$\tilde{S}_j(n) = \frac{F_j^2(n)}{n^{2H}\sigma^2 f(H)} \quad \text{and} \quad \tilde{S}(n) = \frac{F^2(n)}{n^{2H}\sigma^2 f(H)} \tag{7}$$

for $n \in \{1,\dots,N\}$ and $j \in \{1,\dots,[N/n]\}$.

As a consequence, for $n \in \{1,\dots,N\}$, the stationary time series $(\tilde{S}_j(n))_{1\leq j\leq [N/n]}$ satisfy

$$\left\{\begin{array}{lll} \mathbb{E}(\tilde{S}_j(n)) & = & 1+O(\frac{1}{n}) \\ \mathrm{Var}(\tilde{S}_j(n)) & = & \frac{g(H)}{f(H)^2}+O(\frac{1}{n}) \\ \mathrm{Cov}(\tilde{S}_1(n),\tilde{S}_j(n)) & = & \frac{h(H)}{f(H)^2}\cdot\frac{1}{j^{3-2H}} \\ & & \Big(1+O(\frac{1}{n})+O(\frac{1}{j})\Big) \end{array}\right. \tag{8}$$

Under conditions on the asymptotic length n of the windows, one proves a central limit theorem satisfied by the logarithm of the empirical mean $\tilde{S}(n)$ of the random variables $(\tilde{S}_j(n))_{1\leq j\leq [N/n]}$.

Property 3.2: Under the previous assumptions and notations, let $n \in \{1,\dots,N\}$ be such that $N/n \to \infty$ and $N/n^3 \to 0$ when $N \to \infty$. Then

$$\sqrt{[\frac{N}{n}]}\cdot\log(\tilde{S}(n)) \ \xrightarrow[\substack{n\to\infty\\ N\to\infty}]{\mathcal{L}}\ \mathcal{N}(0,\gamma^2(H))),$$

where $\gamma^2(H) > 0$ depends only on H.

This result can be obtained for different lengths of windows satisfying the conditions $N/n \to \infty$ and $N/n^3 \to 0$. Let $(n_1,\dots,n_m)$ be such different window lengths. Then, one can write for N and n_i large enough

$$\log(\tilde{S}(n_i)) \simeq \frac{1}{\sqrt{[N/n_i]}}\cdot\varepsilon_i \quad \Longrightarrow$$

$$\log(F(n_i)) \simeq H\cdot\log(n_i)+\frac{1}{2}\log(\sigma^2 f(H))+\frac{1}{\sqrt{[N/n_i]}}\cdot\varepsilon_i,$$

with $\varepsilon_i \sim \mathcal{N}(0,\gamma^2(H))$. As a consequence, a linear regression of $\log(F(n_i))$ on $\log(n_i)$ provides an estimation of H. More precisely,

Proposition 3.3: Under the previous assumptions and notations, let $n \in \{1,\dots,N\}$, $m \in \mathbb{N}^*\setminus\{1\}$ and $r_1 < \dots < r_m \in \{1,\dots,[N/n]\}^m$ be such that $N/n \to \infty$ and $N/n^3 \to 0$ when $N \to \infty$ with $n_i = r_i n$ for each i. Let $\widehat{H}$ be the estimator of H from the linear regression of $\log(F(r_i\cdot n))$ on $\log(r_i\cdot n)$, *i.e.*

$$\widehat{H} = \frac{\sum_{i=1}^m(\log(F(r_i\cdot n))-\overline{\log(F)})(\log(r_i\cdot n)-\overline{\log(n)})}{\sum_{i=1}^m(\log(r_i\cdot n)-\overline{\log(n)})^2}.$$

Then $\widehat{H}$ is a consistant estimator of H such that

$$\mathbb{E}[(\widehat{H}-H)^2] \leq C(H,m,r_1,\dots,r_m)\frac{1}{[N/n]} \tag{9}$$

with $C(H,m) > 0$.

Remark 3.4: More precisely, it could be possible to show a central limit theorem for $\widehat{H}$, with a convergence rate of $\sqrt{[N/n]}$. Unfortunately, the proof of such a result requires the asymptotic development of $\mathrm{Cov}(\tilde{S}_i(n_k),\tilde{S}_j(n_\ell))$, which is more than complicated, for obtaining a multidimensional central limit theorem for $(\log(\tilde{S}(n_1)),\dots,\log(\tilde{S}(n_m))$.

IV. Extension of the results for a general class a long-range dependence process

Let $\{Y(k), k \in \mathbb{N}\}$ be a stationary zero mean long-range dependant process with Hurst parameter $H \in]\frac{1}{2},1[$. More precisely, let $r_Y(k)$ be the autocorrelation function of this process and assume that there exists a slowly varying function $L(k)$ such that :

$$r_Y(k) \sim k^{2H-2}L(k) \ \text{, as } k \to \infty. \tag{10}$$

Under different additional assumptions on Y, Davydov (1970), Taqqu (1975), Dobrushin and Major (1979), Giraitis and Surgailis (1989) and others authors have studied the asymptotic behavior of the Donsker line and obtained the following convergence,

$$\big(L(n)^{-\frac{1}{2}}n^{-H}\sum_{i=1}^{[nt]}Y(i)\big)_{t>0} \xrightarrow[n\to\infty]{\mathcal{D}} \big(\sigma\cdot B_H(t)\big)_{t>0}, \qquad (11)$$

with $\sigma > 0$ and B_H a fractional Brownian motion. More precisely,

Theorem 4.1: (Davydov, Taqqu, Dobrushin, Major, Giraitis and Surgailis) Let $Y = \{Y(k),\, k \in \mathbb{N}\}$ be a stationary zero mean long-range dependant process satisfying assumption (10). Then, if :

- Y is a linear process ($Y(k) = \sum_{i=-\infty}^{\infty} a_i\,\xi_{k-i}$ for $k \in \mathbb{N}$ with (a_k) a sequence of real numbers and (ξ_n) a sequence of zero mean i.i.d.r.v.) or a polynomial of a linear process,
- Y is a function of a Gaussian process with Hermite rank $r = 1$,

then (11) holds, and the convergence takes place in the Skorohod space.

In such a case, roughly speaking, the aggregated process $(X(k))$ has nearly the same behavior than a fractional Brownian motion and the previous asymptotic results of the DFA method can be applied. But propositions 3.1 and 3.3 can not be proved under so general assumptions. Indeed, the proofs of such results use a very precise expression of the covariance and a restricted version of assumption (10) is necessary. Hence, the covariance r_Y of the stationary process Y is now supposed to satisfy $r_Y \in \mathcal{H}(H,\beta,C)$ with

$$\mathcal{H}(H,\beta,C) = \Big\{r_Y,\ r_Y(k) = C\cdot k^{2H-2}\big(1+O(1/k^{\beta})\big) \text{ when } k\to\infty\Big\},$$

with $1/2 < H < 1$, $C > 0$ and $\beta > 0$. In such semi-parametric frame, the previous proofs are still valuable and :

Theorem 4.2: Let $Y = \{Y(k),\, k \in \mathbb{N}\}$ be a Gaussian stationary zero mean long-range dependant process with covariance $r_Y \in \mathcal{H}(H,\beta,C)$. Then, Property 3.1 holds with the addition of $O(1/n^{\beta})$ in each expansion. Moreover, if $N = o\big(n^{\max(2\beta+1,3)}\big)$, Property 3.2 and Proposition 3.3 hold.

As a consequence of this theorem, if $0 < \beta \le 1$, the DFA method provide a semi-parametric estimator of H with the well-known minimax rate of convergence for the Hurst parameter in this semi-parametric setting (see for instance Giraitis *et al.*, 1997), *i.e.*

$$\limsup_{N\to\infty}\ \sup_{r_Y\in\mathcal{H}(H,\beta,C)} N^{2\beta/(1+2\beta)}\mathbb{E}[(\widehat{H}-H)^2] < +\infty.$$

However, if $\beta \ge 1$, this result is replaced by $\limsup_{N\to\infty}\ \sup_{r_Y\in\mathcal{H}(H,\beta,C)} N^{2/3}\mathbb{E}[(\widehat{H}-H)^2] < +\infty$ (it is such a case of FGN or Gaussian FARIMA(p,d,q)).

V. Cases of particular trended long-range dependent processes

In this Section, two general examples of trended long-range dependent processes are considered and it is proved that DFA method in such cases provides biased and unusable estimation of the Hurst parameter.

Let $Y = \{Y(k),\, k \in \mathbb{N}\}$ be a Gaussian stationary zero mean long-range dependant process satisfying assumption (12) (for instance, Y is a FGN) and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a deterministic function. From Lemma 2.3, it is obvious that $n \in \{1,\cdots,N\}$,

$$\mathbb{E}(F^2_{Y+f}(n)) = \mathbb{E}(F^2_Y(n)) + \mathbb{E}(F^2_f(n)). \qquad (12)$$

Moreover, denote respectively $F^2_{Y,j}$ and $F^2_{f,j}$ the DFA function of Y and f relating to window $j \in \{1,\ldots,[\frac{N}{n}]\}$. Then, with few changes in the proof of Lemma 2.3,

$$\mathbb{E}(F^2_{Y+f,j}(n)) = \mathbb{E}(F^2_{Y,j}(n)) + \mathbb{E}(F^2_{f,j}(n)). \qquad (13)$$

Case of power law and polynomial trends

First, assume that it exists $\lambda > 0$ and $a \in \mathbb{R}$ such that

$$f(t) = a(t^{\lambda+1} - (t-1)^{\lambda+1}), \quad \text{for } t \ge 1.$$

Then, the associated integrated function is

$$g(k) = \sum_{i=1}^{k} f(i) = a k^{\lambda+1}.$$

For this kind of trend,

Property 5.1: For $f(t) = a(t^{\lambda+1} - (t-1)^{\lambda+1})$, with $\gamma(a,N,\lambda)$ a real number depending only on a, N and λ, $\log F_f(n) \simeq 2\log n + \gamma(a,N,\lambda)$ for $n \to \infty$.

Thus, it appears that a linear regression of $\log F_f(n_i)$ and $\log(n_i)$ for different values of n_i will provide a slope 2 for any $\lambda > 0$.

Proof: In the j-th window, with $j \in \{1,\ldots,[N/n]\}$, consider E_j the vector subspace de£ned above and de£ne the vector $G^{(j)} = a((1+n(j-1))^{\lambda+1},\ldots,(nj)^{\lambda+1})'$. We have

$$F^2_{f,j}(n) = \frac{1}{n}\left(G^{(j)'}\cdot G^{(j)} - G^{(j)'}\cdot P_{E_j}\cdot G^{(j)}\right)$$

An explicit asymptotic expansion (in n and N) of this partial DFA function can be obtained by approximating sums by integrals. Then,

$$F^2_{f,j}(n) = a^2 n^{2\lambda+2}\Big(1+O(\frac{1}{n})\Big)\Big(\int_0^1\int_0^1 (x+j-1)^{2\lambda+2} - (4-6(x+y)+12xy)(x+j-1)^{\lambda+1}(y+j-1)^{\lambda+1}dxdy\Big)$$

Moreover, using Taylor expansion in j up to order 3, one obtains

$$F^2_{f,j}(n) = \alpha(a,\lambda)\cdot n^{2\lambda+2}j^{2\lambda-2}\Big(1+O(\frac{1}{n})+O(\frac{1}{j})\Big), \qquad (14)$$

and it implies that the DFA function relating to f can be written like

$$F_f^2(n) = \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} F_{f,j}^2(n)$$
$$= \beta(a,\lambda) \cdot n^4 N^{2\lambda-2} \Big(1 + O(\frac{1}{n}) + O(\frac{1}{[\frac{N}{n}]})\Big),$$

with $\alpha(a,\lambda)$, $\beta(a,\lambda)$ two positive numbers depending only on a and λ. □

For illustrating this result (see Figure 3), several simulations have been made for various values of $\lambda > 0$, a and $(n_1,\dots,n_m)$. The presented results exhibit the relation between $\log F_f(n_i)$ and $\log(n_i)$, that is nearly linear with a slope of the adjustment linear line estimated at 2 like it was theoretically proved.

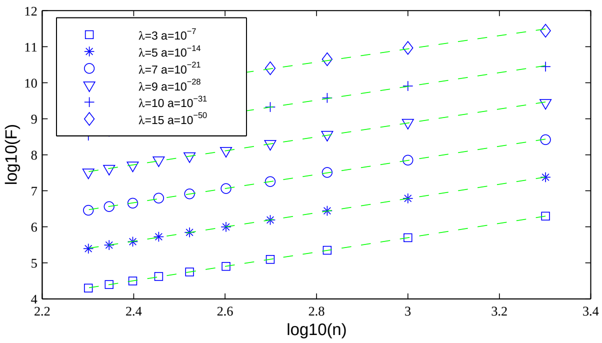

Fig. 3. Relation between $\log F_f(n_i)$ and $\log n_i$ in the case of power law trend

This result can be also used for deducing similar results for polynomial trends.

Property 5.2: Assume that it exists $p \in \mathbb{N}^*$ and a family $(a_j)_{0\le j\le p}$ with $a_p \neq 0$ such that for $k \in \mathbb{N}$, $f(k) = a_p k^p + \dots + a_0$. Then, $\Longrightarrow \log F_{a_pk^p+\dots+a_0}(n) \simeq 2\log n + \gamma(a_p, N)$ for $n \to \infty$.

Proof: Indeed, ,

$$f(k) = a_p k^p + \dots + a_0 \quad \Longrightarrow$$
$$g(k) = \sum_{i=1}^{k} f(i) = b_{p+1}k^{p+1} + \dots + b_0,$$

with $b_{p+1} \neq 0$, *i.e.* the associated integrated function is also a polynomial function. From the expression of the partial DFA function and with the asymptotic expansion (14) depending on the degree λ, for large enough n and N,

$$F^2_{a_pk^p+\dots+a_0,j}(n) = F^2_{a_pk^p,j}(n)\Big(1 + O(\frac{1}{n}) + O(\frac{1}{j})\Big)$$

(the degree of the partial DFA function of a_pk^p is greater than the others). This approximation leads to the following expression of the DFA function of a polynomial function,

$$F^2_{a_pk^p+\dots+a_0}(n) = \beta(b_{p+1}) \cdot n^4 N^{2\lambda-2}\Big(1 + O(\frac{1}{n}) + O(\frac{1}{[\frac{N}{n}]})\Big). \square$$

Using relations (12) and (13), the previous results for trends can be used for deducing the behavior of the DFA function of trended long range dependent processes. Hence, in both the previous cases of trends, it exists $C > 0$ such that

$$\mathbb{E}(F^2_{Y+f}(n)) = C \cdot n^4 N^{2\lambda-2}\Big(1 + O(\frac{1}{n}) + O(\frac{1}{[\frac{N}{n}]})\Big) + \sigma'^2 f(H) \cdot n^{2H}\Big(1 + O(\frac{1}{n^{\min(1,\beta)}})\Big) \simeq C \cdot n^4 N^{2\lambda-2}.$$

Hence, it is clear that the trend is dominant at large n and the graph tracing the relation between $\log F_{Y+f}(n_i)$ and $\log n_i$ for different power law trends and different coefficients H confirms this (the estimated slope is always close to 2).

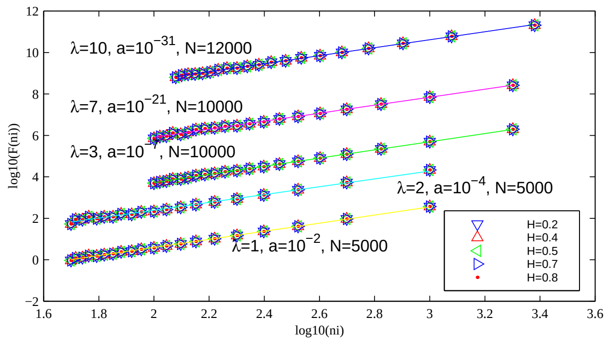

Fig. 4. Relation between $\log F_{Y+f}(n_i)$ and $\log n_i$ in the case of power law trend

Case of a piecewise constant trend

Assume now that f is a step function of the form $f(t) = \sum_{i=0}^{m-1} a_i \mathbb{1}_{]t_i,t_{i+1}]}$ with $t_0 = 0$, $t_m = N$ and $m \in \mathbb{N}^*$. The associated integrated series is

$$g(k) = \sum_{i=0}^{m-1}\Big(\sum_{s=0}^{i}(a_{s-1}-a_s)t_s + a_ik\Big)\mathbb{1}_{]t_i,t_{i+1}]} \text{ with } a_{-1} = 0$$

For $j \in \{1,\dots,[N/n]\}$, the partial DFA function $F^2_{fj}(n)$ is null except if there exist i_p with $p \in \{1,\dots,r\}$ and $(r, i_r) \in \{1,\dots,m-1\}^2$ such that $t_{i_p} \in [(j_p-1)n+\tau n, j_pn-\tau n]$ with $\tau \in]0, \frac{1}{2}[$. In such case, we calculate the partial DFA function:

$$F^2_{f,j_p}(n) = \frac{1}{n}\sum_{k=1}^{n}(g(k+(j_p-1)n) - \widehat{g}_n(k+(j_p-1)n))^2$$
$$= \frac{1}{n}\Big(G^{(j_p)'} \cdot P_{E^\perp_{j_p}} \cdot G^{(j_p)}\Big)$$

If we consider the first window, the partial DFA function can be undervalued by :

$$F^2_{f,1}(n) \geq \frac{1}{n}\Big(\sum_{k=1}^{\tau n}(g(k)-\widehat{g}_n(k))^2 + \sum_{k=n-\tau n}^{n}(g(k)-\widehat{g}_n(k))^2\Big)$$

where the $n \times 1$ vector $\big(g(k) - \widehat{g}_n(k)\big)_{1 \le k \le n} = P_{E_1^\perp} \cdot G^{(1)}$ with :

$$G^{(1)} = \big(a_0 \cdot 1, \ldots, a_0 \cdot t_1, (a_0 - a_1)t_1 + a_1 \cdot (t_1 + 1), \ldots, (a_0 - a_1)t_1 + a_1 \cdot n\big)'$$

Then,

$$\sum_{k=1}^{\tau n} (g(k) - \widehat{g}_n(k))^2 = \Big(J_{\tau n} \cdot P_{E_1^\perp} \cdot G^{(1)}\Big)' \cdot \Big(J_{\tau n} \cdot P_{E_1^\perp} \cdot G^{(1)}\Big)$$

where $J_{\tau n}$ is a square matrix of order n with ones in the τn first terms of the diagonal and zeros elsewhere. When we approximate sums by integrals, this expression can be written like :

$$\sum_{k=1}^{\tau n} (g(k) - \widehat{g}_n(k))^2 = n^3 \Big(\int_0^\tau \Big(\int_0^1 a_0 y - \big(a_0 x \cdot \mathbb{1}_{x \le \frac{t_1}{n}} + (a_1 x + (a_0 - a_1)\frac{t_1}{n})\mathbb{1}_{x > \frac{t_1}{n}}\big)(4 - 6(x+y) + 12xy)dx\Big)^2 dy\Big) \cdot \Big(1 + O(\frac{1}{n})\Big)$$

For $\tau \in]0, \frac{1}{2}[$, the second term can be developed in the same way while replacing $J_{\tau n}$ by $J_{n - \tau n}$ which is a square matrix of order n with ones in the $n - \tau n$ last terms of the diagonal and zeros elsewhere. Then, this term can be approximate by :

$$\sum_{k=n-\tau n}^{n} (g(k) - \widehat{g}_n(k))^2 = n^3 \Big(\int_{1-\tau}^1 \Big(\int_0^1 (a_0 - a_1)\frac{t_1}{n} + a_1 y - \big(a_0 x \cdot \mathbb{1}_{x \le \frac{t_1}{n}} + (a_1 x + (a_0 - a_1)\frac{t_1}{n})\mathbb{1}_{x > \frac{t_1}{n}}\big)(4 - 6(x+y) + 12xy)dx\Big)^2 dy\Big)\Big(1 + O(\frac{1}{n})\Big)$$

Then after the development of the two terms, we deduce that it exists a positive number $c(a_0, \ldots, a_{i_p}, t_{i_p}, \tau)$ such as the partial DFA function in the j_p-th window where $t_{i_p} \in [(j_p - 1)n + \tau n, j_p n - \tau n]$, for $p \in \{1, \ldots, r\}$, can be written, for n large enough, like :

$$F^2_{f, j_p}(n) \ge c(a_0, \ldots, a_{i_p}, t_{i_p}, \tau) n^2. \qquad (15)$$

Then if we suppose that it exists only one change point or a definite number of windows $j_1, \ldots, j_r$, it exists $c'(a_0, \ldots, a_{i_r}, t_{i_1}, \ldots, t_{i_r}, \tau) > 0$ such as the DFA function relating to f is :

$$F_f^2(n) = \frac{1}{[\frac{N}{n}]} \sum_{j=j_1}^{j_r} F^2{}_{f,j}(n) \ge c'(a_0, \ldots, a_{i_r}, t_{i_1}, \ldots, t_{i_r}, \tau) n^3 N^{-1} \Big(1 + O(\frac{1}{n})\Big)$$

Then (see Figure 5), for different values $(n_1, \ldots, n_m)$, the graph tracing the relation between $\log F_f(n_i)$ and $\log(n_i)$, shows a slope estimated at $\frac{3}{2}$.
If we consider the signal formed by the superposition between trend and a long range dependent process, we point out that $\mathbb{E}(F_Y^2(n)) = \sigma'^2 f(H) \cdot n^{2H}\Big(1 + O(\frac{1}{n^{\min(1,\beta)}})\Big)$, we can deduce, according to the previous conditions on n and N ($N/n \to \infty$ and $N = o(n^{\min 3, 2\beta+1})$, that the trend is dominant for large n.

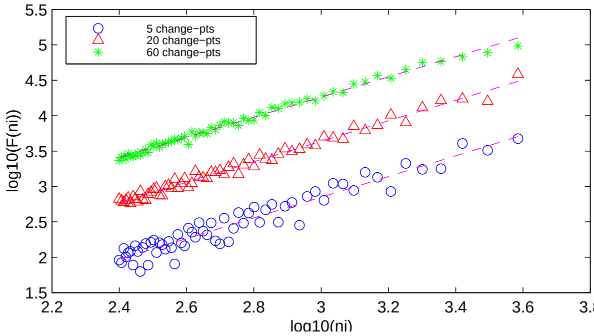

Fig. 5. Relation between $\log F_f(n_i)$ and $\log n_i$ in the case of trend with change points

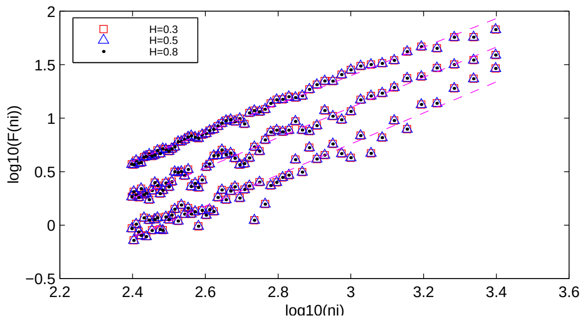

Fig. 6. Relation between $\log F_{f+Y}(n_i)$ and $\log n_i$ in the case of trend with change points

VI. Conclusion

In the semi-parametric frame of long memory stationary process, we showed, using the DFA method, that the estimator of the long range dependance parameter is convergent with a reasonable convergence rate. However, in numerous cases of trended long range dependent process (with perhaps the only exception of a constant trend), this estimator does not converge. The DFA method is therefore not all a robust method and should not be applied for trended processes. In the case of polynomial, the wavelet based method is method provides a better estimator of the Hurst parameter, with appropriated number of vanishing wavelet moments (see for instance Abry *et al.*, 1998 or Veitch and Abry, 1999).

Appendix I

Proof of Property 3.1: 1. From the proof of Lemma 2.2 and with its notations, one obtains

$$F_1^2(n) = \frac{1}{n}(X^{(1)} - P_{E_1} \cdot X^{(1)})' \cdot (X^{(1)} - P_{E_1} \cdot X^{(1)}) = \frac{1}{n}\Big(X^{(1)\prime} \cdot X^{(1)} - X^{(1)\prime} \cdot P_{E_1} \cdot X^{(1)}\Big).$$

As a consequence,

$$\mathbb{E}(F_1^2(n)) = \frac{1}{n}\Big(\text{trace}(\Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n)\Big),$$

where Σ_n is the covariance matrix of $X^{(1)}$ and is such that

$$\Sigma_n = \text{Cov}(X_i, X_j)_{1 \le i,j \le n} = \frac{\sigma^2}{2}\big(|i|^{2H} + |j|^{2H} - |i-j|^{2H}\big)_{1 \le i,j \le n}$$

But, $\text{trace}(\Sigma_n) = \sigma^2 \sum_{i=1}^n |i|^{2H} = \sigma^2 n^{2H+1} \big(\frac{1}{n}\sum_{i=1}^n \big|\frac{i}{n}\big|^{2H}\big)$ $= \sigma^2 n^{2H+1} \big(\int_0^1 x^{2H} dx + O(\frac{1}{n})\big)$. Therefore, in one hand,

$$\text{trace}(\Sigma_n) = \frac{\sigma^2}{2H+1}\, n^{2H+1} \cdot \Big(1 + O(\frac{1}{n})\Big). \qquad (16)$$

In the other hand, it is well known that P_{E_1} is a $(n\times n)$ square matrix such that

$$P_{E_1} = \frac{2}{n(n-1)}\Big((2n+1) - 3(i+j) + 6\,\frac{i\cdot j}{1+n}\Big)_{1\le i,j\le n}$$

Then, after some straightforward computations, we obtain the formula

$$\text{trace}(P_{E_1}\cdot \Sigma_n) = \frac{\sigma^2 n^{2H+1} n^2}{n(n-1)} \sum_{p=1}^n \sum_{q=1}^n \Big[\frac{1}{n^2}\Big((2+\frac{1}{n}) - 3\cdot \frac{p+q}{n} + \frac{6p\cdot q}{n(1+n)}\Big)\Big(\big|\frac{q}{n}\big|^{2H} + \big|\frac{p}{n}\big|^{2H} - \big|\frac{q-p}{n}\big|^{2H}\Big)\Big]$$

In order to clarify the formula, we approximate these sums by integrals

$$\text{trace}(P_{E_1}\cdot \Sigma_n) = \sigma^2 n^{2H+1}\cdot \Big(1+O(\frac{1}{n})\Big)\cdot \int_0^1\!\!\int_0^1 \Big[\big(2-3(x+y)+6xy\big)\big(x^{2H}+y^{2H}-|x-y|^{2H}\big)\Big]dxdy$$

After the calculation of this integral and a simpli£cation with formula (16), we get the result

$$\text{trace}(\Sigma_n) - \text{trace}(P_{E_1}\cdot\Sigma_n) = \sigma^2 f(H)\cdot n^{2H+1}\cdot \Big(1+O(\frac{1}{n})\Big)$$

and therefore the formula of $\mathbb{E}(F_1^2(n))$.

2. From the previous notations and the property of the trace of a product of matrix,

$$\text{Var}(F_1^2(n)) = \frac{1}{n^2}\Big[\mathbb{E}(X^{(1)'}\cdot P_{E_1^\perp}\cdot X^{(1)}\cdot X^{(1)'}\cdot P_{E_1^\perp}\cdot X^{(1)}) - \Big(\mathbb{E}(X^{(1)'}\cdot P_{E_1^\perp}\cdot X^{(1)})\Big)^2\Big]$$

$$= \frac{1}{n^2}\Big[\text{trace}(\Sigma_n\cdot\Sigma_n) - \text{trace}(P_{E_1}\cdot\Sigma_n\cdot\Sigma_n)\Big] \quad (17)$$

The development of the £rst term provides the following asymptotic expansion

$$\text{trace}(\Sigma_n\cdot\Sigma_n) = \frac{\sigma^4}{4}\sum_{i=1}^n\sum_{p=1}^n \big(|i|^{2H}+|p|^{2H}-|i-p|^{2H}\big)^2 = \frac{\sigma^4}{4} n^{4H+2}\Big(1+O(\frac{1}{n})\Big)\int_0^1\!\!\int_0^1 \big(|x|^{2H}+|y|^{2H}-|x-y|^{2H}\big)^2 dxdy$$

The calculation of this integrals provides the following simpli£ed expression

$$\text{trace}(\Sigma_n\cdot\Sigma_n) = \frac{\sigma^4}{4}\, n^{4H+2}\Big(1+O(\frac{1}{n})\Big)\cdot \Big[\frac{1}{4H+1} + \frac{1}{(4H+1)(4H+2)} - 2\frac{(\Gamma(2H+1))^2}{\Gamma(4H+3)}\Big] \quad (18)$$

The same development can be done for the second term

$$\text{trace}(P_{E_1}\cdot\Sigma_n\cdot\Sigma_n) = \frac{\sigma^4}{2}\, n^{4H+2}\Big(1+O(\frac{1}{n})\Big)\cdot \int_0^1\!\!\int_0^1\!\!\int_0^1 \Big(|x|^{2H}+|y|^{2H}-|y-x|^{2H}\Big)\Big(|x|^{2H}+|z|^{2H}-|x-z|^{2H}\Big)\cdot\Big(2-3(y+z)+6yz\Big)\,dx\,dy\,dz$$

After the computation of this last integral, and using relations (17) and (18)

$$\Big[\text{trace}(\Sigma_n\cdot\Sigma_n) - \text{trace}(P_{E_1}\cdot\Sigma_n\cdot\Sigma_n)\Big] = \sigma^4\cdot g(H)\, n^{4H+2}\Big(1+O(\frac{1}{n})\Big)$$

with, $g(H) = \frac{1}{2}\Big(-\frac{(16H^2+24H+17)(\Gamma(2H+1))^2}{(4H+5)\Gamma(4H+4)} + \frac{H+1}{(2H+1)(4H+1)} + \frac{7H+3}{2(2H+1)^2(H+1)} - \frac{3}{2(H+1)^2} + \frac{3(4H+3)}{2(2H+1)^2(H+1)^2(4H+5)} - \frac{4}{(2H+1)^2(4H+3)}\Big)$.

Then, using the relations (17), one obtains $\text{Var}\big(F_1^2(n)\big) = \sigma^4\cdot g(H)\cdot n^{4H}\Big(1+O(\frac{1}{n})\Big)$.

3. An asymptotic expansion of the covariance between two DFA functions in two suf£ciently far windows can be provided. Indeed

$$\text{Cov}(F_1^2(n), F_j^2(n)) = \frac{1}{n^2}\,\text{Cov}\Big((X^{(1)}-\widehat{X}^{(1)})'\cdot(X^{(1)}-\widehat{X}^{(1)}), (X^{(j)}-\widehat{X}^{(j)})'\cdot(X^{(j)}-\widehat{X}^{(j)})\Big)$$

$$= \frac{1}{n^2}\,\Big(\text{trace}\Big(\Sigma^{(1,j)}\cdot\Sigma^{(1,j)}\Big) - \text{trace}\Big(P_{E_1}\cdot\Sigma^{(1,j)}\cdot\Sigma^{(1,j)}\Big)\Big),$$

because $P_{E_1^\perp} = P_{E_j^\perp}$ and with $\Sigma^{(1,j)}$ the covariance matrix $\mathbb{E}(X^{(1)}\cdot X^{(j)},) = (\sigma^{(1,j)}_{k,k'})_{1\le k,k'\le n}$. As usual, this formula can be developed

$$\text{Cov}(F_1^2(n), F_j^2(n)) = \frac{1}{n^2}\Big(\sum_{k=1}^n\sum_{k'=1}^n \sigma^{(1,j)}_{k,k'}\cdot\sigma^{(1,j)}_{k',k} - \sum_{i=1}^n\sum_{k'=1}^n\sum_{k=1}^n p_{i,k}\cdot\sigma^{(1,j)}_{k,k'}\cdot\sigma^{(1,j)}_{k',i}\Big),$$

with

$$\sigma^{(1,j)}_{k,k'} = \frac{\sigma^2}{2}\big(|k+nj|^{2H} + |k'|^{2H} - |k-k'+nj|^{2H}\big)_{1\le k,k'\le n}$$

and with $P_{E_1} = (p_{i,j})_{1\le i,j\le n}$ such that

$$p_{i,j} = \frac{2}{n(n-1)}\Big((2n+1) - 3(i+j) + 6\,\frac{i\cdot j}{1+n}\Big).$$

Now, one considers the asymptotic expansion of this formula when n is large enough

$$\text{Cov}(F_1^2(n), F_j^2(n)) = \frac{\sigma^4}{4}\, n^{4H}\Big(1+O(\frac{1}{n})\Big)\Big(\int_0^1\!\!\int_0^1 \big(|x+j|^{2H} + y^{2H} - |x-y+j|^{2H}\big)\big(|y+j|^{2H} + x^{2H} - |y-x+j|^{2H}\big)dxdy - \int_0^1\!\!\int_0^1\!\!\int_0^1 (4-6(x+z)+12xz)\big(|x+j|^{2H} + y^{2H} - |x-y+j|^{2H}\big)\big(|y+j|^{2H} + z^{2H} - |y-z+j|^{2H}\big)\,dx\,dy\,dz\Big)$$

For obtaining an asymptotic expansion of this formula when j is large enough (*i.e.* both windows are taken away one of the other one), a Taylor expansion in j up to order 3 is necessary. After calculation of integrals and simpli£cation, we get the result. $\square$

Proof of Property 3.2: One divides the proof in 3 steps:

• Step 1: one proves that $[N/n]\cdot \mathrm{Var}(\tilde S(n)) \to \gamma^2(H)$, where $\gamma^2(H)$ depends only on H, when $[N/n]\to\infty$. Indeed,

$$\mathrm{Var}(\tilde S(n)) = \frac{1}{[\frac{N}{n}]^2}\sum_{j=1}^{[\frac{N}{n}]}\sum_{j'=1}^{[\frac{N}{n}]}\mathrm{Cov}(\tilde S_j(n),\tilde S_{j'}(n))$$

$$= \frac{1}{[\frac{N}{n}]}\mathrm{Var}(\tilde S_j(n)) + \frac{2}{[\frac{N}{n}]^2}\sum_{j=1}^{[\frac{N}{n}]}\big([\frac{N}{n}]-j\big)\mathrm{Cov}(\tilde S_1(n),\tilde S_j(n))$$

from the stationarity.

However, with properties (8), one deduces that when $[N/n]\to\infty$, $\sum_{j=1}^{[\frac{N}{n}]}\mathrm{Cov}(\tilde S_1(n),\tilde S_j(n))$ and $\sum_{j=1}^{[\frac{N}{n}]} j\cdot\mathrm{Cov}(\tilde S_1(n),\tilde S_j(n))$ converge, because it exists $C \geq 0$ such that $\big|\mathrm{Cov}(\tilde S_1(n),\tilde S_j(n))\big|\leq C\cdot j^{2H-3}$ and $0<H<1$.

Therefore, it exists $\gamma^2(H)$ depending only on H such that

$$\lim_{[N/n]\to\infty}[N/n]\cdot\mathrm{Var}(\tilde S(n)) = \gamma^2(H). \qquad (19)$$

• Step 2: the proof of a central limit theorem for $\tilde S(n)$ when $[N/n]\to\infty$ can be obtained from the same method than in the proof of the Proposition 2.1 of Bardet (2000) (the Theorem 3 of Soulier, 2000, leads to the same result).

Indeed, $\tilde S(n) = \dfrac{1}{n^{2H+1}\sigma^2 f(H)\cdot[N/n]}\displaystyle\sum_{i=1}^{n\cdot[N/n]} Z_i^2$, where the zero-mean Gaussian vector $Z=(Z_1,\dots,Z_{n\cdot[N/n]})$ has the covariance matrix $P\cdot\Sigma\cdot P$, where P is a diagonal block matrix with each block constituted with (n,n) matrix $P_{E_1^\perp}$ and Σ is the covariance matrix of a FBM times series (each (n,n) block is $\Sigma^{(i,j)}$ with the previous notations). Using a Lindeberg condition, $\tilde S(n)$ satis£es the following central limit theorem

$$\sqrt{[N/n]}\cdot\Big(\tilde S(n)-\mathbb{E}(S(n))\Big)\ \xrightarrow[{[N/n]\to\infty}]{\mathcal{L}}\ \mathcal{N}(0,\gamma^2(H)), \qquad (20)$$

if $\lambda = \|P\cdot\Sigma\cdot P\|$, the supremum of the eigenvalues of the symmetric matrix $P\cdot\Sigma\cdot P$, is such that

$$\lambda = o\Big(\frac{1}{\sqrt{[N/n]}}\Big). \qquad (21)$$

But, following the proof of the Proposition 2.1 of Bardet (2000),

$$\lambda \leq \frac{1}{[N/n]}\max_{i\in\{1,\dots,[N/n]\}}\left(\sum_{j=1}^{[N/n]}\sqrt{\mathrm{Cov}\big(\tilde S_i(n),\tilde S_j(n)\big)}\right)$$

$$\lambda \leq \frac{1}{[N/n]}\left(\sum_{j=1}^{[N/n]}\sqrt{\mathrm{Cov}\big(\tilde S_1(n),\tilde S_j(n)\big)}\right)$$

$$\leq C(H)\cdot\frac{1}{[N/n]}\sum_{j=1}^{[N/n]}\sqrt{j^{2H-3}} \qquad \text{third line of (8)}$$

$$\leq C(H)\cdot[N/n]^{H-3/2}.$$

Therefore (21) and (20) are proved .

• Step 3: Now, $\mathbb{E}\big(\tilde S(n)\big) = 1+O\big(\frac{1}{n}\big)$ for n large enough. Then, if $\sqrt{[N/n]}\cdot\frac{1}{n}\to 0$, that is $N/n^3\to 0$,

$$\sqrt{[N/n]}\cdot\Big(\tilde S(n)-1\Big)\ \xrightarrow[{[N/n]\to\infty}]{\mathcal{L}}\ \mathcal{N}(0,\gamma^2(H))).$$

The classical Delta method allows the passage between a central limit theorem for $\tilde S(n)$ and central limit theorem for $\log(\tilde S(n))$ (thanks to the regularity properties of the function logarithm). $\square$

Proof of Proposition 3.3: It is possible to write $\widehat H = (1,0)\cdot(Z'\cdot Z)^{-1}\cdot Z'\cdot F$, where Z is the $(n,2)$ matrix such that $P_{E_1^\perp} = Z\cdot(Z'\cdot Z)^{-1}\cdot Z'$. and $F=(\log(F(n_1)),\dots,\log(F(n_m)))'$. Then,

$$\begin{aligned}&\mathrm{Var}(\widehat H)\\ &=(1,0)\cdot(Z'\cdot Z)^{-1}\cdot Z'\cdot\mathrm{Cov}(F,F)\cdot Z\cdot(Z'\cdot Z)^{-1}\cdot(1,0)'\\ &\leq\|(1,0)\cdot(Z'\cdot Z)^{-1}\cdot Z'\|^2\cdot\|\mathrm{Cov}(F,F)\|^2\\ &\leq\|(1,0)\cdot(Z'\cdot Z)^{-1}\cdot Z'\|^2\cdot 2m\cdot\gamma^2(H).\end{aligned}$$

Like, $\|(1,0)\cdot(Z'\cdot Z)^{-1}\cdot Z'\|$ only depends on $r_1,\cdot,r_n$, the proof of Proposition 3.3 is completed. $\square$

Proof of Theorem 4.2: From the assumptions on Y and r_Y, if $i\geq j\geq 1$,

$$\mathrm{Cov}(X_i,X_j) = \sum_{k=1}^{i}\sum_{\ell=1}^{j}\mathrm{Cov}(Y_k,Y_\ell)$$

$$=\sum_{k=1}^{i}(i-k)r_Y(k)+\sum_{k=1}^{j}(j-k)r_Y(k)-\sum_{k=1}^{i-j}(i-j-k)r_Y(k).$$

As a consequence, for all $(i,j)\in\{1,\dots,n\}^2$,

$$\mathrm{Cov}(X_i,X_j) = C\cdot\Big(\int_0^1(1-u)u^{2H-2}du\Big)\cdot$$
$$\Big(i^{2H}\big(1+O(\frac{1}{i^{\min(\beta,1)}})\big)+j^{2H}\big(1+O(\frac{1}{j^{\min(\beta,1)}})\big)-|i-j|^{2H}$$
$$.\qquad \big(1+O(\frac{1}{(1+|i-j|)^{\min(\beta,1)}})\big)\Big)$$

Now, this covariance can be used in every place of the proofs, replacing the previous one. This implies

$$
\begin{aligned}
1.\ \mathbb{E}(F_1^2(n)) &= \sigma'^2 f(H)\cdot n^{2H}\Big(1+O\big(\frac{1}{n^{\min(\beta,1)}}\big)\Big),\\
2.\ \mathrm{Var}\big(F_1^2(n)\big) &= \sigma'^4 g(H)\cdot n^{4H}\Big(1+O\big(\frac{1}{n^{\min(\beta,1)}}\big)\Big),\\
3.\ \mathrm{Cov}(F_1^2(n),F_j^2(n)) &= \sigma'^4 h(H)\cdot n^{4H}\cdot j^{2H-3}\Big(1+\\
&\qquad +O\big(\frac{1}{n^{\min(\beta,1)}}\big)+O\big(\frac{1}{j}\big)\Big),
\end{aligned}
$$

with $\sigma'^2 = 2C\cdot\Big(\int_0^1 (1-u)u^{2H-2}du\Big)$. The proofs of property 3.2 and proposition 3.3 are the same than in the case of FGN. □

REFERENCES

[1] Abry, P., Veitch, D. and Flandrin, P. *Long-range dependence: revisiting aggregation with wavelets.* J. Time Ser. Anal. 19, no. 3, 253-266, 1998.

[2] Absil P.A., Sepulchre R., Bilge A., Gérard P., *Nonlinear analysis of cardiac rhythm ¤uctuations using DFA method*, J. Physica A : Statistical mechanics and its applications, 235-244, 1999.

[3] Arcones M., *Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors.*, Ann. Probab. 22, 2242-2274, 1994.

[4] Bardet J.M., *Statistical study of the wavelet analysis of fractional Brownian motion*, IEEE Trans. Inform. Theory. 48, 991-999, 2002.

[5] Bardet J.M., Lang, G., Oppenheim, G., Philippe, A. and Taqqu, M.S. *Generators of long-range dependent processes : A survey.* In *Long-range Dependence : Theory and Applications*, Birkhauser (2002).

[6] Chen Z., Ivanov P.C., Hu K., Stanley H.E, *Effect of nonstationarities on detrended ¤uctuation analysis*, Physical Review E, Vol. 65, 041107, 2002.

[7] Delignières D., *L'analyse des processus stochastiques*, "Sport, Performance, Santé", EA 2991, Université Montpellier I, janvier 2001.

[8] Gao J.B., Cao Y., Lee J.M, *Principal component analysis of* $1/\mathrm{f}^\alpha$ *noise*, Physics Letters, A 314, 392-400, 2003.

[9] Giraitis, L., Robinson, P. and Samarov, A., *Rate optimal semi-parametric estimation of the memory parameter of the Gaussian time series with long range dependence*, J. Time Ser. Anal., 18, 49-61, 1997.

[10] Hu K., Ivanov P.C., Chen Z., Carpena P., Stanley H.E, *Effect of trends on detrended ¤uctuation analysis*, Physical Review E, Vol. 64, 011114, 2001.

[11] Hwa R.C., Ferree T.C.,*Fluctuation analysis of human electroencephalogram*, Nonlinear phenomena in complex systems, 302-307, 2002.

[12] Kantelhardt J.W., Ashkenazy Y., Ivanov P.C., Bunde A., Havlin S., Penzel T., Peter J.H., Stanley H.E., *Characterization of sleep stages by correlations in the magnitude and sign of heartbeat increments*, Physical Review E, Vol. 65, 051908, 2002.

[13] Kantelhardt J.W., Koscielny-Bunde E., Rego H.A.H., Havlin S., Bunde A., *Detecting Long- range Correlations with Detrended Fluctuation Analysis*, Physica A, 295, 441-454, 2001.

[14] Karasik R., Sapir N., Ashkenazy Y., Ivanov P.C., Dvir I., Lavie P., Havlin S., *Correlation differences in heartbeat ¤uctuations during rest and exercise*, Physical Review E 66, 062902, 2002.

[15] Martinis M., Knezevic A., Krstacic G., Vargovic E., *Changes in the Hurst exponent of heartbeat intervals during physical activities*, Physics 0212029, 2002.

[16] Masugi, M., *Detrended ¤uctuation analysis of IP-network traf£c using a two-dimensional topology map*, Phys. A 337, no. 3-4, 664–678, 2004.

[17] Nagarajan R., Kavasseri R.G, *Minimizing the effect of sinusoidal trends in detrended ¤uctuation analysis*, (in press) International Journal of Bifurcations and Chaos, Vol. 15, No. 2, 2005.

[18] Peng C.K., Buldyrev S.V., Havlin S., Simons M., Stanley H.E., Goldberger A.L., *Mosaic organization of DNA nucleotides*, Physical Review E, Vol. 49, 1685-1689, 1994.

[19] Peng C.K., Hausdorff J.M., Goldberger A.L., *Fractal mechanisms in neural control: Human heartbeat and gait dynamics in health and disease*. In: Walleczek J, ed. Nonlinear Dynamics, Self-Organization, and Biomedicine. Cambridge University Press, 1999.

[20] Peng C.K., Havlin S., Stanley H.E., Goldberger A.L., *Quanti£cation of scaling exponents and crossover phenomena in nonstationary heartbeat time series*, Chaos 5, 82, 1995.

[21] Samorodnitsky, G. and Taqqu M.S., *Stable non-Gaussian Random Processes*, Chapman and Hall, 1994.

[22] Soulier, P., *Moment bounds and central limit theorem for functions of Gaussian vectors*, Statist. Probab. Lett. 54, 193-203, 2001.

[23] Taqqu, M.S., *Weak Convergence to Fractional Brownian Motion and to Rosenblatt Process*, Zeit. Wahr. verw. Geb., 31, 287-302, 1975.

[24] Taqqu M.S., Teverovsky V., Willinger W., *Estimators for long-range dependence: an empirical study*, Fractals, Vol. 3, No. 4, 785-788, 1995.

[25] Veitch, D., Abry, P., *A wavelet-based joint estimator of the parameters of long-range dependence.* IEEE Trans. Inform. Theory, Vol. 45, No. 3, 878–897, 1999