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Log-regularized periodogram regression

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Abstract

We present an estimation method based on log-periodogram regression for general non fractional stationary processes. The consistency and the asymptotic normality of the estimators are established, provided that the periodogram is regularized. The results are extended to the non gaussian case and compared with Whittle-type estimates. Simulation experiments illustrate our results.

Keywords: Periodogram, Regression, Whittle.

1 Introduction

Among methods for estimation of stationary processes, the well-known procedure of Whittle leads to strong consistent, asymptotically normal estimator. Besides, it is asymptotically efficient in the Gaussian case. This procedure has been adapted to strongly dependent stationary time series by Fox and Taqqu (1986) and Dahlhaus (1989). As an alternative to the Whittle (pseudo) likelihood, Taniguchi (1979) defines the D_2 measure, equal to the integral of the squared difference between the logarithms of two densities. Then, the estimators are the values that minimize the distance between the parametric density and the periodogram. He proves the consistency, asymptotic normality and some robustness properties of these estimates, under summability conditions on the autocovariance sequence.

We consider here another method based on regression on log-regularized periodogram. We differ from Taniguchi (1979) in that we use here a discrete average of the log periodogram instead of a continuous integral.

Kashyap and Eom (1988), Geweke and Porter-Hudak (1983), have proposed in the fractional framework an estimation by regression on the log periodogram. Their results have been confirmed by Hassler (1993) in the Gaussian non long memory case ; but the obtained estimates are not efficient. We have adapted this idea for an application to non fractional models. Besides, we regularize the periodogram in order to obtain efficiency. The advantage is that the specification of the model can be less precise; in particular, the exponent parameter if any can be estimated alone. Besides, optimizations on the Whittle contrast are often computationally very slow, whereas this method is immediately faster, as soon as there is an exponent parameter.

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We do not consider here fractional models, as ARIMA(p,d,q) or GARMA processes studied by Gray, Zhang and Woodward (1989), for which the spectral density may be unbounded or have zeros. But the models we study may nevertheless involve an exponent parameter: think for instance of the model $(1-\alpha B)^d X_t = \varepsilon_t, |\alpha| < 1, |d| < \frac{1}{2}$.

The model is the following:

$$X_t = A(B)\,\varepsilon_t,\quad (\varepsilon_t) \text{ i.i.d. } (0,1),\ A(B) = \sum_{j\geq 0} a_j B^j,\ \ a_0 = 1,\ \ \sum_{j\geq 0} a_j^2 < \infty. \tag{1}$$

observed on $t = 1, \cdots, n$, where B is the usual lag operator $(BX_t = X_{t-1})$. (1) can be rewritten: $X_t = \sum_{j\geq 0} a_j \varepsilon_{t-j}$.

Let $X_n(\lambda) = \sum_{t=1}^{n} X_t\ e^{it\lambda}, \varepsilon_n(\lambda) = \sum_{t=1}^{n} \varepsilon_t\ e^{it\lambda}$ be the discrete Fourier transforms of X and ε observed on $\{1,\ldots,n\}$, and $I_{nX}(\lambda) = \frac{1}{n}|X_n(\lambda)|^2,\quad I_{n\varepsilon}(\lambda) = \frac{1}{n}|\varepsilon_n(\lambda)|^2$ the periodograms of X and ε, $J_{nX}(\lambda) = |A(e^{i\lambda})|^2\ I_{n\varepsilon}(\lambda) = 2\pi F(\lambda) I_{n\varepsilon}(\lambda)$ that we call the pseudo-periodogram of X, where F is the spectral density of X.

We have:

$$\ln J_{nX}(\lambda) = \ln 2\pi F(\lambda)\ +\ \ln I_{n\varepsilon}(\lambda) \tag{2}$$

The procedure "estimates" the parameters by considering regression (2) at frequencies $\lambda_k = \frac{2\pi k}{n},\ k = 0, \ldots \frac{n}{2}$. This is the idea proposed by Kashyap and Eom (1988) for an ARIMA$(0,d,0)$.

Section 2 studies the estimation of parametric Gaussian models (1) based first on the pseudo-periodogram. In order to keep efficiency results, we replace the pseudo-periodogram J_{nX} by its "regularized" version and consider regression (2) consequently modified. We recall results about the approximation of I_{nX} by J_{nX}; we can then study the regression procedure based on the true periodogram. This second stage is important because J_{nX} is unobservable. The proofs of the results are gathered at the end of the section. Then section 3 studies the comparison of the log periodogram regression and the Whittle estimators, under the Gaussian assumption. Section 4 gives extension of the convergence and asymptotic normality results in the non Gaussian case. Lastly, section 5 gives some simulation results.

2 Gaussian regression

2.1 Pseudo Periodogram Regression

We consider the model (1) with

(H$_\varepsilon$) ε i.i.d. $\mathcal{N}(0,1)$,

(H$_X$) the spectral density parametrization $\theta \to F(\theta, .)$ is injective, $\theta \in \Theta$, the compact set of $\mathbb{R}^p$ of the parameters and $\int_{-\pi}^{\pi} \ln F(\theta_0, \lambda) d\lambda = 0$ where θ_0 is the true value of the parameter.

We consider Fourier frequencies $\lambda_k = \frac{2\pi k}{n}$, where $1 \leq k \leq n/2$. Assume $n = mK$ (m even) and let us divide n in m blocks, each of length K. Therefore, $\{1, \cdots, \frac{n}{2}\} = \bigcup_{l=0}^{\frac{m}{2}-1} \mathcal{I}_l$, with $\mathcal{I}_l = \{lK+1, \cdots, (l+1)K\}$.
The midpoint of $\mathcal{I}_l$ is $lK + \frac{K+1}{2}$ and we denote $\tilde{F}(\theta, l) = F(\theta, \lambda_{lK+\frac{K+1}{2}})$. We have:

$$J_{nX}(\lambda_k) = 2\pi \tilde{F}(\theta, l) I_{n\epsilon}(\lambda_k) + 2\pi (F(\theta, \lambda_k) - \tilde{F}(\theta, l)) I_{n\epsilon}(\lambda_k) \tag{3}$$

Then we average (3) on the frequencies of $\mathcal{I}_l$, and take the logarithm. Let $\bar{J}_{nX}(l) = \frac{1}{K} \sum_{k \in \mathcal{I}_l} J_{nX}(\lambda_k)$ and $\bar{I}_{n\epsilon}(l) = \frac{1}{K} \sum_{k \in \mathcal{I}_l} I_{n\epsilon}(\lambda_k)$. The regression equation is:

$$\ln \bar{J}_{nX}(l) = \tilde{g}(\theta, l) + W_l + \ln(1 + F_l), \quad l \in A_m \tag{4}$$

with $\tilde{g}(\theta, l) = \ln 2\pi \tilde{F}(\theta, l)$, $W_l = \ln \bar{I}_{n\epsilon}(l)$, $F_l = \frac{\Delta_l}{\tilde{F}(\theta, l) \bar{I}_{n\epsilon}(l)}$, $\Delta_l = \frac{1}{K} \sum_{k \in \mathcal{I}_l} (F(\theta, \lambda_k) - \tilde{F}(\theta, l)) I_{n\epsilon}(\lambda_k)$, $A_m = \{l; l = 0, \ldots, \frac{m}{2} - 1\}$.
Assuming the two following conditions,

(C_1): For all $x \in [0, \pi]$ and $\theta \in \Theta$, F is differentiable w.r.t. x and of class C^1 (i.e. differentiable with continuous derivative) w.r.t. θ and there exist b and B such that:

$$F(\theta, x) \geq b > 0, \quad |\frac{\partial F}{\partial x}(\theta, x)| \leq B < \infty \quad \text{and} \quad \|\frac{\partial F}{\partial \theta}(\theta, x)\| \leq B < \infty$$

where $\|.\|$ denotes the Euclidian norm on $\mathbb{R}^p$.

$$(C_2): \frac{\sqrt{K}}{m} \to 0 \quad \textit{and} \quad \frac{m}{K} \to 0 \,; K \geq 9$$

we obtain consistency and asymptotic normality for the least square estimator $\hat{\theta}_n$ derived from (4).

Theorem 1 *Assume (H_ϵ), (H_X). Under (C_1) and (C_2), the least square estimator $\hat{\theta}_n$ is strongly consistent.*

Theorem 2 *Under the assumptions of Theorem 1, and if F is of class C^2 w.r.t. θ,*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \underset{n \to \infty}{\longrightarrow} \mathcal{N}(0, I(\theta_0)^{-1})$$

where $I(\theta) = \frac{1}{2\pi} \int_0^\pi \frac{\partial}{\partial \theta} g(\theta, \lambda) \frac{\partial}{\partial \theta} g(\theta, \lambda)' \, d\lambda$, where $g(\theta, x) = \ln F(\theta, x)$, and u' denotes the transpose of u.

Remark: The asymptotic variance with J instead of $\bar{J}$ would be $(\pi^2/6) I(\theta_0)^{-1}$ because of the asymptotic variance of the $I_\epsilon(\lambda_k)$ (i.i.d, $\mathcal{E}xp(1)$). This is the reason for the regularization.

2.2 True Periodogram regression

Let $\delta_n(\lambda) = I_{nX}(\lambda) - J_{nX}(\lambda) = \frac{1}{n}\left(\left|\sum_{t=1}^{n} X_t e^{i\lambda t}\right|^2 - |A(e^{i\lambda})|^2 \left|\sum_{t=1}^{n} \varepsilon_t e^{i\lambda t}\right|^2\right)$. We first recall a result proved by Priestley (1981), Theorem 6.2.2 p424:

$$\begin{cases} \textit{Let } b \textit{ be a positive constant.} \\ \textit{If } \sum_j j^b |a_j| < +\infty, \textit{ then } I\!E|\delta_n(\lambda)|^2 = O(\frac{1}{n^{2b}}), \textit{ uniformly in } \lambda. \end{cases}$$

Let: $\bar{\delta}_n(l) = \frac{1}{K}\sum_{k \in \mathcal{I}_l} \delta_n(\lambda_k)$. Then: $\bar{I}_{nX}(l) = \bar{J}_{nX}(l) + \bar{\delta}_n(l)$. As $\bar{I}_{nX}(l) \geq 0$, $\bar{J}_{nX}(l) \geq 0$, we can take a log transform and write: $\ln(\bar{I}_{nX}(l)) = \ln(\bar{J}_{nX}(l)) + \ln\left(1 + \frac{\bar{\delta}_n(l)}{\bar{J}_{nX}(l)}\right)$.

Then we find the new regression equation:

$$\ln \bar{I}_{nX}(l) = \tilde{g}(\theta, l) + W_l' + U_l, \quad l \in A_m \tag{5}$$

with still $\tilde{g}(\theta, l) = \ln 2\pi \tilde{F}(\theta, l)$, $W_l' = \ln \bar{I}_{n\varepsilon}(l) + \ln(1 + F_l)$ and now $U_l = \ln\left(1 + \frac{\bar{\delta}_n(l)}{\bar{J}_{nX}(l)}\right)$.

To control the order of $\bar{\delta}_n(l)$, we make the assumption:

$$(\mathrm{C}_3) \quad \exists a > 0 \text{ so that } \sum_j j^{\frac{1}{2}+a}|a_j| < +\infty.$$

Then we have the following results:

Theorem 3 *Assume (H_ε), (H_X), (C_1), (C_2) and (C_3). Then the least square estimate $\tilde{\theta}_n$ of the regression (5) is strongly consistent.*

Theorem 4 *Under the assumptions of Theorem 3, and if F is of class C^2 w.r.t. θ, the least square estimate $\tilde{\theta}_n$ of the regression (5) satisfies:*

$$\sqrt{n}(\tilde{\theta}_n - \theta_0) \underset{n\to+\infty}{\longrightarrow} \mathcal{N}\left(0, I(\theta_0)^{-1}\right)$$

where $I(\theta)$ is defined in Theorem 2.

Note that the result of Priestley can be applied to non constant frequencies λ_k (see the proof in Priestley (1981)). The condition (C_3) allows all ARMA(p,q) models but implies that the processes considered here are non fractional in the usual understanding: indeed, in the ARIMA(p,d,q) case, we know from Hassler (1993) that $a_j \sim Cj^{d-1}$, C constant, when $|d| < \frac{1}{2}$. But some models which involve an exponent parameter such as the FRAC(1) presented in the numerical examples satisfy the assumption (C_3) and therefore the assumptions of the former theorem.

2.3 Proofs of Theorems

We will denote $U^{(i)}$ (or $V^{(i)}$) the i^{th} derivative of the contrast process U (or V) w.r.t. θ, $i = 1, 2$.

2.3.1 Proof of Theorem 1:

Let $U_{m,K}(\theta) = \frac{2}{m} \sum_{l \in A_m} (\ln \bar{J}_{nX}(l) - \tilde{g}(\theta, l))^2$ be the OLS contrast. We verify the criteria of strong consistency given by Hardouin (1992), generalizing those of Dacunha-Castelle and Duflo (1986), Theorem 3.2.8 p126, in the weak sense. See also Guyon (1995) Theorem 3.4.1 and Amemiya (1986) Theorem 4.1.1.

We recall $\theta_0 \in \Theta$ which is compact under $(\mathbf{H}_X)$. The criterion is the following:

A) $\liminf_{n \to \infty} U_{m,K}(\theta) - U_{m,K}(\theta_0) \geq K(\theta, \theta_0)$ P_{θ_0} $a.s.$ where $K(\theta, \theta_0)$ is a contrast function, $K(\theta, \theta_0) = 0$ if and only if $\theta = \theta_0$.

B) $\theta \to U_{m,K}(\theta)$ and $\theta \to K(\theta, \theta_0)$ are continuous.

C) Let $W_{m,K}(\eta) = \sup_{\theta, \theta' \in \Theta} |U_{m,K}(\theta) - U_{m,K}(\theta')|$ with $||\theta - \theta'|| < \eta$. Then there exists a sequence ε_k decreasing to zero such that $\mathbb{P}_{\theta_0}\left(\lim \sup_{m,K \to \infty} W_{m,K}(\frac{1}{k}) > \varepsilon_k\right) = 1$.

A: $U_{m,K}(\theta) = \frac{2}{m} \sum_{l \in A_m} \left(\ln \bar{I}_{n\varepsilon}(l) + \ln(1 + F_{\theta_0,l}) + \tilde{g}(\theta_0, l) - \tilde{g}(\theta, l)\right)^2$ where we denote $F_{\theta,l}$ instead of F_l to precise the dependence on the parameter. Then,

$$\frac{2}{m} \sum_{l \in A_m} (\tilde{g}(\theta_0, l) - \tilde{g}(\theta, l))^2 \underset{m \to \infty}{\longrightarrow} \frac{1}{\pi} \int_0^{\pi} (\ln F(\theta_0, x) - \ln F(\theta, x))^2 \, dx := K(\theta_0, \theta)$$

and the convergence is uniform in K. So, we are going to show that

$$\frac{2}{m} \sum_{l \in A_m} \left(\ln \bar{I}_{n\varepsilon}(l) + \ln(1 + F_{\theta_0,l})\right)^2 \underset{m,K \to \infty}{\longrightarrow} 0 \quad P_{\theta_0} a.s.$$

and this will give the announced result using the Cauchy-Schwarz inequality.

- First, $\left|\lambda_k - \lambda_{lK + \frac{K+1}{2}}\right| \leq \left|\lambda_{(l+1)K} - \lambda_{lK+\frac{K+1}{2}}\right| \leq \frac{\pi K}{n} = \frac{\pi}{m}$, for $k \in \mathcal{I}_l$ and $l \in A_m$, and

$$\tilde{F}(\theta, l) - F(\theta, \lambda_k) = \frac{\partial F}{\partial x}(\theta, l^*) \, (\lambda_k - \lambda_{lK+\frac{K+1}{2}}), \quad \text{where } \left|l^* - \lambda_{lK+\frac{K+1}{2}}\right| < \left|\lambda_k - \lambda_{lK+\frac{K+1}{2}}\right|.$$

Hence $|\tilde{F}(\theta, l) - F(\theta, \lambda_k)| \leq \frac{B\pi}{m}$ under $(\mathbf{C}_1)$, and then $|F_{\theta,l}| \leq \frac{B\pi}{bm} < \frac{1}{2}$ for $m > m_0 = \frac{2B\pi}{b}$.

Finally, as $|\ln(1+x)| \leq 2|x|$ on $[-\frac{1}{2}; \frac{1}{2}]$, $|\ln(1 + F_{\theta,l})| \leq 2|F_{\theta,l}| \leq \frac{2B\pi}{bm} a.s.$ for $m > m_0$.

Then, $|\frac{2}{m} \sum_{l \in A_m} \ln(1 + F_{\theta_0,l})^2| \leq (\frac{2b\pi}{bm})^2 \underset{m,K \to \infty}{\longrightarrow} 0$ i.e. $\frac{2}{m} \sum_{l \in A_m} \ln(1 + F_{\theta_0,l})^2 \underset{m,K \to \infty}{\longrightarrow} 0 \quad P_{\theta_0} a.s.$

- Again with the Cauchy-Schwarz inequality, it is sufficient to prove now that:

$$\frac{2}{m}\sum_{l\in A_m}\left(\ln \bar{I}_{n\epsilon}(l)\right)^2 \to 0 \quad P_{\theta_0} a.s. \tag{6}$$

Let $X_m = \frac{2}{m}\sum_{l\in A_m}\left(\left(\ln \bar{I}_{n\epsilon}(l)\right)^2 - \mathbb{E}\left[\left(\ln \bar{I}_{n\epsilon}(l)\right)^2\right]\right)$. $\mathbb{E}[X_m^2] = \frac{2}{m}\mathrm{Var}\left[\left(\ln \bar{I}_{n\epsilon}(l)\right)^2\right]$.

As $|\ln \frac{x}{y}| = |\ln(1+\frac{x-y}{y})| = |\ln(1+\frac{y-x}{x})| \leq \frac{|x-y|}{\inf(x,y)} \leq \frac{|x-y|}{x} + \frac{|x-y|}{y}$ for all $x, y > 0$,

$|\ln \bar{I}_{n\epsilon}(l)| \leq |\bar{I}_{n\epsilon}(l) - 1|\left(\frac{1}{\bar{I}_{n\epsilon}(l)} + 1\right)$, $\mathbb{E}\left[\left(\ln \bar{I}_{n\epsilon}(l)\right)^2\right] \leq \left(\mathbb{E}\left[\left(\bar{I}_{n\epsilon}(l)-1\right)^4\right]\mathbb{E}\left[\left(\frac{1}{\bar{I}_{n\epsilon}(l)}+1\right)^4\right]\right)^{\frac{1}{2}}$,

and $\mathbb{E}\left[\left(\ln \bar{I}_{n\epsilon}(l)\right)^4\right] \leq \left(\mathbb{E}\left[\left(\bar{I}_{n\epsilon}(l)-1\right)^8\right]\mathbb{E}\left[\left(\frac{1}{\bar{I}_{n\epsilon}(l)}+1\right)^8\right]\right)^{\frac{1}{2}}$. We use now the following lemma:

Lemma 1 *Under (H_ϵ),*

$$\mathbb{E}\left[\frac{1}{\bar{I}_{n\epsilon}(l)^{2p}}\right] < \infty \text{ for } K \geq 2p+1, \quad \mathbb{E}\left[\left(\bar{I}_{n\epsilon}(l)-1\right)^4\right] = O(\frac{1}{K^2}), \quad \mathbb{E}\left[\left(\bar{I}_{n\epsilon}(l)-1\right)^8\right] = O(\frac{1}{K^4})$$

Then, $\mathrm{Var}\left[\left(\ln \bar{I}_{n\epsilon}(l)\right)^2\right] = O(\frac{1}{K^2})$, and $\mathbb{E}[X_m^2] = O(\frac{1}{mK^2}) = o(\frac{1}{m^3})$ with $(\mathbf{C}_2)$.

So that $\mathbb{P}(|X_m| > \epsilon) \leq \frac{\mathrm{Var}[X_m]}{\epsilon^2} \leq o(\frac{1}{m^3\epsilon})$ and this is summable in m independently of K.
Finally, $X_m \underset{m\to\infty}{\longrightarrow} 0 \quad P_{\theta_0} a.s.$ independently of K, and $X_m \underset{n\to\infty}{\longrightarrow} 0 \quad P_{\theta_0} a.s.$

We have now $\frac{2}{m}\sum_{l\in A_m}\ln \bar{I}_{n\epsilon}(l)^2 = X_m + \frac{2}{m}\sum_{l\in A_m}\mathbb{E}\left[\left(\ln \bar{I}_{n\epsilon}(l)\right)^2\right]$. The second term on the right hand side is of order $O(\frac{1}{K})$ which tends to zero when $n \to \infty \quad P_{\theta_0} a.s.$. We get (6).

B: clear.

C: $U_{m,K}(\theta) - U_{m,K}(\theta') = \Delta^1_{m,K} + \Delta^2_{m,K}$ where

$$\Delta^1_{m,K} = \frac{2}{m}\sum_{l\in A_m}\left(2\tilde{g}(\theta_0,l) - \tilde{g}(\theta,l) - \tilde{g}(\theta',l)\right)\left(\tilde{g}(\theta',l) - \tilde{g}(\theta,l)\right)$$

$$\Delta^2_{m,K} = \frac{4}{m}\sum_{l\in A_m}\left(\ln \bar{I}_{n\epsilon}(l) + \ln(1+F_{\theta_0,l})\right)\left(\tilde{g}(\theta',l) - \tilde{g}(\theta,l)\right)$$

Let $\varepsilon(\eta) = \sup_{\substack{\theta,\theta'\in\Theta \\ ||\theta-\theta'||<\eta}} |\tilde{g}(\theta',l) - \tilde{g}(\theta,l)|$. Then $\varepsilon(\eta) \underset{\eta\to 0}{\longrightarrow} 0$ by uniform continuity of $\tilde{g}$ on a compact set. And let $M = \sup_{\theta\in\Theta}|\tilde{g}(\theta,l)|$. We obtain: $|\Delta^1_{m,K}| \leq 4M\varepsilon(\eta)$ and

$$|\Delta^2_{m,K}| \leq \underbrace{\frac{4}{m}\sum_{l\in A_m}\left(|\ln \bar{I}_{n\epsilon}(l)| + \ln(1+F_{\theta_0,l})\right)}_{\to 0 \quad P_{\theta_0} a.s. \text{ as previously}} \times\varepsilon(\eta) \leq \frac{\varepsilon(\eta)}{2} \text{ if } m, K \text{ great enough.}$$

Finally, $\mathbb{P}_{\theta_0}\left(\limsup_{m,K\to\infty}\sup_{\substack{\theta,\theta'\in\Theta \\ ||\theta-\theta'||<\eta}}|\Delta_{m,K}| > \frac{\varepsilon(\eta)}{2}\right) = 1. \quad \blacksquare$

2.3.2 Proof of Theorem 2:

We prove that we can replace the residual $W_l + \ln(1+F_l)$ by $\bar{I}_{n\varepsilon}(l) - 1$.

• $0 = \sqrt{n}U^{(1)}_{m,K}(\theta_0) + U^{(2)}_{m,K}(\theta_n^\star)\sqrt{n}(\hat{\theta}_n - \theta_0)$ with $||\theta_n^\star - \theta_0|| < ||\hat{\theta}_n - \theta_0||$. As $\hat{\theta}_n$ is strongly consistent, the limit law of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is given by $-[U^{(2)}_{m,K}(\theta_0)]^{-1}\sqrt{n}U^{(1)}_{m,K}(\theta_0)$.

• Study of $U^{(1)}_{m,K}(\theta_0) = -\frac{4}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}(W_l + \ln(1+F_l))$; $\sqrt{n}U^{(1)}_{m,K}(\theta_0) = T_1 + T_2$ where:

$$T_1 = -\frac{4\sqrt{n}}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}(\bar{I}_{n\varepsilon}(l) - 1 + \ln(1+F_l)) \quad (7)$$

$$T_2 = -\frac{4\sqrt{n}}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}(\ln\bar{I}_{n\varepsilon}(l) - (\bar{I}_\varepsilon(l) - 1)). \quad (8)$$

$I_{n\varepsilon}(\lambda_k) \sim \mathcal{E}xp(1)$ are i.i.d so the $e_l = \bar{I}_\varepsilon(l) - 1 + \ln(1+F_l)$ are independent too, and $\mathrm{Var}\bar{I}_{n\varepsilon}(l) = \frac{1}{K}$. Therefore: $\mathrm{Var}\ T_1 = \frac{16n}{m^2}\sum_{l\in A_m}\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)'\ \mathrm{Var}\ e_l$. Then,

$$\begin{aligned}\mathrm{Var}T_1 &= \frac{16n}{m^2}\sum_{l\in A_m}\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)' \Big(\mathrm{Var}(\bar{I}_\varepsilon(l) - 1) \\ &\qquad + 2\mathrm{Cov}(\bar{I}_\varepsilon(l) - 1, \ln(1+F_l)) + \mathrm{Var}\ln(1+F_l)\Big) \\ &= \frac{16n}{m^2K}\sum_{l\in A_m}\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)' \Big(1 + 2K\mathrm{Cov}(\bar{I}_\varepsilon(l) - 1, \ln(1+F_l) + K\mathrm{Var}\ln(1+F_l)\Big)\end{aligned}$$

But $K\mathrm{Var}\ln(1+F_l) \le K\frac{4B^2\pi^2}{b^2m^2}$ *a.s.* for $m > m_0$ (see Proof of Theorem 1); as $\frac{K}{m^2} \to 0$ with $(\mathbf{C}_2)$, we have:

$$\lim_{n\to\infty}\mathrm{Var}T_1 = \lim_{m\to\infty}\frac{16}{m}\sum_{l\in A_m}\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)\left(\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\right)' = 16I(\theta_0).$$

Applying the CLT to T_1 yields to: $T_1 = -\frac{4}{\sqrt{m}}\sum_{l\in A_m}\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}(\sqrt{K}e_l) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, 16I(\theta_0))$.

• We have now to verify that $T_2 \xrightarrow{P} \vec{0}$ (p-dimensional null vector) i.e:

$$\frac{4\sqrt{n}}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}(\theta_0,l)}{\partial\theta}\left(\ln\bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l) - 1)\right) \xrightarrow{P} \vec{0}.$$

We study this term's limit in L^1 and need the following lemma:

Lemma 2 *Under* $(\boldsymbol{H_\varepsilon})$, $||\ln\bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l) - 1)||_1 = O(\frac{1}{K})$, *where* $||.||_1$ *denotes the* L^1*-norm.*

Then:

$$\left\| \frac{4\sqrt{n}}{m} \sum_{l\in A_m} \frac{\partial \tilde{g}(\theta_0,l)}{\partial \theta} \left(\ln \bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l)-1)\right) \right\|_1 \leq \frac{4\sqrt{n}}{m} \sum_{l\in A_m} \left\| \frac{\partial \tilde{g}(\theta_0,l)}{\partial \theta} \right\| \frac{C}{K} \leq \frac{4BC}{b}\sqrt{\frac{m}{K}} \xrightarrow{(C_2)} 0$$

that is $T_2 \xrightarrow{P} \vec{0}$.

• Study of $U^{(2)}_{m,K}$: Finally, we consider $U^{(2)}_{m,K}(\theta_0) = -\frac{4}{m}\sum_{l\in A_m}\left(\frac{\partial^2\tilde{g}(\theta_0,l)}{\partial\theta^2}W'_l - \left(\frac{\partial \tilde{g}(\theta_0,l)}{\partial\theta}\right)\left(\frac{\partial \tilde{g}(\theta_0,l)}{\partial\theta}\right)'\right)$, where $W'_l = \ln \bar{I}_{n\varepsilon}(l) + \ln(1+F_l)$ and $\frac{\partial^2\tilde{g}}{\partial\theta^2}$ is the $p\times p$ Hessian matrix. As the W'_l are independent, L^2, with asymptotic mean zero, the Strong Law of Large Numbers gives:

$$\frac{4}{m}\sum_{l\in A_m}\frac{\partial^2\tilde{g}(\theta_0,l)}{\partial\theta^2}W'_l \xrightarrow[m\to\infty]{a.s.} 0_p$$

where 0_p is the $p\times p$ null matrix and the convergence is to be understood term by term. Then, $U^{(2)}_{m,K}(\theta_0)$'s limit in probability is that of $\frac{4}{m}\sum_{l\in A_m}\left(\frac{\partial \tilde{g}(\theta_0,l)}{\partial\theta}\right)\left(\frac{\partial \tilde{g}(\theta_0,l)}{\partial\theta}\right)'$ i.e. $4I(\theta_0)$. ∎

2.3.3 Proof of Theorem 3:

We follow the proof of Theorem 1. We study now the contrast $V_{m,K}(\theta) = \frac{2}{m}\sum_{l\in A_m}\left(\ln \bar{I}_{nX}(l) - \tilde{g}(\theta,l)\right)^2$.

• The study of **A** is the same, but we have to control another term: $\frac{2}{m}\sum_{l\in A_m}U_l^2 = \frac{2}{m}\sum_{l\in A_m}\ln\left(1+\frac{\bar{\delta}_n(l)}{\bar{J}_{nX}(l)}\right)^2$. With $|U_l| \leq \frac{|\bar{\delta}_n(l)|}{\bar{I}_{nX}(l)} + \frac{|\bar{\delta}_n(l)|}{\bar{J}_{nX}(l)}$, we have

$$\mathbb{E}[U_l^2] \leq \mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{I}_{nX}(l)^2}\right] + \mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{J}_{nX}(l)^2}\right] + 2\left(\mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{I}_{nX}(l)^2}\right]\mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{J}_{nX}(l)^2}\right]\right)^{\frac{1}{2}}$$

First, $\mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{I}_{nX}(l)^2}\right] \leq \left(\mathbb{E}[\bar{\delta}_n(l)^4]\mathbb{E}\left[\frac{1}{\bar{I}_{nX}(l)^4}\right]\right)^{\frac{1}{2}}$. Looking at the proof of Theorem 6.2.2 of Priestley (1981), and at equation 6.2.32 p424, we see that we can easily extend his result to:

$$\mathbb{E}[\delta_n(\lambda)^4] = O(n^{-2-4a})\text{uniformly in } \lambda$$

under (C_3). So we deduce, still under (C_3), that $\mathbb{E}[\bar{\delta}_n(l)^4] = O(n^{-2-4a})$; next, we use the following lemma:

Lemma 3 *Let $(Z_k^n)_k$ be a sequence of independent variables of law $\mathcal{E}xp(\mu_k)$ and $\bar{Z}_K^n = \frac{1}{K}\sum_{k=1}^{K} Z_k^n$, $K \geq 9$. Then, $\mathbb{E}\left[\left(\frac{1}{\bar{Z}_K^n}\right)^2\right] \leq 8\,(\sup \mu_k)^2$, and $\mathbb{E}\left[\left(\frac{1}{\bar{Z}_K^n}\right)^4\right] \leq 4\,(\sup\mu_k)^4$.*

We apply this result to $Z_k^n = I_{nX}(\frac{k}{n})$ which are asymptotically independent and of law $\mathcal{E}xp$ with mean $F(\frac{k}{n})$ (see e.g. Davis and Jones (1968)), and then $\mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{I}_{nX}(l)^2}\right]$ is of order $O(n^{-1-2a})$.

It remains $\mathbb{E}\left[\frac{\bar{\delta}_n(l)^2}{\bar{J}_{nX}(l)^2}\right] \leq \left(\mathbb{E}[\bar{\delta}_n(l)^4]\mathbb{E}[\frac{1}{\bar{J}_{nX}(l)^4}]\right)^{\frac{1}{2}}$. Then, $\mathbb{E}\left[\frac{1}{\bar{J}_{nX}(l)^4}\right] \leq \frac{1}{b^4}\mathbb{E}\left[\frac{1}{\bar{I}_{n\varepsilon}(l)^4}\right] < \infty$ with Lemma 1 and $(\mathbf{C}_1)$.

So we obtain $\mathbb{E}[U_l^2] = O(n^{-1-2a})$. Then $\mathbb{P}\left(\frac{2}{m}\sum_{l\in A_m} U_l^2 > \varepsilon\right) \leq \frac{\mathbb{E}[\frac{2}{m}\sum U_l^2]}{\varepsilon} \leq \frac{C}{\varepsilon}n^{-1-2a}$ which is summable in m uniformly in K with $(\mathbf{C}_2)$, so $\frac{2}{m}\sum_{l\in A_m} U_l^2 \to 0 \;\; P_{\theta_0} a.s.$

- Conditions B and C are clearly fullfilled. ■

2.3.4 Proof of Theorem 4:

The law of $\sqrt{n}(\tilde{\theta}_n - \theta_0)$ is given by $-[V_{m,K}^{(2)}(\theta_0)]^{-1}\sqrt{n}V_{m,K}^{(1)}(\theta_0)$, with Theorem 3. $V_{m,K}^{(1)}(\theta_0) = -\frac{4}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)(W_l' + U_l)$ gives: $\sqrt{n}V_{m,K}^{(1)}(\theta_0) = T_1 + T_2 - \frac{4\sqrt{n}}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)U_l$, with T_1 and T_2 given by (7) and (8). Then their limits are known: $T_1 \xrightarrow{\mathcal{L}} \mathcal{N}(0, 16I(\theta_0))$ and $T_2 \overset{L^1 \text{ and } P}{\longrightarrow} \vec{0}$.

We can also write the last term of $V_{m,K}^{(1)}(\theta_0)$:

$$T_3 := -\frac{4\sqrt{n}}{m}\sum_{l\in A_m}\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)\ln\left(\frac{\bar{I}_{nX}(l)}{\bar{J}_{nX}(l)}\right) \tag{9}$$

As $|\ln\frac{x}{y}| \leq \frac{|x-y|}{x} + \frac{|x-y|}{y}$ for all $x,y > 0$, $\mathbb{E}[||T_3||] \leq \mathbb{E}[T_4] + \mathbb{E}[T_5]$, where

$$T_4 = \frac{4\sqrt{n}}{m}\sum_{l\in A_m}\left\|\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)\right\|\left(\frac{|\bar{\delta}_n(l)|}{\bar{J}_{nX}(l)}\right) \tag{10}$$

and

$$T_5 = \frac{4\sqrt{n}}{m}\sum_{l\in A_m}\left\|\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)\right\|\left(\frac{|\bar{\delta}_n(l)|}{\bar{I}_{nX}(l)}\right). \tag{11}$$

$(\mathbf{C}_3)$ implies, with the result of Priestley (1981) that $\mathbb{E}[\bar{\delta}_n(l)^2] \leq \frac{C}{n^{1+2a}}$. Moreover, since $\mathbb{E}\left[\frac{1}{\bar{I}_{n\varepsilon}(l)^2}\right] = \frac{1}{(1-\frac{1}{K})(1-\frac{2}{K})} \leq 4$ for $K \geq 4$,

$$\mathbb{E}[T_4] \leq \frac{4\sqrt{n}}{m}\sum_{l\in A_m}\frac{B}{b}\frac{1}{b}\left(\mathbb{E}[\bar{\delta}_n(l)^2]\mathbb{E}\left[\frac{1}{\bar{I}_{n\varepsilon}(l)^2}\right]\right)^{\frac{1}{2}} \leq \frac{4\sqrt{n}}{m}m\frac{B}{b^2}\left(4\frac{C}{n^{1+2a}}\right)^{\frac{1}{2}} = \frac{8B\sqrt{C}}{b^2}\frac{1}{n^a},$$

which implies that $\mathbb{E}[T_4] \underset{n\to+\infty}{\longrightarrow} 0$.

We study T_5 in a same way and then, $\mathbb{E}[T_5] \leq \frac{4B\sqrt{C}}{b}\frac{1}{n^a}\left(\mathbb{E}\left[\frac{1}{\bar{I}_{nX}(l)^2}\right]\right)^{\frac{1}{2}}$. Then we apply Lemma 3 to $I_{nX}(\lambda_k)$ again and this implies that $\mathbb{E}[T_5] \underset{n\to+\infty}{\longrightarrow} 0$.

Therefore, we have: $\sqrt{n}V^{(1)}_{m,K}(\theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 16I(\theta_0))$.

$$\text{Next, } V^{(2)}_{m,K}(\theta_0) = \underbrace{-\frac{4}{m}\sum_{l\in A_m}\frac{\partial^2\tilde{g}}{\partial\theta^2}(\theta_0,l)W_l'}_{\xrightarrow{L^1} o_p \text{ as in Theorem 2}} \underbrace{-\frac{4}{m}\sum_{l\in A_m}\frac{\partial^2\tilde{g}}{\partial\theta^2}(\theta_0,l)U_l}_{\xrightarrow{L^1} o_p \text{ as } T_3} + \underbrace{\frac{4}{m}\sum_{l\in A_m}\left(\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)\right)\left(\frac{\partial\tilde{g}}{\partial\theta}(\theta_0,l)\right)'}_{\xrightarrow{a.s.} 4I(\theta_0)}.$$

So that $V^{(2)}_{m,K}(\theta_0) \overset{L^1 \text{ and } P}{\longrightarrow} 4I(\theta_0)$.

Collecting the terms gives the announced limit law for $\sqrt{n}(\tilde{\theta}_n - \theta_0)$. ■

Proof of Lemma 1: f_K the density of $\bar{I}_{n\epsilon}(l)$ is equal to : $f_K(z) = \frac{K^K}{(K-1)!}z^{K-1}e^{-Kz}1_{z\geq 0}$.

Let q an integer, $q \geq 1-K$;

$$\mathbb{E}[\bar{I}_{n\epsilon}(l)^q] = \frac{1}{K^q(K-1)!}\int_0^\infty u^{q+K-1}e^{-u}du = \frac{\Gamma(q+K)}{K^q(K-1)!} = \frac{(q+K-1)(q+K-2)\dots(K+1)}{K^{q-1}}.$$

Then, we calculate $\mathbb{E}[\bar{I}_{n\epsilon}(l)^{-2p}] = \frac{K^{2p}}{(K-1)(K-2)\dots(K-2p)}$ for $K \geq 1+2p$,

$\mathbb{E}[(\bar{I}_{n\epsilon}(l)-1)^4] = \frac{3}{K^2} + \frac{6}{K^3}$ and $\mathbb{E}[(\bar{I}_{n\epsilon}(l)-1)^8] = \frac{105}{K^4} + \frac{2380}{K^5} + \frac{7308}{K^6} + \frac{5040}{K^7}$. ■

Proof of Lemma 2: • We have $\mathbb{E}[\bar{I}_{n\epsilon}(l)] = 1$, $\mathrm{Var}\bar{I}_{n\epsilon}(l) = \frac{1}{K}$ and the density of $\bar{I}_{n\epsilon}(l)$ is given above.

$$\begin{aligned}\mathbb{E}\left[|\ln \bar{I}_{n\epsilon}(l) - (\bar{I}_{n\epsilon}(l)-1)|\right] &= \int_0^\infty |\ln z - (z-1)|f_K(z)dz = \int_0^\infty \big(-\ln z + (z-1)\big)f_K(z)dz \\ &= -\int_0^\infty \ln z f_K(z)dz\end{aligned}$$

• $\int_0^\infty \ln z.z^{K-1}e^{-Kz}dz = \int_0^\infty \ln\frac{u}{K}\left(\frac{u}{K}\right)^{K-1}e^{-u}\frac{du}{K} = \frac{1}{K^K}[J_{K-1} - \ln K.(K-1)!]$
where $J_K = \int_0^\infty \ln u.u^K e^{-u}du$.

• We find $J_K = KJ_{K-1} + (K-1)! = K!\sum_{i=1}^{K}\frac{1}{i} + J_0K!$.

Then $J_0 = \int_0^\infty \ln xe^{-x}dx = -\nu$ and $\lim_{n\to\infty}\ln n - (1+..+\frac{1}{n}) = -\nu$ where ν is the Euler constant.

So finally $\mathbb{E}\left[|\ln \bar{I}_{n\epsilon}(l) - (\bar{I}_{n\epsilon}(l)-1)|\right] = -\sum_{i=1}^{K-1}\frac{1}{i} - J_0 + \ln K = O(\frac{1}{K})$. ■

Proof of Lemma 3:

$$\mathbb{E}\left(\frac{1}{\bar{Z}_K^n}\right)^2 = \mathbb{E}\frac{K^2}{\left(\sum_{k=1}^{K} Z_k^n\right)^2} = \int\ldots\int \frac{K^2}{\left(\sum_{k=1}^{K} z_k\right)^2}\mu_1\ldots\mu_K.exp\{-\sum_{k=1}^{K}\mu_k z_k\}1_{z_1\geq 0}\ldots 1_{z_K\geq 0}dz_1\ldots dz_K$$

$$= K^2\int\ldots\int \frac{1}{\left(\sum_{k=1}^{K} y_k/\mu_k\right)^2}exp\{-\sum_{k=1}^{K} y_k\}1_{y_1\geq 0}\ldots 1_{y_K\geq 0}dy_1\ldots dy_K$$

$$\leq (\sup\mu_k)^2\mathbb{E}\left[\frac{1}{\bar{I}_{n\epsilon}(l)^2}\right] \leq (\sup\mu_k)^2\frac{K^2}{(K-1)(K-2)}$$

and $\frac{K^2}{(K-1)(K-2)} \leq 8$ as soon as $K \geq 3$.

In the same way, $\mathbb{E}\left[\left(\frac{1}{\bar{Z}_K^n}\right)^4\right] \leq (\sup\mu_k)^4\mathbb{E}\left[\frac{1}{\bar{I}_{n\epsilon}(l)^4}\right] \leq (\sup\mu_k)^4\frac{K^4}{(K-1)(K-2)(K-3)(K-4)}$ which is less than $4(\sup\mu_k)^4$ as soon as $K \geq 9$. ∎

3 Comparison with Whittle's estimate

A well known procedure for estimation is Whittle's. Under rather general hypotheses it leads to strongly consistent, asymptotically normal and asymptotically efficient estimates, **in the Gaussian case.**

Let us compare those two methods: Whittle's and the Log-Regularized Periodogram, which we will denote now by LRP. Let $\tilde{\theta}_n$ denote the LRP estimator and $\hat{\theta}_n$ Whittle's one. The asymptotic variance of $\sqrt{n}\hat{\theta}_n$ is also $I^{-1}(\theta_0)$. We show that the estimators are in fact asymptotically equivalent under the previous assumptions and the following:

(C_4): The second and third partial derivatives of $F(\theta,x)$ w.r.t. x and θ exist, are continuous and bounded.

Theorem 5 *Assume (H_ϵ), (H_X).*

Under (C_1), (C_2), (C_3) and (C_4), Whittle's $\hat{\theta}_n$ and LRP $\tilde{\theta}_n$ estimators are asymptotically equivalent: the two methods are asymptotically similar.

Note that, in fact, the estimation is performed in two steps. If $\theta = (\sigma^2,\theta^\star)$, $\theta^\star$ is estimated first, using that $F(\theta,x) = \sigma^2F(\theta_1,x)$ with $\theta_1 = (1,\theta^\star)$. The fisrt step gives then the two estimates $\hat{\theta}_n^\star$ and $\tilde{\theta}_n^\star$. The second step gives estimates of σ^2 with:

$$\begin{cases} \hat{\sigma}_n^2 = (2/n)\sum_{k\in A_n} I_{nX}(\frac{k}{n})F^{-1}(\hat{\theta}_n^\star,\frac{k}{n}) \\ \tilde{\sigma}_n^2 = exp\left\{(2/m)\sum_{l\in A_m}\ln\left(\bar{I}_{nX}(l)\bar{F}^{-1}(\tilde{\theta}_n^\star,l)\right)\right\} \end{cases}$$

Such a procedure is known to be consistent under regularity assumptions on the contrast, condition which are satisfied here provided that the first step estimators are consistent (see Hardouin (1992) and Guyon (1995)).

Proof of Theorem 5: Let $A_n = \{k, \lambda_k = 2\pi k/n \in [0,\pi]\}$.

Whittle's estimator is obtained by minimization of

$$W_n(\theta) = (4/n) \sum_{k\in A_n} \left(\ln F(\theta,\lambda_k) + I_{nX}(\lambda_k)F^{-1}(\theta,\lambda_k)\right),$$

and we have:

$$0 = \sqrt{n}W_n^{(1)}(\theta_0) + W_n^{(2)}(\hat{\theta}_n^\star)\ \sqrt{n}(\hat{\theta}_n - \theta_0) \text{ with } \|\hat{\theta}_n^\star - \theta_0\| < \|\hat{\theta}_n - \theta_0\|$$

On the other hand, we consider the contrast:

$$V_{m,K}(\theta) = (2/m) \sum_{l\in A_m} (\ln \bar{I}_{nX}(l) - \tilde{g}(\theta,l))^2$$

in the LRP method, and we also have:

$$0 = \sqrt{n}V_{m,K}^{(1)}(\theta_0) + V_{m,K}^{(2)}(\tilde{\theta}_n^\star)\ \sqrt{n}(\tilde{\theta}_n - \theta_0) \text{ with } \|\tilde{\theta}_n^\star - \theta_0\| < \|\tilde{\theta}_n - \theta_0\|$$

Then to prove Theorem 5 it is sufficient to show that:

$$\|\sqrt{n}\ W_n^{(1)}(\theta_0) - \sqrt{n}V_{m,K}^{(1)}(\theta_0)\| \underset{n\to\infty}{\longrightarrow} 0 \text{ in } P_{\theta_0} - \text{ probability} \tag{12}$$

$$W_n^{(2)}(\hat{\theta}_n^\star) \text{ and } V_{m,K}^{(2)}(\tilde{\theta}_n^\star) \text{ have the same limit in probability} \tag{13}$$

• Let us denote $h(\theta,x) = (\frac{\partial F}{\partial\theta}/F)(\theta,2\pi x)$. We have:

$$\begin{aligned}\sqrt{n}W_n^{(1)}(\theta_0) &= \frac{4}{\sqrt{n}}\sum_{k\in A_n} h(\theta_0,\tfrac{k}{n}) - \frac{4}{\sqrt{n}}\sum_{k\in A_n} I_{nX}(\lambda_k)\ (\frac{\partial F}{\partial\theta}/F^2)(\theta_0,\lambda_k)\\ &= \frac{4}{\sqrt{n}}\sum_{k\in A_n} h(\theta_0,\tfrac{k}{n}) - \frac{4}{\sqrt{n}}\sum_{k\in A_n} I_{n\varepsilon}(\lambda_k)\ h(\theta_0,\tfrac{k}{n})\\ &\quad -\frac{4}{\sqrt{n}}\sum_{k\in A_n}\delta_n(\theta_0,\lambda_k)\ (\frac{\partial F}{\partial\theta}/F^2)(\theta_0,\lambda_k)\end{aligned}$$

With Priestley's (1981) Theorem 6.22, $(\mathbf{C}_3)$ and $(\mathbf{C}_1)$, we know that there exists $a>0$ such that $\mathbb{E}\left\|\delta_n(\theta_0,\lambda_k)\frac{\partial F}{\partial\theta}/F^2(\theta_0,\lambda_k)\right\| \le (CB)/(n^{1/2+a}b^2)$ and so, the third term of $\sqrt{n}W_n^{(1)}(\theta_0)$ tends to $\vec{0}$ in L^1.

On the other hand, $\sqrt{n}V_{m,K}^{(1)}(\theta_0) = T_1 + T_2 + T_3$ with T_i defined by (7), (8), (9).

We know from Theorem 4 that $T_1 \xrightarrow{\mathcal{L}} \mathcal{N}(0, 16I(\theta_0))$ where the part depending on $\ln(1+F_l)$ has null weight on the asymptotic variance and that $T_2, T_3 \overset{L^1 \text{ and } P}{\longrightarrow} \vec{0}$.

Thus, the main term of T_1 is $-(4\sqrt{n}/m)\sum_{l\in A_m}\left(\bar{I}_{n\varepsilon}(l)-1\right)(\frac{\partial F}{\partial \theta}/F)(\theta_0, 2\pi(lK+\frac{K+1}{2})/n)$. Besides, under ($\mathbf{C}_2$) and ($\mathbf{C}_4$), $(lK+\frac{K+1}{2})/n = (l+1/2)/m + 1/2n$ can be replaced by $(l+1/2)/m$. So we want now to compare:

$$\sqrt{n}W_n'^{(1)}(\theta_0) = -\frac{4}{\sqrt{n}}\sum_{k\in A_n}(I_{n\varepsilon}(\lambda_k)-1)\,h(\theta_0,\tfrac{k}{n}) \text{ and } \sqrt{n}V_{m,K}'^{(1)}(\theta_0) = -\frac{4\sqrt{n}}{m}\sum_{l\in A_m}\left(\bar{I}_\varepsilon(l)-1\right)h(\theta_0,\tfrac{l+1/2}{m})$$

The Taylor's expansion gives:

$$h(\theta_0,\tfrac{k}{n}) = h(\theta_0,\tfrac{l+1/2}{m}) + (\tfrac{k}{n}-\tfrac{l+1/2}{m})\tfrac{\partial h}{\partial x}(\theta_0,\tfrac{l+1/2}{m}) + \tfrac{1}{2}(\tfrac{k}{n}-\tfrac{l+1/2}{m})^2\,\tfrac{\partial^2 h}{\partial x^2}(\theta_0,\tilde{x}_{k,l})$$

with $|\tilde{x}_{k,l}-\frac{l+1/2}{m}| < |\frac{k}{n}-\frac{l+1/2}{m}|$.

Then $-\sqrt{n}\,W_n'^{(1)}(\theta_0)/4 + \sqrt{n}\,V_{m,K}'^{(1)}(\theta_0)/4 = A_1 + A_2 + A_3/2$, with:

$$\begin{cases} A_1 = (\sqrt{n}/m)\sum_{l\in A_m} h(\theta_0,\frac{l+1/2}{m}) - (1/\sqrt{n})\sum_{k\in A_n} h(\theta_0,\frac{k}{n}), \\ A_2 = (1/\sqrt{n})\sum_{l\in A_m}\frac{\partial h}{\partial x}(\theta_0,\frac{l+1/2}{m})\sum_{k\in\mathcal{I}_l}(\frac{k}{n}-\frac{l+1/2}{m})\,I_{n\varepsilon}(\lambda_k), \\ A_3 = (1/\sqrt{n})\sum_{l\in A_m}\sum_{k\in\mathcal{I}_l}(\frac{k}{n}-\frac{l+1/2}{m})^2\,I_{n\varepsilon}(\lambda_k)\frac{\partial^2 h}{\partial x^2}(\theta_0,\tilde{x}_{k,l}) \end{cases}$$

We have:

- $A_1 = (1/\sqrt{n})\left\{\sum_{l\in A_m}\sum_{k\in\mathcal{I}_l}\left(h(\theta_0,\frac{l+1/2}{m}) - h(\theta_0,\frac{k}{n})\right)\right\} = -(2m\sqrt{n})^{-1}\sum_{l\in A_m}\frac{\partial h}{\partial x}(\theta_0,\frac{l+\frac{1}{2}}{m}) + O(n^{-1/2})$

because $\sum_{k\in\mathcal{I}_l}(\frac{k}{n}-\frac{l+1/2}{m}) = 1/2m$. Then $A_1 \to \vec{0}$.

- We know that the $I_{n\varepsilon}(\lambda_k)$ are i.i.d $\mathcal{E}xp(1)$. Then $\mathbb{E}\|A_2\| \to 0$ still with $\sum_{k\in\mathcal{I}_l}(\frac{k}{n}-\frac{l+1/2}{m}) = 1/2m$ and with ($\mathbf{C}_4$): $\mathrm{Var}\|A_2\| \leq (C/n)\sum_{l\in A_m}\sum_{k\in\mathcal{I}_l}(\frac{k}{n}-\frac{l+1/2}{m})^2$.

But $|\frac{k}{n}-\frac{l+1/2}{m}| \leq 1/2m$ so $\mathrm{Var}\,\|A_2\| \leq (CmK)/(8nm^2) \to 0$ so that $A_2 \to \vec{0}$ in P_{θ_0}-probability.

- In the same way, $A_3 \to \vec{0}$ in P_{θ_0}-probability.

So that finally we get (12).

• $\hat{\theta}_n^\star$ tends to θ_0 ($\hat{\theta}_n$ is consistent) so we can consider

$$\begin{aligned} W_n^{(2)}(\hat{\theta}_n^\star) &= (4/n)\sum_{k\in A_n}\frac{\partial h}{\partial \theta}(\theta_0,\tfrac{k}{n}) - (4/n)\sum_{k\in A_n} I_{n\varepsilon}(\lambda_k)\left(\frac{\partial h}{\partial \theta}(\theta_0,\tfrac{k}{n}) - h(\theta_0,\tfrac{k}{n})h(\theta_0,\tfrac{k}{n})'\right) \\ &\quad -(4/n)\sum_{k\in A_n}\delta_n(\theta_0,\lambda_k)\frac{\partial}{\partial\theta}\left((\frac{\partial F}{\partial\theta}/F^2)(\theta_0,\lambda_k)\right) \end{aligned}$$

As previously, we use Theorem 6.22 of Priestley (1981): there exists $a > 0$ such that $\mathbb{E}\,[\delta_n^2(\theta_0, \lambda_k)] \le Cn^{-1-2a}$ and assumptions (C_2) and (C_4) imply that all terms in the matrix $\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial\theta}/F^2\right)$ are bounded.
Then $\mathbb{E}W_n^{(2)}(\hat{\theta}_n^\star)$ tends to

$$4\int_0^{1/2}\frac{\partial h}{\partial\theta}(\theta_0,x) - 4\int_0^{1/2}\left(\frac{\partial h}{\partial\theta}(\theta_0,x) - h(\theta_0,x)h(\theta_0,x)'\right)dx = 4\int_0^{1/2} h(\theta_0,x)h(\theta_0,x)'dx = 4I(\theta_0).$$

Besides, $\mathrm{Var}W_n^{(2)}(\hat{\theta}_n^\star) \sim (16/n)\int_0^{1/2}(\frac{\partial h}{\partial\theta}(\theta_0,x) - h(\theta_0,x)h(\theta_0,x)')^2 dx \underset{n\to\infty}{\longrightarrow} 0_p$.
In the same way, we can consider

$$V_{m,K}^{(2)}(\tilde{\theta}_n^\star) \sim -(4/m)\sum_{l\in A_m}(\bar{I}_{n\varepsilon}(l) - 1)\frac{\partial h}{\partial\theta}(\theta_0, \tfrac{l+1/2}{m}) + (4/m)\sum_{l\in A_m} h(\theta_0, \tfrac{l+1/2}{m})h(\theta_0, \tfrac{l+1/2}{m})'$$

Then $\mathbb{E}V_{m,K}^{(2)}(\tilde{\theta}_n^\star)$ tends to $4\int_0^{1/2} h(\theta_0,x)h(\theta_0,x)'\,dx$ and $\mathrm{Var}V_{m,K}^{(2)}(\tilde{\theta}_n^\star) \le C/(mK)$, term by term.
This gives (13). ∎

4 The non-Gaussian case.

We assume now that:

$$(\mathrm{H'}_\varepsilon)\quad \begin{cases} \varepsilon_t \text{ are i.i.d } (0,1) \text{ and their eighth moment } \mu_8 \text{ exists} \\ \mathbb{E}|\frac{1}{\bar{I}_{n\varepsilon}(l)}|^4 < +\infty \text{ for } K \ge 5 \end{cases}$$

and:

$$(\mathrm{H'}_X):\quad (\mathrm{H}_X) \text{ and } \mathbb{E}|\frac{1}{\bar{I}_{nX}(l)}|^2 < +\infty \text{ for } K \ge 5.$$

Let us recall equation (5): $\ln \bar{I}_{nX}(l) = \tilde{g}(\theta, l) + W_l' + U_l, \qquad l \in A_m$.

Theorem 6 *Assume (H'_X), (H'_ε) and (C_1), (C_2) and (C_3).*
(i) The least square estimate $\bar{\theta}_n$ of the regression (5) is weakly consistent.
(ii) If moreover F is of class C^2 w.r.t. θ,

$$\sqrt{n}\,(\bar{\theta}_n - \theta_0) \underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0, I(\theta_0)^{-1}J(\theta_0)I(\theta_0)^{-1})$$

with $J(\theta) = \frac{\mu_4 - 3}{4\pi^2}\Big[\int_0^\pi \frac{\partial g}{\partial\theta}(\theta,x)\,dx\Big]\Big[\int_0^\pi \frac{\partial g}{\partial\theta}(\theta,x)\,dx\Big]' + I(\theta)$, *where $I(\theta)$ is defined in Theorem 2.*

Remark: Looking at the proofs of the Theorems 1 and 3, the assumptions do not provide sufficient control to allow an almost sure consistency. However, the strong consistency could be obtained with stronger hypotheses: **(H'$_X$)** and **(H'$_\varepsilon$)**, (C$_1$), (C$_2$), (C$_3$), $\mathbb{E}\left[\frac{1}{\bar{I}_{n\varepsilon}(l)^8}\right] < \infty$ and $\mathbb{E}\left[(\bar{I}_{n\varepsilon}(l)-1)^8\right] = O(\frac{1}{K^4})$, and $\mathbb{E}\left[\frac{1}{\bar{I}_{nX}(l)^4}\right] < \infty$.

Besides, the convergence of the estimator based on the Pseudo periodogram can also be proved: **(H$_X$)**, **(H'$_\varepsilon$)**, (C$_1$), (C$_2$) ensure weak consistency and if we add $\mathbb{E}\left[\frac{1}{\bar{I}_{n\varepsilon}(l)^8}\right] < \infty$ and $\mathbb{E}\left[(\bar{I}_{n\varepsilon}(l)-1)^8\right] = O(\frac{1}{K^4})$, we will obtain the strong consistency.

Proof of Theorem 6:
(i) For weak consistency, the criteria **A**, **B**, **C** must be replaced by **A'**, **B**, **C'**, where **A'** and **C'** are as **A** and **C** with convergence in probability instead of almost sure.
From the proof of Theorem 1, we find $\frac{2}{m}\sum_{l\in A_m} \ln \bar{I}_{n\varepsilon}(l)^2 \xrightarrow[n\to\infty]{L^1} 0$ instead of (6) (almost sure) because we only can check:

Lemma 4 *$\mathbb{E}[\ln \bar{I}_{n\varepsilon}(l)^2]$ is of order $O(\frac{1}{K})$.*

From the proof of Theorem 3, we also have to control $\frac{2}{m}\sum_{l\in A_m} U_l^2$, but with no result on $\mathbb{E}\left[\frac{1}{\bar{I}_{nX}(l)^4}\right]$.

For all $x, y > 0$, $\quad |\ln \frac{x}{y}| \le 2\left(\frac{\sqrt{|x-y|}}{\sqrt{x}} + \frac{\sqrt{|x-y|}}{\sqrt{y}}\right)$.
Indeed, $|\ln \frac{x}{y}| = 2|\ln\sqrt{\frac{x}{y}}| \le 2\left(\frac{|\sqrt{x}-\sqrt{y}|}{\sqrt{x}} + \frac{|\sqrt{x}-\sqrt{y}|}{\sqrt{y}}\right)$ and $|\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|}$. Thus,

$$\begin{aligned}
\mathbb{E}[U_l^2] &\le 4\mathbb{E}\left[\left(\sqrt{\bar{\delta}_n(l)}\left(\frac{1}{\sqrt{\bar{I}_{nX}(l)}} + \frac{1}{\sqrt{\bar{J}_{nX}(l)}}\right)\right)^2\right] \\
&\le 4\left(\mathbb{E}\left[\frac{\bar{\delta}_n(l)}{\bar{I}_{nX}(l)}\right] + \mathbb{E}\left[\frac{\bar{\delta}_n(l)}{\bar{J}_{nX}(l)}\right] + 2\mathbb{E}\left[\frac{|\bar{\delta}_n(l)|}{\sqrt{\bar{I}_{nX}(l)\bar{J}_{nX}(l)}}\right]\right) \\
&\le 4\left(\left(\mathbb{E}\left[\bar{\delta}_n(l)^2\right]\mathbb{E}\left[\frac{1}{\bar{I}_{nX}(l)^2}\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\bar{\delta}_n(l)^2\right]\mathbb{E}\left[\frac{1}{\bar{J}_{nX}(l)^2}\right]\right)^{\frac{1}{2}} + 2\mathbb{E}\left[\frac{|\bar{\delta}_n(l)|}{\sqrt{\bar{I}_{nX}(l)\bar{J}_{nX}(l)}}\right]\right).
\end{aligned}$$

The order of this last expression is given by the one of $\sqrt{\mathbb{E}\left[\bar{\delta}_n(l)^2\right]}$ which is not summable but tends to zero.

That finally gives $\frac{2}{m}\sum_{l\in A_m} U_l^2 \to 0$ in P_{θ_0} -probability.

Thus, we get the convergence in L^1 (and so in probability) of the contrast process towards the same contrast function as in **A**, which implies **A'**.

Also, we supply criterion **C** by **C'**:

$$\lim_{m,K\to\infty} \mathbb{P}_{\theta_0}(W_{m,K}(\frac{1}{k}) \geq \varepsilon_k) = 0$$

which will be obtained in a same way.

(ii) For sake of simplicity, we give the proof of (ii) for θ scalar. The extension to the vectorial case is straightforward.

Looking at the proof of Theorem 4, as the result of Priestley is given without Gaussianity assumption but needs only independence and equidistribution with eighth moment, we see that under $(\mathbf{H'}_\varepsilon)$, $(\mathbf{H'}_X)$ and $(\mathbf{C}_3)$, we can report to regression (4). This means that, keeping the notations of Theorem 4, $\sqrt{n}V^{(1)}_{m,K}(\theta) = T_1 + T_2 + T_3$, the previous remark implies $T_3 \xrightarrow{L^1} 0$.

• We have $\bar{I}_{n\varepsilon}(l) - 1 = n^{-1}\sum_{t=1}^{n}(\varepsilon_t^2 - 1) + n^{-1}\sum_{\substack{s,t=1\\ s\neq t}}^{n} \varepsilon_t\varepsilon_s \, a_l(t-s)$ with $a_l(j) = K^{-1}\sum_{k\in\mathcal{I}_l}\cos 2\pi kj/n$.

Then $\mathrm{Cov}(\bar{I}_{n\varepsilon}(l) - 1 \ , \ \bar{I}_{n\varepsilon}(l') - 1) = (\mu_4 - 3)/n + \Delta_{l,l'}$ with $\Delta_{l,l'} = (2/n^2)\sum_{s,t=1}^{n} a_l(t-s)\, a_{l'}(t-s)$.

We know that in the Gaussian case, this is zero if $l \neq l'$ and K^{-1} if $l = l'$. Then $\Delta_{l,l'} = \delta_{l,l'}/K$. Therefore,

$$\mathrm{Cov}(\bar{I}_{n\varepsilon}(l) - 1 \ , \ \bar{I}_{n\varepsilon}(l') - 1) = \frac{\mu_4 - 3}{n} + \frac{\delta_{l,l'}}{K} \tag{14}$$

We need now the following lemma which is the analogue of Lemma 2:

Lemma 5 *Under* $(\mathbf{H'}_\varepsilon)$, $\left\|\ln \bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l) - 1)\right\|_1 = O(\frac{1}{K})$

Thus, we can consider again $\bar{I}_{n\varepsilon}(l) - 1$ in place of $\ln \bar{I}_{n\varepsilon}(l)$ under $(\mathbf{C}_1)$, $(\mathbf{C}_2)$ and $(\mathbf{H'}_\varepsilon)$. So that, $T_2 \xrightarrow{P} 0$.

• Let h be a continuous function defined on $[-\pi, \pi]$, symmetric and Riemann square integrable. Let $L_n^{(m)} = (2\sqrt{n}/m)\sum_{l\in A_m} h(\frac{2\pi l}{m})\, (\bar{I}_{n\varepsilon}(l) - 1)$.

According to (14), $\mathrm{Var}L_n^{(m)}$ tends to $4\Delta^2(h) = 4\left[(\mu_4 - 3)(\frac{1}{2\pi}\int_0^\pi h(x)dx)^2 + \frac{1}{2\pi}\int_0^\pi h^2(x)dx\right]$.

Applying this result to $T_1 = -2L_n^{(m)} - (4\sqrt{n}/m)\sum_{h\in A_m} h(\frac{2\pi l}{m})\ln(1+F_l)$ with $h(\frac{2\pi l}{m}) = \frac{\partial\tilde{g}}{\partial\theta}(\theta_0, l)$ and as we know from the proof of Theorem 2 that the second term has no weight in the asymptotic variance, we have $\lim_{n\to\infty}\mathrm{Var}T_1 = 16J(\theta_0)$. The following weak convergence result:

Lemma 6 $L_n^{(m)} \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0, 4\Delta^2(h))$.

implies the convergence in law of T_1 to $\mathcal{N}(0, 16J(\theta_0))$, as its second term tends to 0 in probability. On the other hand, $\mathbb{E}[V_{m,K}^{(2)}(\theta_0)] \underset{n\to\infty}{\longrightarrow} 4I(\theta_0)$ and Var $V_{m,K}^{(2)}(\theta_0)$ tends to zero with (14).

This ensures Theorem 6. ■

Proof of Lemma 4: We still write $\bar{I}_{n\varepsilon}(l) - 1 = n^{-1}\sum_{t=1}^{n}(\varepsilon_t^2 - 1) + n^{-1}\sum_{\substack{s,t=1\\ s\neq t}}^{n} \varepsilon_t\varepsilon_s\, a_l(t-s)$. Then we calculate $\left(\bar{I}_{n\varepsilon}(l) - 1\right)^4$. Since the ε are independent variables $(0,1)$, we find that the order of the expectation will be given by terms involving the square of $\Delta_{l,l}$:

$$\Delta_{l,l}^2 = \left((2/n^2)\sum_{s,t=1}^{n} a_l(t-s)^2\right)^2 = \frac{1}{K^2}.$$

Then we get the result with **(H'$_\varepsilon$)** and $\mathbb{E}[\ln \bar{I}_{n\varepsilon}(l)^2] \leq \left(\mathbb{E}[(\bar{I}_{n\varepsilon}(l)-1)^4]\mathbb{E}[(\frac{1}{\bar{I}_{n\varepsilon}(l)}+1)^4]\right)^{\frac{1}{2}}$. ■

Proof of Lemma 5:

- $\mathbb{E}\left|\ln \bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l) - 1)\right| = \mathbb{E}\left|\left(\ln \bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l) - 1)\right) 1_{\bar{I}_{n\varepsilon}(l)\geq\frac{1}{2}}\right|$
$+\mathbb{E}\left|\left(\ln \bar{I}_{n\varepsilon}(l) - (\bar{I}_{n\varepsilon}(l) - 1)\right) 1_{\bar{I}_{n\varepsilon}(l)<\frac{1}{2}}\right|$

Because $g(z) = z - 1 - \ln z - (z-1)^2 \leq 0$ on $[\frac{1}{2}, \infty[$, the first term is less or equal to $2\mathbb{E}\left(\bar{I}_{n\varepsilon}(l) - 1\right)^2$ which is of order $O(\frac{1}{K})$ with (14).

- Let $Z = \dfrac{1}{\bar{I}_{n\varepsilon}(l)}$. Then the second term, say E_2, is equal to $\mathbb{E}\left(\frac{1}{Z} - 1 + \ln Z\right) 1_{Z>2} = \mathbb{E}\left(\frac{Z\ln Z - Z + 1}{Z}\right) 1_{Z>2}$.

Now we use $g(z) = z\ln z - z + 1 - z(z-1) \leq 0$ for $z \geq 1$ and then $E_2 \leq \mathbb{E}[(Z-1)1_{Z>2}]$. So that we have:

$$\begin{aligned} E_2 &\leq \mathbb{E}\left[\left(\frac{1}{\bar{I}_{n\varepsilon}(l)} - 1\right) 1_{\bar{I}_{n\varepsilon}(l)<\frac{1}{2}}\right] \leq \mathbb{E}\left[\left(\frac{1-\bar{I}_{n\varepsilon}(l)}{\bar{I}_{n\varepsilon}(l)}\right)^2\right]^{\frac{1}{2}} \mathbb{P}\left[\bar{I}_{n\varepsilon}(l) < \frac{1}{2}\right]^{\frac{1}{2}} \\ &\leq \mathbb{E}\left[\left(\bar{I}_{n\varepsilon}(l) - 1\right)^4\right]^{\frac{1}{4}} \mathbb{E}\left[\frac{1}{\bar{I}_{n\varepsilon}(l)^4}\right]^{\frac{1}{4}} \mathbb{P}\left[\bar{I}_{n\varepsilon}(l) < \frac{1}{2}\right]^{\frac{1}{2}} \end{aligned}$$

Under **(H'$_\varepsilon$)**, $\mathbb{E}\left[\dfrac{1}{\bar{I}_{n\varepsilon}(l)^4}\right]$ is constant; then $\mathbb{P}\left[\bar{I}_{n\varepsilon}(l) < \frac{1}{2}\right] \leq \mathbb{P}\left[\left|\bar{I}_{n\varepsilon}(l) - 1\right| > \frac{1}{2}\right] \leq 4\mathbb{E}\left[\left(\bar{I}_{n\varepsilon}(l) - 1\right)^2\right]$ which is of order $\dfrac{1}{K}$ with (14), and we know from the proof of Lemma 4 that $\mathbb{E}\left[\left(\bar{I}_{n\varepsilon}(l) - 1\right)^4\right] = O\left(\frac{1}{K^2}\right)$ which ends the proof. ■

Proof of Lemma 6: - We have $L_n^{(m)} = \sqrt{n}\left[\mathcal{A}_m(0)(C_n(0) - 1) + 2\sum_{j=1}^{n-1}\mathcal{A}_m(j)(1 - j/n)C_n(j)\right]$ with $\mathcal{A}_m(j) = (2/m)\sum_{l\in A_m} h(\frac{2\pi l}{m})a_l(j)$, and $C_n(j) = (n-j)^{-1}\sum_{t=1}^{n-j}\varepsilon_t\varepsilon_{t+j}$, $j = 0$ *to* $n-1$.

Then Var $L_n^{(m)} = (\mu_4 - 1)\,\mathcal{A}_m^2(0) + 4\sum_{j=1}^{n-1}(1 - j/n)\,\mathcal{A}_m^2(j)$ which tends to $4\Delta^2(h)$ with (14).

Note that we can also write

$$\mathrm{Var} L_n^{(m)} = 4(\mu_4 - 1)\left(m^{-1}\sum_{l\in A_m} h(\tfrac{2\pi l}{m})\right)^2 + 4\left[m^{-1}\sum_{l\in A_m} h^2(\tfrac{2\pi l}{m}) - 2\left(m^{-1}\sum_{l\in A_m} h(\tfrac{2\pi l}{m})\right)^2\right].$$

- $\forall j \in \{0,1,\ldots,n-1\}$, $\mathcal{A}_m(j) \underset{m\to\infty}{\longrightarrow} \mathcal{A}(j) = \frac{1}{\pi}\int_0^\pi h(x)\ \cos jx\ dx.$

Let $\mathcal{B}_m(j) = \sqrt{1-j/n}\mathcal{A}_m(j)$. Note that, for any fixed j, $j < n$, $\mathcal{B}_m(j) \underset{m\to+\infty}{\longrightarrow} \mathcal{A}(j)$.

Let $L_n^{(\infty)} = \sqrt{n}\Big(\mathcal{A}(0)(C_n(0)-1) + 2\sum_{j=1}^{n-1}\mathcal{A}(j)\sqrt{1-j/n}C_n(j)\Big)$.

We also have Var $L_n^{(\infty)} = (\mu_4-1)\mathcal{A}^2(0) + 4\sum_{j=1}^{n-1}\mathcal{A}^2(j)$.

From $\frac{1}{\pi}\int_0^\pi h^2(x)\ dx = \mathcal{A}^2(0) + 2\sum_{j\ge 1}\mathcal{A}^2(j)$, we obtain Var $L_n^{(\infty)} \underset{n\to\infty}{\longrightarrow} 4\Delta^2(h)$.

- So we have a sequence of $l^2(\mathbb{N})$ $\mathcal{B}_m = (\mathcal{B}_m(j)\ ,\ j = 0,\ldots,n-1)$, $\mathcal{B}_m(j) = 0$ if $j \ge n$, such that

$$\|\mathcal{B}_m\|^2 \underset{m\to\infty}{\longrightarrow} C(h) = \frac{1}{2\pi}\int_0^\pi h^2(x)\ dx + 2\Big(\frac{1}{2\pi}\int_0^\pi h(x)\ dx\Big)^2,$$

and one element of $l^2(\mathbb{N})$, $\mathcal{A} = (\mathcal{A}(j)\ ,\ j\ge 0)$ such that $\|\mathcal{A}\|^2 = C(h)$.
Moreover, $\mathcal{B}_m(j) \to \mathcal{A}(j)$ for every fixed j. Using the following result:

Lemma 7 *$\mathcal{B}_m \underset{n\to\infty}{\longrightarrow} \mathcal{A}$ in $l^2(\mathbb{N})$*

$$\text{We have: } \mathrm{Var}(L_n^{(m)} - L_n^{(\infty)}) = (\mu_4-1)(\mathcal{B}_m(0)-\mathcal{A}(0))^2 + 4\sum_{j=1}^{n-1}(\mathcal{B}_m(j)-\mathcal{A}(j))^2$$
$$\le \sup\{\mu_4 - 1\ ,\ 4\}\ \|\mathcal{B}_m - \mathcal{A}\|^2 \quad \text{which tends to zero.}$$

- Finally, let

$$Y_{n,p} = \sqrt{n}\Big(\mathcal{A}(0)(C_n(0)-1) + 2\sum_{j=1}^{p}\mathcal{A}(j)\sqrt{1-j/n}C_n(j)\Big).$$

Because of the normal convergence of the $(\sqrt{n}(C_n(0)-1)\ ,\ \sqrt{n}C_n(j)\ ,\ j\ge 1)$, as stated in Corollary 8.4.1 of Theorem 8.4.2 of Anderson (1971), we know that $Y_{n,p} \xrightarrow{\mathcal{L}} Y_p \sim \mathcal{N}(0,\sigma_p^2)$, with $\sigma_p^2 \to 4\Delta^2(h)$; then $Y_p \xrightarrow{\mathcal{L}} Y \sim \mathcal{N}(0, 4\Delta^2(h))$. As $\mathrm{Var}(L_n^\infty - Y_{n,p}) = 4\sum_{j>p}\mathcal{A}^2(j) \to 0$ uniformly in n when $p\to+\infty$, Proposition 6.3.9 in Brockwell and Davis (1991) implies that $L_n^{(\infty)} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4\Delta^2(h))$. ∎

Proof of Lemma 7: We have

$$\|\mathcal{B}_m - \mathcal{A}\|^2 = \sum_{j=0}^{\infty}(\mathcal{B}_m(j)-\mathcal{A}(j))^2 = \sum_{j=0}^{n-1}\mathcal{B}_m(j)^2 + \sum_{j=0}^{\infty}\mathcal{A}(j)^2 - 2\sum_{j=0}^{n-1}\sqrt{1-j/n}\mathcal{A}_m(j)\mathcal{A}(j).$$

We know that

$$\lim_{n\to+\infty}\sum_{j=0}^{n-1}\mathcal{B}_m(j)^2 = C(h) \quad \text{and} \quad \sum_{j=0}^{\infty}\mathcal{A}(j)^2 = C(h).$$

We have to prove that the cross product term has the same limit. For that purpose, we write

$$\mathcal{A}(j) = \frac{1}{\pi}\int_0^{\pi} h(x)\cos(jx)dx = \frac{1}{\pi}\int_0^{\pi n} h(\tfrac{u}{n})\cos(u\tfrac{j}{n})du/n,$$

so that

$$\sum_{j=0}^{n-1}\sqrt{1-\tfrac{j}{n}}\mathcal{A}_m(j)\mathcal{A}(j) = \frac{2}{\pi m K}\sum_{l\in A_m}\sum_{k\in\mathcal{I}_l} h(\tfrac{2\pi l}{m})\int_0^{\pi n} h(\tfrac{u}{n})\left(\frac{1}{n}\sum_{j=0}^{n-1}\sqrt{1-\tfrac{j}{n}}\cos(2\pi k\tfrac{j}{n})\cos(u\tfrac{j}{n})\right)du$$

$$\frac{1}{n}\sum_{j=0}^{n-1}\sqrt{1-\tfrac{j}{n}}\cos(2\pi k\tfrac{j}{n})\cos(u\tfrac{j}{n}) = \int_0^1\sqrt{1-y}\cos(2\pi k y)\cos(uy)dy + \tfrac{1}{2n} + O(n^{-2}),$$

with the trapezoidal rule, and we find that

$$\lim_{n\to+\infty}\sum_{j=0}^{n-1}\sqrt{1-j/n}\mathcal{A}_m(j)\mathcal{A}(j) = 2\left(\frac{1}{2\pi}\int_0^{\pi} h(x)dx\right)^2 + \lim_{n\to+\infty} T_n$$

where

$$T_n = (2/\pi m K)\sum_{l\in A_m}\sum_{k\in\mathcal{I}_l} h(\tfrac{2\pi l}{m})\int_0^{\pi n} h(\tfrac{u}{n})\left(\int_0^n\sqrt{1-z/n}\cos(\tfrac{uz}{n})\cos(\tfrac{2\pi k z}{n})dz/n\right)du$$

with the change of variable $y = z/n$.

$$T_n = \frac{2}{\pi}\int_0^n\sqrt{1-z/n}(1/mK)\sum_{l\in A_m} h(\tfrac{2\pi l}{m})\sum_{k\in\mathcal{I}_l}\cos(\tfrac{2\pi k z}{n})\int_0^{\pi} h(v)\cos(vz)dv\; dz$$

with the change of variable $v = u/n$.

Then as

$$(1/mK)\sum_{l\in A_m} h(\tfrac{2\pi l}{m})\sum_{k\in\mathcal{I}_l}\cos(\tfrac{2\pi z k}{n}) = \frac{1}{2\pi}\int_0^{\pi} h(x)\cos(zx)dx + O(n^{-1}),$$

we find that

$$\lim_{n\to+\infty} T_n = \lim_{n\to+\infty} 4\int_0^n\sqrt{1-z/n}\left(\frac{1}{2\pi}\int_0^{\pi} h(x)\cos(zx)dx\right)^2 dz.$$

Then the Plancherel formula for continuous Fourier Transform applied to $\tilde{h}(x) = h(x)1_{[-\pi,\pi]}$ gives

$$4\int_0^{+\infty}\left(\frac{1}{2\pi}\int_0^{\pi} h(x)\cos(ux)dx\right)^2 du = \frac{1}{2\pi}\int_0^{\pi} h(u)^2 du.$$

Then as $\sqrt{1-z/n}$ is dominated by 1, Lebesgue Theorem implies that

$$\lim_{n\to+\infty}\sum_{j=0}^{n-1}\sqrt{1-j/n}\mathcal{A}_m(j)\mathcal{A}(j) = 2\left(\frac{1}{2\pi}\int_0^{\pi} h(x)dx\right)^2 + \frac{1}{2\pi}\int_0^{\pi} h^2(x)dx \quad \blacksquare$$

5 Simulation results

We give simulation results obtained both in the Gaussian and the non Gaussian (ε uniform $\mathcal{U}(0,1)$) cases, and with both Whittle and LRP methods.

The process we consider is a non-long memory process we call FRAC(1) because of an apparent fractional feature: $(1-\alpha L)^d X_t = \varepsilon_t$, $|\alpha| < 1$, ε_t iid $(0, \sigma^2)$. The spectral density can be written here $f(\lambda; d, \alpha) = \sigma_\varepsilon^2 |1 - \alpha e^{i\lambda}|^{-2d} = (1 + \alpha^2 - 2\alpha\cos\lambda)^{-d}\sigma_\varepsilon^2$.

From model (1), we generate truncated series $X_t = \sum_{j=0}^{T} a_j \varepsilon_{t-j}$ with $T = 8000$ (instead of $T = +\infty$), the a_j's being computed recursively (see Goncalves (1987)); more precisely, for an MA process with transfer function $A(z) = \sum_{j=0}^{+\infty} a_j z^j$, let B and C be polynomials with no common roots such that: $\frac{1}{A(z)}\frac{dA(z)}{dz} = \frac{B(z)}{C(z)}$, $B(z) = b_0 + b_1 z + \ldots + b_{K-1}z^{K-1}$ and $C(z) = c_0 + c_1 z + \ldots + c_K z^K, c_K \neq 0$. Then the a_j's satisfy the equation:

$$(c_0 j)a_j + (-b_0 + c_1(j-1))a_{j-1} + \ldots + (-b_{K-1} + c_K(j-K))a_{j-K} = 0.$$

We have, for the FRAC(1) process: $\frac{1}{A(z)}\frac{dA(z)}{dz} = \frac{\alpha d}{1-\alpha z}$, where $A(z) = (1-\alpha z)^{-d}$; so we obtain $a_0 = 1$, $a_j = \alpha(j-1+d)a_{j-1}/j$, $\quad j \geq 1$.

Then we estimate the parameters d, α and σ_ε^2 by the two methods. For instance, Whittle's contrast can here be written:

$$L_n^W = \ln\sigma_\varepsilon^2 + \frac{1}{2\pi\sigma_\varepsilon^2}\int_{-\pi}^{\pi} I_{nX}(\lambda)|1-\alpha e^{-i\lambda}|^{2d}d\lambda = \ln\sigma_\varepsilon^2 + \frac{1}{\sigma_\varepsilon^2}\int_{-1/2}^{1/2} I_{nX}(2\pi\omega)|1-\alpha e^{-2i\pi\omega}|^{2d}d\omega,$$

so that $\hat{d}_W$ and $\hat{\alpha}_W$ minimize the integral term, and then $\hat{\sigma}^2_{\varepsilon W} = \int_{-1/2}^{1/2} I_{nX}(2\pi\omega)\times$ $|1 - \hat{\alpha}_W e^{-2i\pi\omega}|^{2\hat{d}_W}d\omega$.

On the other hand, we obtain $\hat{d}_{LRP}$, $\hat{\alpha}_{LRP}$ and $\hat{\sigma}^2_{\varepsilon LRP}$ from the following regression equation:

$$\ln \bar{I}_{nX}(l) = \ln\sigma_\varepsilon^2 + d\tilde{g}(\alpha, l) + e_l \qquad l = 0, \ldots, m/2 - 1$$

where here $\tilde{g}(\alpha, l) = -\ln|1 + \alpha^2 - 2\alpha\cos 2\pi(lK + \frac{K+1}{2}/n|)$.
α is sampled by 10^{-2} steps; for each value we calculate d and choose $(\hat{\alpha}_{LRP}, \hat{d}_{LRP}, \ln\hat{\sigma}^2_{\varepsilon LRP})$ which minimize $\sum e_l^2$.

Table 1 gives the results obtained from 100 simulations, $n = 4096$, $m = K = 64$. Note that here, the choice $m = K$ is a limiting case that is clearly not fulfilling condition (C_2) and that the regressions are performed with only about 30 data.

Finally, we would further point out that we have also applied the "log-periodogram" regression (LP) proposed by Kashyap and Eom (1988); we obtained good results, also illustrating the $\pi^2/6$ ratio between the LRP (and Whittle) and LP estimators' variances.

	FRAC(1)					
	Gaussian			Non Gaussian ($\mathcal{U}(0,1)$)		
	W	LP	LRP	W	LP	LRP
$d = 0.30$	0.2980	0.3005	0.2981	0.3000	0.2999	0.3013
DSE $\times 10^3$	1.41	2.38	1.46	8.38	16.5	8.20
α =0.80	0.8022	0.8063	0.8029	0.7975	0.7968	0.7961
DSE $\times 10^3$	3.90	6.20	4.01	2.88	5.41	2.85
$\sigma^2 = 1$	1.0014	1.0027	0.9933	1.0009	1.0036	0.9936
DSE $\times 10^4$	2.11	7.66	5.33	1.87	4.88	2.44
$d = 0.45$	0.4689	0.4719	0.4696	0.4535	0.4441	0.4548
DSE $\times 10^3$	7.64	10.4	7.91	4.41	9.20	4.73
α =0.60	0.5940	0.5867	0.5941	0.6051	0.6202	0.6044
DSE $\times 10^3$	6.94	12.1	7.22	6.01	10.2	6.24
$\sigma^2 = 1$	1.0017	1.0007	0.9944	0.9997	1.0005	0.9925
DSE $\times 10^4$	4.01	6.89	4.73	2.02	5.13	2.74

Table 1: Simulation results for a FRAC(1),
W: Whittle method, LP: Log-periodogram method, LRP: Log Regularized Periodogram method.
$n = 4096, m = K = 64$, 100 simulations.

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