THE COVARIANCE EXTENSION PROBLEM

José Marcano
Universidad de Carabobo

María D. Morán & José R. León
Universidad Central de Venezuela

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A common problem in practice, is to obtain, as a result of any collecting data process in time series studies, a finite complex sequence $\{c_k\}_{k=-p}^{p}$ $p \in \mathbb{N}$ and try to know when such a sequence is the first covariance function (or matrix) coefficients. The mathematical formulation is as follow: "Under what conditions over $\{c_k\}_{k=p}^{p}$ there is at least a mesure (or a matrix mesure) on the unit circle $\mathbb{T}$ μ such that

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} \mu(dx) \qquad k=-p,\dots,p \qquad (1)"$$

In such a case we have that $c_{-k} = \overline{c_k}$ $k \in \{-p,\dots,p\}$. This problem has a long history.

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In 1911, Toeplitz dealed with the case that the sequence is of the form $\{c_k\}_{k=0}^{\infty}$, he proved that if the solution exists it is unique. Problem (1), can be seen as a generalization of Toeplitz's problem, but now the solution, if it exists does not has to be unique. Several works since then dealed with this problem, using mainly Operator Theory.

In 1988 Dym and in 1989 Woederman described partially the solutions.

Since the solution of (1) is not unique, it is important to find the one which maximizes the Burg an Krein entropy functionals:

$$\varepsilon(f) = \frac{1}{2\pi}\int_0^{2\pi} \log f(e^{it})\, dt$$

$$\varepsilon_a(f) = \frac{1}{2\pi}\int_0^{2\pi} \log f(e^{it}) \frac{1-|a|^2}{|1-a\bar{e}^{it}|^2}\, dt \quad |a|<1,\ a\in\mathbb{C}$$

In that follows we study these two problems, both in the onedimensional case as in the vectoriel case, using a model of Operator Theory named "Arov-Grossman model".

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We associate to a given finite set $\{c_k\}_{k=0}^{p}$, that can been thought as the first p autocorrelation coefficients of a second order centered stationary process $\{X_n\}_{n\in\mathbf{Z}}$, an isometry V acting on a Hilbert space, and that some unitary extension U of V, generates a process X such that the spectrum f verifies:

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(e^{it}) \,, \quad k=-p,\dots,p \quad (2)$$

The Arov – Grossman's model provides a description of all different spectrum f of X verifying (2).

This description is given by the 1–1 correspondence between such set and a subset of the open unitary ball of $H^\infty(\mathbb{D})$, the set of all analytic and essentially bounded functions defined on $\mathbb{D} = \{z\in\mathbb{C} : |z|\leq 1\}$.

Organization of the talk:

1. Onedimensional problem. Arov-Grossman representation.
2. Levinson's & Schur's algorithm.

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3. Burg & Krein maximun entropy problem in between with Covariane extension problem.

4. Multidimensional covariance matrix extension problem. Burg's and Krein entropy and their solutions.

5. Finals remarks.

A sequence $\{c_k\}_{k=-p}^{p} \subset \mathbb{C}$ is said to be strictly definite positive (sdp) if and only if

$$\sum_{n=0}^{p} \sum_{m=0}^{p} \lambda_n \overline{\lambda_m} C_{n-m} > 0 , \quad \{\lambda_n\}_{n=0}^{p} \subset \mathbb{C} - \{0\} \quad (3)$$

If $\{c_k\}_{k=-p}^{p}$ is sdp of complex numbers, we can introduce an inner product in E_p, the trigonometric polynomials of degree $k \leq p$, let $e_k(\zeta) := \zeta^k$ for $f = \sum_{k=0}^{p} a_k e_k$ and $g = \sum_{k=0}^{p} b_k e_k$

$$\langle f, g \rangle_p = \sum_{n=0}^{p} \sum_{m=0}^{p} a_n \overline{b_m} C_{m-n}$$

As a cousequence of (3) $(E_p, \langle , \rangle_p)$ is a $(p+1)$-dimensional Hilbert space. We define $\Gamma_p : (E_p, \langle , \rangle) \to (E_p, \langle , \rangle)$ by

$$\langle \Gamma_p f, g \rangle := \langle f, g \rangle_p \qquad f, g \in E_p$$

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Γ_p is a linear operator and $\|\Gamma_p\| \leq 1$. We can conclude the following classical lemma.

Lemma 1: Let $D_p = \text{Span}\{e_k\}_{k=0}^{p-1}$, $R_p = \text{Span}\{e_k\}_{k=1}^{p}$ be subspaces of E_p and set $V_p : D_p \to R_p$ defined by

$(V_p f)(\zeta) = \zeta f(\zeta)$ $\zeta \in T$, $f \in D_p$. Then

(a) V_p is an isometry acting on the space $(E_p, \langle , \rangle_p)$

(b) The orthogonal complement of D_p, $N_p = E_p \ominus D_p$ and the orthogonal complement of R_p, $M_p = E_p \ominus R_p$ have dimension one. Futhermore, N_p and M_p are spanned by:

$$n_p(\zeta) := \frac{\Gamma_p^{-1} e_p}{\|\Gamma_p^{-1} e_p\|_p} \quad \text{and} \quad m_p(\zeta) := \frac{\Gamma_p^{-1} e_0}{\|\Gamma_p^{-1} e_0\|_p}$$

respectively.

(Now if $P_{N_p}^{E_p}$ denotes the orthogonal projection $E_p \to N_p$)

(c) $P_{N_p}^{E_p} e_p = \dfrac{n_p}{\widehat{n_p(p)}} = (1 - P_{D_p}^{E_p}) e_p$ $\quad \widehat{n}_p(p)$ is the p-th Fourier coefficient of

Proof:

(a) If $\Gamma_p f = 0 \Rightarrow \langle f, f \rangle_p = 0 \Rightarrow f = 0$ Γ_p is injective hence inversible.

(b) $N_p = E_p \ominus D_p$ if $f \in N_p$ and $k = \{0, \dots, p-1\}$ we have

$$\langle \Gamma_p f, e_k \rangle = \langle f, e_k \rangle_p = 0 \Rightarrow \Gamma_p f = \lambda e_p \Rightarrow f = \lambda \Gamma_p^{-1} e_p$$

it yields $N_p = \text{span}\left\{ \dfrac{\Gamma_p^{-1} e_p}{\|\Gamma_p^{-1} e_p\|_p} \right\} = \text{span}\{n_p\}$

The same thing can be proved for M_p.

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(c) $\hat{m}_p(p) = \left\langle \frac{\Gamma_p^{-1} e_p}{\|\Gamma_p^{-1} e_p\|_p}, e_p \right\rangle = \|\Gamma_p^{-1} e_p\|_p$

$P^{E_p}_{\mathcal{H}_p} e_p = \langle n_p, e_p \rangle_p n_p = \langle \Gamma_p n_p, e_p \rangle n_p = \frac{n_p}{\|\Gamma_p^{-1} e_p\|_p} = \frac{n_p}{\hat{n}_p(p)}$

The following lemma establishes a connection between n_p and m_p and also shows where the zeros of both functions lie

Lemma 2: On the same hypothesis of lemma 1, we have:

(a) $\Gamma_p^{-1} e_p = e_p \overline{\Gamma_p^{-1} e_0}$

(b) $n_p = e_p \overline{m_p}$ that is $n_p = \overline{\widehat{m(p)}} + \overline{\widehat{m_p(p-1)}} e_1 + \dots + \overline{\widehat{m_p(0)}} e_p$

where $\hat{m}_p(j)$ is the j-th m_p coefficient.

(c) All the zeros of $n_p(z)$ and $m_p(z)$ lie in $|z|<1$ and $|z|>1$ respectively

Proof:

(a) $\Gamma_p^{-1} e_0 = \sum_{n=0}^{p} a_n e_n$ so $\langle \Gamma_p^{-1} e_0, e_k \rangle_p = \langle e_0, e_k \rangle = \delta_p(k) = \langle \overline{e_p \Gamma_p^{-1}(e_0)}, e_k \rangle_p$

(b) Is a consequence of the definitions.

(c) Suppose that γ is a zero of n_p, there exists $S_{p-1} \in E_{p-1}$ such that $n_p(z) = (z-\gamma) S_{p-1}(z)$

or equivalently

$$n_p(z) + \gamma S_{p-1}(z) = z S_{p-1}(z)$$

Since $n_p \perp E_{p-1}$ and V_p is an isometry

$\|n_p + \gamma S_{p-1}\|_p^2 = \langle n_p + \gamma S_{p-1}, n_p + \gamma S_{p-1} \rangle_p = \|n_p\|_p^2 + |\gamma|^2 \|S_{p-1}\|_p^2$

which yields

$$\|n_p\|_p^2 + |\gamma|^2 \|S_{p-1}\|_p^2 = \|z S_{p-1}(z)\|_p = \|V_p S_{p-1}\|_p^2 = \|S_{p-1}\|_p^2$$

whence $1 - |\gamma|^2 = \frac{1}{\|S_{p-1}\|_p} > 0$ as required. The result for m_p is similar.

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2. Levinson's and Shur's algorithms.

Proposition 3: For each $p \in \mathbb{N}$, $p \geq 2$ there exists γ_p such that

$$\frac{n_p}{\widehat{n_p(p)}} = \frac{\zeta\, n_{p-1}}{\widehat{n}_{p-1}(p-1)} - \gamma_p \frac{m_{p-1}}{\widehat{m_{p-1}(p-1)}}$$

$$\frac{m_p}{\widehat{m_p(p)}} = \frac{m_{p-1}}{\widehat{m_{p-1}(p-1)}} - \bar{\gamma}_p \frac{\zeta\, n_{p-1}}{\widehat{n_{p-1}(p-1)}} \qquad (4)$$

where $\frac{n_1}{\widehat{n_1(1)}} = e_1 e_0 - \bar{e}_1 e_0$, $\quad m_1 = e_1 \bar{n}_1$. Futhermore

$$|\widehat{m_p}(p)|^2 = \frac{|\widehat{n_{p-1}(p-1)}|^2}{1-|\gamma|^2}$$

Proof: To obtain the last equality we can rewrite (4) as

$$\frac{n_p}{\widehat{n_p(p)}} + \gamma \frac{m_{p-1}}{\widehat{m_{p-1}(p-1)}} = \frac{\zeta\, n_{p-1}}{\widehat{n_{p-1}(p-1)}}$$

thus

$$\left\| \frac{n_p}{\widehat{n}_p(p)} + \gamma_p \frac{m_{p-1}}{\widehat{m}_{p-1}(p-1)} \right\|_p^2 = \left\| V_p \frac{m_{p-1}}{\widehat{m_{p-1}}(p-1)} \right\|_p^2$$

by using the fact that V_p is an isometry and $n_p \perp m_{p-1}$ we find

$$\frac{1}{|\widehat{n}_p(p)|^2} = \frac{1-|\gamma|^2}{|\widehat{n}_{p-1}(p-1)|^2}$$

We have the orthogonal decomposition

$D_p = E_{p-1} = D_{p-1} \oplus N_{p-1} = R_{p-1} \oplus M_{p-1} = V_p D_{p-1} \oplus M_{p-1}$

(recall $V_p : D_p \to R_p$)

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hence

$$P^{E_p}_{D_p} e_p = P^{E_p}_{D_p}\left(V_p\, P^{E_{p-1}}_{D_{p-1}} e_{p-1} + V_p\, P^{E_{p-1}}_{N_{p-1}} e_{p-1}\right).$$

But $P^{E_p}_{D_p} V_p\, P^{E_{p-1}}_{N_{p-1}} e_{p-1} = P^{E_p}_{M_{p-1}} V_p \langle n_{p-1}, e_{p-1}\rangle_{p-1}\, n_{p-1} = \langle n_{p-1}, e_{p-1}\rangle_{p-1} P^{E_p}_{M_{p-1}} V_p n_{p-1}$

$$= \langle n_{p-1}, e_{p-1}\rangle_{p-1} \langle m_{p-1}, V_p n_{p-1}\rangle_p\, m_{p-1}$$

then

$$P^{E_p}_{D_p} e_p = V_p\, P^{E_{p-1}}_{D_{p-1}} e_{p-1} + \langle n_{p-1}, e_{p-1}\rangle_{p-1} \langle m_{p-1}, V_p n_{p-1}\rangle_p\, m_{p-1}$$

Thus

$$e_p - P^{E_p}_{D_p} e_p = V_p e_{p-1} - V_p\, P^{E_{p-1}}_{D_{p-1}} e_{p-1} - \frac{\langle m_{p-1}, V_p n_{p-1}\rangle_p}{\widehat{m_{p-1}}(p-1)}\, m_{p-1}$$

$$\frac{n_p}{\widehat{m_p}(p)} = \frac{\zeta\, n_{p-1}}{\widehat{n_{p-1}}(p-1)} - \frac{\gamma_p\, m_{p-1}}{\widehat{m_{p-1}}(p-1)}$$

where $\gamma_p = \langle m_{p-1}, V_p n_{p-1}\rangle_p$ and $e_p - P^{E_p}_{D_p} e_p = \frac{n_p}{\widehat{m_p}(p)}$ the other recursion follows from $n_p = e_p\, \overline{m_p}$.

Remark: The coefficients γ_p are called the Schur parameters, indeed if $G_p(z) := \frac{n_p(z)}{m_p(z)}$ we can write

$$\boxed{G_p(z) = \frac{z\, n_{p-1} - \gamma_p\, m_{p-1}}{m_{p-1} - \overline{\gamma_p}\, z\, n_{p-1}} = \frac{z\, G_{p-1}(z) - \gamma_p}{1 - \overline{\gamma_p}\, z\, G_{p-1}(z)}}$$

3. Covariance extension problem description of all the solutions.

$E_p = \{\text{trigonometric polynomials degree} \le p\}$

$V_p(e_k) = \zeta\, e_k(\zeta) \quad (k < p) \qquad V_p : D_p \to R_p$ is an isometry

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$N_p = E_p \ominus D_p$ and $M_p = E_p \ominus R_p$

The we define $U : F \to F$ as a minimal unitary extension of V

if U is unitary F is a Hilbert space and:

(i) $E_p \subset F$ (ii) $U\big|_{D_p} = V_p$ (iii) $F = \underset{n \in \mathbb{Z}}{\operatorname{span}}\, U^n(E_p)$.

The Arov & Grossman functional model establishes the existence of a bijection between the unitary extensions of the isometry $V_p : D_p \to R_p$ and the bounded analytical functions $\Theta : D \to L(N_p, M_p)$, ($L(N_p, M_p)$ are the linear operator between N_p and M_p), defined as

$$\Theta(z) = P^F_{M_p} U (I - z P^F_{F \ominus E_p} U)^{-1}\Big|_{N_p} \qquad \|\Theta\|_\infty \le 1.$$

In fact N_p and M_p have dimension one and since $L(N_p, M_p) \cong \mathbb{C}$ we shall associate to each Θ a function $H : D \to \mathbb{C}$ analytical and with $\|H\|_\infty \le 1$. We have the following

Proposition: Given $H \in H^\infty(D)$ such that $\|H\|_\infty \le 1$, let μ^H be the spectral mesure of U_H the minimal extension of $V_p : D_p \to R_p$ associated to H.

If $\mu_H(t) = \langle \mu^H e_0, e_0 \rangle$ then μ_H verifies:

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$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu_H(t)=\frac{2zH(z)}{m_p(z)-zH(z)n_p(z)}\,\frac{z^p}{m_p(z)}+\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,\frac{1}{|m_p(e^{it})|^2}\,dt$$

Proof: It follows of algebraic manipulations of $H(z)$ and the propierties of the spectral mesure of U.

This proposition provides "the universal formula" for all of spectral densities whose first coefficients are $\{c_k\}_{k=0}^{p}$.

Corollary: Given $H\in H^\infty(D)$ with $\|H\|_\infty\leq 1$, let μ_H the mesure defined before. The mesure μ_H is absolutely continuous with respect to the Lebesgue mesure on T, with density f_H given by

$$f_H(\zeta)=\frac{1}{|m_p(\zeta)|^2}\,\mathrm{Re}\left[\frac{m_p(\zeta)+\zeta H(\zeta)n_p(\zeta)}{m_p(\zeta)-\zeta H(\zeta)n_p(\zeta)}\right]$$

if and only if $\lambda\{\zeta\in T:\ |H(\zeta)|=1\}=0$ and $\dfrac{1}{m_p-\zeta H n_p}\in L^2$

Proof: It can be seen that $m_p(z)-zH(z)n_p(z)\neq 0,\ z\in D$

$\Rightarrow \{\zeta\in T:\ m(\zeta)-\zeta H(\zeta)=0\}$ has Lebesgue mesure zero.

Therefore, from the last proposition

$$\lim_{z\to\zeta}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\frac{e^{it}+z}{e^{it}-z}\,d\mu_H(t)=\frac{1}{|m_p(\zeta)|^2}+\mathrm{Re}\left[\frac{2H(\zeta)\zeta^{p+1}}{m_p(\zeta)[m_p(\zeta)-\zeta H n_p(\zeta)]}\right]$$

$$=\frac{1}{|m_p(\zeta)|^2}\,\mathrm{Re}\left[\frac{m_p(\zeta)+\zeta H(\zeta)n_p(\zeta)}{m_p(\zeta)-\zeta H(\zeta)m_p(\zeta)}\right]=\frac{1-|H(\zeta)|^2}{|m_p(\zeta)-\zeta H(\zeta)n_p(\zeta)|^2}$$

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From the fact $1-\|H\|_\infty^2 \le 1-|H|^2 \le 1$ we get $f_H \in L^1$

$\Leftrightarrow \frac{1}{m_p - \xi H m_p} \in L^2$

Theorem: Let $p \in \mathbb{N}$ and $\{C_n\}_{n=-p}^{p}$, the following conditions are equivalents

(i) $\sum_n \sum_m \lambda_n \bar{\lambda}_m C_{n-m} > 0$, $\{\lambda_n\}_{n=0}^{p} \subseteq \mathbb{C} - \{0\}$

(ii) there exists a positive Lebesgue's integrable function such that $C_k = \hat{f}(k)$ $k=-p,\dots,p$.

Moreover, given $H \in H^\infty(\mathbb{D})$ such that $\|H\|_\infty \le 1$, the set $\{\xi \in T : H(\xi)=1\}$ has Lebesgue mesure zero and $\frac{1}{m_p - \xi H n_p}$, we define

$$f_H(\xi) = \frac{1-|H(\xi)|^2}{|m_p(\xi) - \xi H(\xi) n_p(\xi)|^2}$$

then

Furthermore, this last formula establishes a bijection between all the power spectrum that solves the covariance extension problem and the $H \in H^\infty$ verifying that the set $\{\xi \in T : |H(\xi)| = 1\}$ has Lebesgue mesure zero and $\frac{1}{m_p - \xi H n_p} \in L^2$.

Finally, the following factorization formula holds:

$$f_H(\xi) = |F_H(\xi)|^2 \text{ for some } F_H \in H^\infty$$

Proof: if (ii) is valid hence (i) is true

Assume (i) holds and $\mathcal{V}_p : D_p \to R_p$. Given H verifying of the theorem let U_H the minimal extension associated to H

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$$c_k = \bar{c}_{-k} = \langle e_0, e_k \rangle_P = \langle e_0, V_P^k e_0 \rangle_P = \langle e_0, U_H^k e_0 \rangle_{F_H}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \, d\mu_H(t)$$

where $\mu_H(t) = \langle \mu^H_{(t)} e_0, e_0 \rangle$. The desired result is a consequence of the precedent corollary. The other statements are easy.

4. Burg's & Krein entropies

We use now the functional model of Arov-Grossman to find the density of a second order stationary process that solves the maximum entropy Burg's problem

Theorem: Let $p \in \mathbb{N}$ and $\{c_k\}_{k=0}^{p}$ be the first $(p+1)$ autocorrelation of a secon order stationary process $X = \{X_k\}_{k \in \mathbb{Z}}$ then the density f_0 of X which maximizes Burg' functional $\varepsilon(f)$ restricted to the conditions

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(e^{it}) \, dt = c_k \qquad k = 0, \dots, p$$

is $$f_0(e^{it}) = \frac{1}{|m_p(e^{it})|^2} = \frac{1}{|n_p(e^{it})|^2} \qquad t \in [0, 2\pi]$$

corresponding "$H \equiv 0$"

Proof: Taking $H \equiv 0$ in the formula

$$f_H(\zeta) = \frac{1}{|m_p(\zeta)|^2} \, \mathrm{Re} \left[\frac{m_p(\zeta) + \zeta H(\zeta) n_p(\zeta)}{m_p(\zeta) - \zeta H(\zeta) n_p(\zeta)} \right]$$

shows that $f_0(e^{it})$ is a solution of the extension problem. futher more:

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$$\frac{1}{2\pi}\int_0^{2\pi}\log f_H(e^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi}\log f_0(e^{it})\,dt + \frac{1}{2\pi}\int_0^{2\pi}\log \mathrm{Re}\left[\frac{m_p + e^{it}H\eta_p}{m - e^{it}H\eta_p}\right]dt$$

From Jensen's inequality and Cauchy's formula we get

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left[\frac{m_p(e^{it}) + e^{it}H(e^{it})\,\eta_p(e^{it})}{m_p(e^{it}) - e^{it}H(e^{it})\,\eta_p(e^{it})}\right]dt$$

$$\leq \log \frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi}\left[\frac{m_p(e^{it}) + e^{it}H(e^{it})\,m_p(e^{it})}{m_p(e^{it}) - e^{it}H(e^{it})\,\eta_p(e^{it})}\right]dt$$

$$= \log \mathrm{Re}(X(z)) = \log 1 = 0$$

were $X(z) = \dfrac{m_p(z) + z\,H(z)\,\eta_p(z)}{m_p(z) - z\,H(z)\,\eta_p(z)}$

Theorem: Let $p \in \mathbb{N}$ and $\{e_k\}_{k=0}^{p}$ as the last theorem. Then the density that solves the Krein entropy maximum is given by taking:

$$H_a(z) = \frac{\bar{a}\,\overline{\eta_p(a)}}{\overline{m_p(a)}}$$

Proof: Define

$$E_p^a = \Gamma_p^{-1}\sum_{k=0}^{p}\bar{a}^k e_k$$

We have for all $P \in E_p$ $\langle P, E_p^a\rangle = P(a)$

The followin Christoffel–Darboux formula is verified

$$E_p^a(e^{i\theta}) = \frac{\overline{m_p(a)}\,m_p(e^{i\theta}) - e^{i\theta}\bar{a}\,\overline{\eta_p(a)}\,\eta_p(e^{i\theta})}{1 - \bar{a}e^{i\theta}}$$

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also $\|E_p^a\|_p^2 = E_p^a(a)$. If we define

$$W_a(e^{i\theta}) = \frac{1 - |H_a(e^{i\theta})|^2}{|m_p(e^{i\theta}) - e^{i\theta} H_a(e^{i\theta})\, n_p(e^{i\theta})|^2}$$

it holds

$$W_a(e^{i\theta}) = \frac{(1-|a|^2)\, \|E_p^a\|_p^2}{|1 - a\bar{e}^{i\theta}|^2\, |E_p^a(e^{i\theta})|^2}$$

Moreover :

$$\exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log\left[\frac{f_H(e^{i\theta})}{W_a(e^{i\theta})}\right] \frac{1-|a|^2}{|1-a\bar{e}^{i\theta}|^2}\, d\theta\right]$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{f_H(e^{i\theta})}{W_a(e^{i\theta})} \frac{1-|a|^2}{|1-a\bar{e}^{i\theta}|^2}\, d\theta$$

$$= \frac{1}{2\pi} \frac{1}{\|E_p^a\|_p^2} \int_0^{2\pi} |E_p^a(e^{i\theta})|^2 f_H(e^{i\theta})\, d\theta = 1$$

Because f_H is a solution to our problem. Hence we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log f_H(e^{i\theta}) \frac{1-|a|^2}{|1-a\bar{e}^{i\theta}|^2}\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \log W_a(e^{i\theta}) \frac{1-|a|^2}{|1-a\bar{e}^{i\theta}|^2}\, d\theta$$

5. *Multidimensional covariance matrix extension problem.*

We consider now the multidimensional analog of the problem studied in the previous sections. Let $\vec{X} = \{\vec{X}_n\}_{n\in\mathbb{Z}}$ be a second order stationary process $\mathbb{R}^q$ valued and centered. We denote

$$R_k = E[X_0 X_k^t] \quad k \in \mathbb{Z}$$

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the autocovariance matrix and $R_0 = I$. There exists a positive matricial mesure F such that

$$R_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\, F(e^{it})\,dt$$

the matrix F is hermitian and positive-definite.

Our first problem is: given a finite sequence $R_0 = I, R_1, \dots, R_k$ of Matrices under which conditions there exists a matricial mesure F such that

$$R_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\, F(e^{it})\,dt \quad k = 0, 1, \dots, p.$$

The other two problems are:
Since the solution of the first problem it is not unique is important to find: the matricial mesures that maximizes

- $\mathcal{E}(F) = \frac{1}{2\pi}\int_0^{2\pi} \log\det F(e^{it})\,dt$ (Burg)
- $h(W;\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|\alpha|^2}{|1-\alpha e^{-it}|^2} \log\det W(e^{it})\,dt$ (Krein)

Verifying also:

$$R_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\, G(e^{it})\,dt$$

in both cases.

The solutions of these problems take the same stebs that in the case $q = 1$.

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- A sequence of matrices is spd if

$$\sum_{n=0}^{p}\sum_{m=0}^{p} \langle R_{m-n} f_n, f_m \rangle_{\mathbb{C}^q} > 0 \quad \{f_k\} \in \mathbb{C}^q - \{\vec{0}\}$$

- Now $E_p(\mathbb{C}^q)$ is the span of $\{e_k E^j\}_{k=0,\dots,p}^{j=1,\dots,q}$ where $e_k(\zeta) = \zeta^k$ and $E^i = (0,\dots,\overset{i}{\overset{\downarrow}{1}},\dots 0)$ the vectorial space $E_p(\mathbb{C}^q)$ form the vectorial trigonometric polynomials of degree $\leq p$.

- We define $\langle \, , \rangle_p$ in $E_p(\mathbb{C}^q)$ as:

$$\left\langle \sum_{n=0}^{p} e_n f_n, \sum_{m=0}^{p} e_m g_m \right\rangle_p = \sum_{n=0}^{p}\sum_{m=0}^{p} \langle R_{m-n} f_n, g_m \rangle_{\mathbb{C}^q}$$

- The operator $\Gamma_p : E_p(\mathbb{C}^q) \to E_p(\mathbb{C}^q)$ can also be defined

$$\Gamma_p(f) = \sum_{k=0}^{p} e_k \sum_{s=0}^{p} R_{k-s} \hat{f}(s); \; f \in E_p(\mathbb{C}^q)$$

where $f = \sum_{s=0}^{p} \hat{f}(k) e_k$, $\hat{f}(k) = \sum_{j=1}^{q} \langle f, e_k E^j \rangle E^j$

Γ_p is a linear operator and also $\langle \Gamma_p f, g \rangle_{L^2_q} := \langle f, g \rangle_p$

L^2_q denotes the set:

$$\left\{ f : \mathbb{T} \to \mathbb{C}^q \text{ s.t. } \frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{it}), f(e^{it}) \rangle_{\mathbb{C}^q} dt < +\infty \right\}$$

As in the classical lemma we have:

- If $D_p := \mathrm{Span}\{e_k E^s\}_{k=0,\dots,p-1}^{s=1,\dots,q}$ $R_p := \mathrm{Span}\{e_k E^s\}_{k=1,\dots,p}^{s=1,\dots,q}$

  and $V_p : D_p \to R_p$ defined as $V_p(e_k E^s) = e_{k+1} E^s$

- V_p is an isometric operator $V_p : E_p \longrightarrow E_p$

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with respect to the inner product $\langle , \rangle_p$

- Γ_p is injective
- The subspaces $N_p = E_p(\mathbb{C}^q) \ominus D_p$ and $M_p = E_p(\mathbb{C}^q) \ominus R_p$ have dimension equal q.

And as before:

$$N_p = \mathrm{Span}\left[\Gamma_p^{-1} e_p E^s\right]_{s=1}^{q} \qquad M_p = \mathrm{Span}\left[\Gamma_p^{-1} e_0 E^s\right]_{s=1}^{q}$$

Let $\{m_p^k\}_{k=1}^q$ and $\{n_p^k\}_{k=1}^q$ the orthonormal vectors obtained via Gram-Schmidt orthonormalization method from:

$$\{\Gamma_p^{-1} e_0 E^k\}_{k=1}^q \quad \text{and} \quad \{\Gamma_p^{-1} e_p E^k\}_{k=1}^q$$

If $n_p^k = (n_p^k(1), \dots, n_p^k(q))^t$ and $m_p^k = (m_p^k(1), \dots, m_p^k(q))^t$

Defining the matrices

$$N_p(\zeta) = \left[n_p^j(i) \right]_{i,j=1,\dots,q} \quad \text{and} \quad M_p(\zeta) = \left[m_p^j(i) \right]_{i,j=1,\dots,q}$$

for all unitary minimal extension of V_p, there is associated an analytical and bounded function $H : D \to M_{q,q}$ (Matrices $q \times q$) such that the corresponding spectral matricial mesure is given by:

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$F_H(\zeta)=$

$(N_p^*)^{-1}\,[(N_p^{-1}M_p-\zeta H)^*]^{-1}(I-H^*H)\,[N_p^{-1}M_p-\zeta H]^{-1}N_p^{-1}$

$H\in H^\infty_{q\times q}(D)$ with $H^*H<I$ $\quad (0<I-H^*H)$

Taking $H\equiv 0$ we obtain $F_B(\zeta)=(M_p^*)^{-1}M_p^{-1}$. We can also write

$$F_H(\zeta)=(M_p^*)^{-1}\,\mathrm{Re}\,[(M_p-\zeta N_p H)^{-1}(M_p+\zeta N_p H)]\,M_p^{-1}$$

and putting $X(\zeta)=(M_p-\zeta N_p H)^{-1}(M_p+\zeta+N_p H)$

we get

$$\frac{1}{2\pi}\int_0^{2\pi}\log\det F_H(e^{it})\,dt=\frac{1}{2\pi}\int_0^{2\pi}\log\det\left(F_B(e^{it})\,\mathrm{Re}\,X(e^{it})\right)dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\log\det \mathrm{Re}\,X(e^{it})\,dt+\frac{1}{2\pi}\int_0^{2\pi}\log\det F_B(e^{it})\,dt$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\log\det F_B(e^{it})\,dt$$

because matricial Jensen inequality gives:

$$\frac{1}{2\pi}\int_0^{2\pi}\log\det[X(e^{it})]\,dt\leq \log\det\left(\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,(X(e^{it}))\,dt\right)$$

$$=\log\det \mathrm{Re}\,\frac{1}{2\pi i}\int_{|z|=1}\frac{X(z)}{z}\,dz$$

$$=0.$$

Finally to solve the maximun entropy problem for Krein entropy, we must define

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$$W_\alpha(e^{it}) = \frac{1-|\alpha|^2}{|e^{it}-\alpha|^2}\left[E_p^\alpha(e^{it})^*\right]^{-1} E_p^\alpha(\alpha)\left[E_p^\alpha(e^{it})\right]^{-1}$$

Where $\alpha \in D$ and

$$E_p^\zeta(z) = \frac{M_p(z)\,M_p^*(\zeta) - z\,N_p(z)\,\bar{\zeta}\,N_p^*(\zeta)}{1 - z\bar{\zeta}}$$

It is now plain to obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\det\left[F_H(e^{it})\right]\frac{1-|\alpha|^2}{|1-\alpha e^{-it}|^2}\,dt$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log\det\left[W_\alpha(e^{it})\right]\frac{1-|\alpha|^2}{|1-\alpha e^{-it}|^2}\,dt$$

The function $W_\alpha(z)$ corresponds to

$$H_\alpha(z) = \bar{\alpha}\,N_p^*(\alpha)\left[M_p^*\right]^{-1}(\alpha)$$