On Intrinsic Autoregressions

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Summary

This talk in honour of Xavier Guyon discusses intrinsic autoregressions as limits of conditional autoregressions and describes some applications to spatial statistics and image analysis (not included in this version).

Non–singular Gaussian distributions

Traditional approach specifies *covariance matrix* V.

Markov random field (MRF) approach specifies *precision matrix* $Q = V^{-1}$.

Equivalently, MRF approach specifies *conditional* means and variances.

Terminology: *auto–Normal scheme* or *conditional autoregression.*

Gaussian conditional autoregressions

Definition. Require $X = (X_1, \dots, X_n)^T$ to have Gaussian full conditionals with

$$\mathrm{E}\,(X_i \,|\, x_{-i}) \;=\; \mu_i + \sum_j \beta_{ij}\,(x_j - \mu_j), \qquad \beta_{ii} \;=\; 0,$$

$$\mathrm{var}\,(X_i \,|\, x_{-i}) \;=\; \kappa_i \;>\; 0,$$

for $i = 1, \dots, n$, subject to compatibility.

Implication. Brook (1964) expansion $\Rightarrow$ X has Gaussian p.d.f.

$$\pi(x) \;\propto\; \exp\{-\tfrac{1}{2}(x-\mu)^T Q\,(x-\mu)\}, \qquad x \in R^n,$$

where $\mu = (\mu_1, \dots, \mu_n)^T$ and Q has elements

$$Q_{ii} \;=\; 1/\kappa_i, \qquad Q_{ij} \;=\; -\beta_{ij}/\kappa_i, \quad i \neq j.$$

$\Rightarrow$ Q must be *symmetric* i.e. $\beta_{ij}\kappa_j \equiv \beta_{ji}\kappa_i$ and *positive definite.*

NB. If $\beta_{ij} \neq 0$, sites i and j are called *neighbours.*

Recall:

$$\mathrm{E}\,(X_i\,|\,x_{-i}) \;=\; \mu_i + \sum_{j\neq i} \beta_{ij}\,(x_j - \mu_j), \qquad\qquad \mathrm{var}\,(X_i\,|\,x_{-i}) \;=\; \kappa_i \;>\; 0,$$

$$\pi(x) \;\propto\; \exp\{-\tfrac{1}{2}(x-\mu)^T Q(x-\mu)\}, \qquad x \in R^n,$$

$$Q_{ii} \;=\; 1/\kappa_i, \qquad Q_{ij} \;=\; -\beta_{ij}/\kappa_i, \quad i \neq j, \qquad \beta_{ij}\kappa_j \;\equiv\; \beta_{ji}\kappa_i$$

Spatial applications

Usually allow $\beta_{ij} \neq 0$ only if i and j are "adjacent" (*neighbours*).

Preference between modelling precision Q or covariance V may depend on context.

Might prefer hybrid approach, modelling some aspects of each matrix (later).

Covariance properties

Take $\mu = 0$ w.l.o.g. and define residuals

$$Z_i \;=\; X_i - \sum_{j\neq i} \beta_{ij} X_j, \qquad i = 1, \ldots, n \qquad \Rightarrow$$

Z_i, Z_j uncorrelated only if $i \neq j$ *and* $\beta_{ij} = 0$ but ...

Z_i's uncorrelated with X_j's for $j \neq i$: hence term "conditional autoregression" OK.

Also: $\sum_j \beta_{ij} X_j$ is *best linear unbiased predictor* (BLUP) of X_i given X_{-i}.

Ex. The simplest conditional autoregression

Suppose X is Gaussian with $\mu = 0$ and

$$\mathrm{E}\,(X_i \,|\, x_{-i}) \;=\; \alpha \sum_j \frac{w_{ij}}{w_{i+}}\, x_j, \qquad \mathrm{var}\,(X_i \,|\, x_{-i}) \;=\; \frac{\kappa}{w_{i+}},$$

(i) $w_{ii} = 0$, (ii) $w_{ij} = w_{ji}$, (iii) $w_{i+} > 0$, (iv) $0 \le \alpha < 1$, (v) $\kappa > 0$

κ redundant but convenient in practice.

$$\pi(x) \;\propto\; \exp\{-\tfrac{1}{2} x^T Q x\}, \qquad x \in \mathcal{R}^n,$$

$$\kappa Q_{ij} \;=\; \begin{cases} w_{i+} & i = j \\ -\alpha w_{ij} & i \neq j \end{cases}$$

$$\kappa x^T Q x \;=\; (1-\alpha) \sum_i w_{i+} x_i^2 \;+\; \alpha \sum_{i<j} w_{ij} (x_i - x_j)^2$$

If any w_{ij}'s are negative, validity must be verified.

Inclusion of a non–zero mean μ is trivial.

Now return to general formulations and ....

Early spatial applications of conditional autoregressions

Human geography.

Cliff and Ord, 1975, 1981, Ch. 4

Besag, 1975

Agricultural field experiments.

Bartlett, 1978

Kempton and Howes, 1981

Martin, 1990

Cressie and Hartfield, 1993

Geographical epidemiology

Clayton and Kaldor, 1987

Marshall, 1991

Mollié and Richardson, 1991

Bernardinelli and Montomoli, 1992

Cressie, 1993, Ch. 7

Astronomy

Molina and Ripley, 1989

Ripley, 1991

Texture and image analysis

Chellappa and Kashyap, 1985

Cohen, Fan and Patel, 1991

Cohen and Patel, 1991

Chellappa, 1985

Jinchi and Chellappa, 1986

Cohen and Cooper, 1987

Simonchy, Chellappa and Lichtenstein, 1989

Zerubia and Chellappa, 1989

Multivariate remote sensing

Kittler and Föglein, 1984

Mardia, 1988

Compromising between Q and V (Dempster, 1972; Besag and Kooperberg, 1995)

Specify conditional independence properties by 0's in Q.

Specify covariances in complementary elements of V.

Unique solution, subject to compatibility, via Dempster's algorithm.

Toy example with 5 sites

$$Q = \begin{pmatrix} ? & ? & 0 & 0 & ? \\ ? & ? & ? & ? & 0 \\ 0 & ? & ? & ? & 0 \\ 0 & ? & ? & ? & ? \\ ? & 0 & 0 & ? & ? \end{pmatrix} \qquad V = \begin{pmatrix} 2 & 1 & ? & ? & 1 \\ 1 & 2 & 1 & 1 & ? \\ ? & 1 & 2 & 1 & ? \\ ? & 1 & 1 & 2 & 1 \\ 1 & ? & ? & 1 & 2 \end{pmatrix}$$

$$\Rightarrow \qquad Q = \begin{pmatrix} 0.789 & -0.289 & 0 & 0 & -0.289 \\ -0.289 & 0.872 & -0.250 & -0.205 & 0 \\ 0 & -0.250 & 0.750 & -0.250 & 0 \\ 0 & -0.205 & -0.250 & 0.872 & -0.289 \\ -0.289 & 0 & 0 & -0.289 & 0.789 \end{pmatrix}$$

$$V = \begin{pmatrix} 2 & 1 & 0.577 & 0.732 & 1 \\ 1 & 2 & 1 & 1 & 0.732 \\ 0.577 & 1 & 2 & 1 & 0.577 \\ 0.732 & 1 & 1 & 2 & 1 \\ 1 & 0.732 & 0.577 & 1 & 2 \end{pmatrix}$$

In practice, specified elements in V are usually estimated from data.

Stationary autoregressions on regular arrays

Applications often involve observations y on a finite rectangular lattice.

Examples include image analysis (pixels) and agricultural experiments (plots).

Basic formulation (extends easily to e.g. Poisson or binomial observations y)

$$y = T\tau + x + z$$

τ	=	"treatment" parameter: primary interest
T	=	design matrix, assigning treatments to sites
x	=	realization of zero–mean spatial process: secondary interest
z	=	residual variation

Bayesian paradigm for x rather natural (but not assumed here – yet!!).

Might interpret x as finite restriction of *stationary* process X on $\mathcal{Z}^2$.

Tools

Autocovariances $\gamma_{st} := \mathrm{E}\,(X_{uv}X_{u+s,v+t})$

Spectral density function $f(\omega_1, \omega_2) := \dfrac{1}{4\pi^2} \displaystyle\sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \gamma_{st} \cos(\omega_1 s + \omega_2 t)$

References

Lévy, 1948; Whittle, 1954; Rosanov, 1967;

Besag, 1974; Cressie, 1993, Ch. 6; Guyon, 1995, Ch. 1.

Stationary autoregressions on regular arrays

Stationary autoregressions $\{X_{uv} : (u,v) \in \mathcal{Z}^2\}$

Defined via the class of spectral density functions:

$$f(\omega_1, \omega_2) \;=\; \frac{\kappa}{4\pi^2}\{1 - \sum_k \sum_l (1 \,-\, \beta_{kl} \cos(\omega_1 k + \omega_2 l)\}^{-1},$$

where (i) $\beta_{00} = 0$, (ii) $\beta_{-k,-l} = \beta_{kl}$, (iii) $\sum_k \sum_l \beta_{kl} \cos(\omega_1 k + \omega_2 l) < 1$.

$$\Rightarrow \qquad \gamma_{st} \;=\; \kappa\, \delta_{st} \,+ \sum_k \sum_l \beta_{kl}\, \gamma_{s-k,t-l}, \qquad\qquad (s,t) \in \mathcal{Z}^2,$$

where $\delta_{st} = 1$ if $(s,t) = (0,0)$ and $\delta_{st} = 0$ otherwise.

$$\Rightarrow \qquad \mathrm{E}\,(X_{uv} \mid \ldots) \;=\; \sum_k \sum_l \beta_{kl}\, x_{u-k,v-l}, \qquad\qquad \mathrm{var}\,(X_{uv} \mid \ldots) \;=\; \kappa.$$

which agrees with the formulation on a finite set.

Also elegant maximum likelihood parameter estimation (cf. Yule–Walker equations).

Inversion formula for γ_{st}'s:

$$\gamma_{st} \;=\; \kappa \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\cos(\omega_1 s + \omega_2 t)\, d\omega_1 d\omega_2}{1 - \sum_k \sum_l \beta_{kl} \cos(\omega_1 k + \omega_2 l)}$$

but not very useful in practice, except for asymptotics.

Ex. First–order conditional autoregression

$$\mathrm{E}\,(X_{uv}\mid\ldots) \;=\; \beta_{10}(x_{u-1,v}+x_{u+1,v}) \;+\; \beta_{01}(x_{u,v-1}+x_{u,v+1}), \qquad |\beta_{10}|+|\beta_{01}|<\tfrac{1}{2}.$$

$$\gamma_{st} \;\equiv\; \kappa\delta_{st} \;+\; \beta_{10}(\gamma_{s-1,t}+\gamma_{s+1,t}) \;+\; \beta_{01}(\gamma_{s,t-1}+\gamma_{s,t+1}) \qquad (\#)$$

$$f(\omega_1,\omega_2) \;\propto\; \{\,1 \;-\; 2\beta_{10}\cos\omega_1 \;-\; 2\beta_{01}\cos\omega_2\,\}^{-1}$$

Explicit formulae for $\gamma_{00},\ \gamma_{10},\ \gamma_{01},\ \gamma_{11},\ \gamma_{-11},\ \gamma_{20},\ \gamma_{02}$ (elliptic functions)

(#) $\Rightarrow$ all γ_{st} in symmetric case $\eta_{10}=\eta_{01}$ with some care!! (Besag, 1981)

Autocorrelations ρ_{st}, for $\beta_{10}=\beta_{01}\approx 0.249993$ so that $\rho_{01}=\rho_{10}=0.75$

						t					
s	0	1	2	3	4	5	6	7	8	9	10
0	1.000	0.750	0.637	0.570	0.523	0.487	0.458	0.434	0.413	0.394	0.378
1	0.750	0.682	0.613	0.560	0.518	0.484	0.456	0.432	0.412	0.393	0.377
2	0.637	0.613	0.576	0.538	0.504	0.475	0.450	0.428	0.408	0.390	0.375
3	0.570	0.560	0.538	0.512	0.486	0.462	0.440	0.420	0.402	0.386	0.371
4	0.523	0.518	0.504	0.486	0.467	0.447	0.429	0.411	0.395	0.380	0.366
5	0.487	0.484	0.475	0.462	0.447	0.432	0.416	0.401	0.387	0.373	0.360
6	0.458	0.456	0.450	0.440	0.429	0.416	0.403	0.390	0.378	0.365	0.354
7	0.434	0.432	0.428	0.420	0.411	0.401	0.390	0.379	0.368	0.357	0.346
8	0.413	0.412	0.408	0.402	0.395	0.387	0.378	0.368	0.358	0.349	0.339
9	0.394	0.393	0.390	0.386	0.380	0.373	0.365	0.357	0.349	0.340	0.331
10	0.378	0.377	0.375	0.371	0.366	0.360	0.354	0.346	0.339	0.331	0.324

NB. $\rho_{10}=\rho_{01}=0.85 \;\;\Rightarrow\;\; \sqrt{s^2+t^2}>2000$ before $\rho_{st}<0.1\,!$

Ex. Second–order conditional autoregression

$$\begin{aligned}\mathrm{E}\,(X_{uv} \mid \ldots) \;&=\; \beta_{10}(x_{u-1,v} + x_{u+1,v}) \;+\; \beta_{10}(x_{u,v-1} + x_{u,v+1}) \\ &\;+\; \beta_{11}(x_{u-1,v-1} + x_{u+1,v+1}) \;+\; \beta_{-11}(x_{u-1,v+1} + x_{v+1,u-1})\end{aligned}$$

$$f(\omega_1, \omega_2) \;\propto\; \{1 - 2\beta_{10}\cos\omega_1 - 2\beta_{01}\cos\omega_2 - 2\beta_{11}\cos(\omega_1 + \omega_2) - 2\beta_{-11}\cos(\omega_1 - \omega_2)\}^{-1}$$

$$|\beta_{10}| + |\beta_{01}| + |\beta_{11}| + |\beta_{-11}| < \tfrac{1}{2} \qquad \Rightarrow \qquad f(\omega_1, \omega_2) > 0$$

Ex. Third–order conditional autoregression

$$\begin{aligned}\mathrm{E}\,(X_{uv} \mid \ldots) \;&=\; \beta_{10}(x_{u-1,v} + x_{u+1,v}) \;+\; \beta_{10}(x_{u,v-1} + x_{u,v+1}) \\ &\;+\; \beta_{11}(x_{u-1,v-1} + x_{u+1,v+1}) \;+\; \beta_{-11}(x_{u-1,v+1} + x_{u+1,v-1}) \\ &\;+\; \beta_{20}(x_{u-2,v} + x_{u+2,v}) \;+\; \beta_{02}(x_{u,v-2} + x_{u,v+2})\end{aligned}$$

Ex. First–order simultaneous equations (Whittle, 1954)

$$X_{uv} \;=\; aX_{u-1,v} \;+\; bX_{u+1,v} \;+\; cX_{u,v-1} \;+\; dX_{v,v+1} \;+\; Z_{uv}$$

Z_{uv}'s independent Gaussian r.v.'s with zero mean (say) and constant variance.

Do not require $a = b$ or $c = d$: $a \leftrightarrow b$, $c \leftrightarrow d$ so non–identifiable.

Special case of third–order conditional autoregression.

Cause of considerable confusion in geographic and econometric literature.

Gaussian intrinsic autoregressions

Recall the simplest conditional autoregression: X is Gaussian with

$$\mathrm{E}\,(X_i\,|\,x_{-i}) \;=\; \alpha \sum_j \frac{w_{ij}}{w_{i+}}\,x_j, \qquad \mathrm{var}\,(X_i\,|\,x_{-i}) \;=\; \frac{\kappa}{w_{i+}},$$

(i) $w_{ii}=0$, (ii) $w_{ij}=w_{ji}$, (iii) $w_{i+}>0$, (iv) $0\le\alpha<1$, (v) $\kappa>0$

$$\pi(x) \;\propto\; \exp\{-\tfrac{1}{2}x^TQ\,x\}, \qquad x\in\mathcal{R}^n,$$

$$\kappa x^TQx \;=\; (1-\alpha)\sum_i w_{i+}x_i^2 \;+\; \alpha\sum_{i<j} w_{ij}(x_i-x_j)^2.$$

If any w_{ij}'s are negative, validity must be checked carefully.

But what if $\alpha=1$?? Interpret in limiting sense: Bayesian posterior OK !

Mean (at least) is undefined. Need to consider *contrasts* :

$$U_k := \textstyle\sum_{i=1}^n A_{ki}X_i \qquad \text{s.t.} \quad A_{k+} \;=\; 0$$

i.e. $U := AX$ s.t. $A1 \;=\; 0.$

OK if all $w_{ij}\ge 0$, with A an $(n-1)\times n$ matrix of rank $n-1$:

$$U \;\sim\; \mathrm{N}\,(0,\, A\,Q^-A^T), \qquad Q^-:\;\; Q\,Q^-Q \;=\; Q.$$

e.g. $U_i \;=\; X_i \;-\; X_{i+1}, \qquad i=1,\ldots,n-1.$

NB. Independent 1–d spatial first–differences proposed by Besag and Kempton (1986).

Gaussian intrinsic autoregressions (Besag, York and Mollié, 1991)

X has Gaussian full conditionals with

$$\mathrm{E}\,(X_i \,|\, x_{-i}) \;=\; \sum_j \frac{w_{ij}}{w_{i+}}\, x_j, \qquad \mathrm{var}\,(X_i \,|\, x_{-i}) \;=\; \frac{\kappa}{w_{i+}},$$

(i) $w_{ii} = 0$, (ii) $w_{ij} = w_{ji}$, (iii) $w_{i+} > 0$, (iv) $\kappa > 0$

Defines an improper p.d.f.

$$\pi(x) \;\propto\; \exp\{-\tfrac{1}{2} x^T Q\, x\}, \qquad x \in \mathcal{R}^n,$$

$$\kappa x^T Q x \;=\; \sum_{i<j} w_{ij}(x_i - x_j)^2, \qquad \mathrm{rank}\, Q \;=\; q \;\le\; n-1.$$

If any w_{ij}'s are negative, validity must be verified.

Mean (at least) is undefined. Need to consider *contrasts* $U := AX$,

where A is $q \times n$ with rows spanning the same q–d subspace of $\mathcal{R}^n$ as rows of Q.

$$\Rightarrow \qquad U \;\sim\; \mathrm{N}\,(0,\, A\, Q^- A^T), \qquad Q^- :\;\; Q\, Q^- Q \;=\; Q.$$

If all $w_{ij} \ge 0$, then $X_i - X_j \;\sim\; \mathrm{N}(0,\, \kappa / w_{ij})$ for $i < j$ s.t. $w_{ij} > 0$, independently, subject to logical constraints over all circuits of the conditional dependence graph.

Applications: geographical epidemiology, agricultural field experiments, Bayesian image analysis, space–time modelling, time series (!), ...

Baseline Gaussian intrinsic autoregression

For each site i, define the *neighbours* ∂i of i symmetrically; e.g. by contiguity.

Define $w_{ij} = 1$ if $j \in \partial i$, else $w_{ij} = 0$. Let m_i = number of neighbours of i.

$$\Rightarrow \qquad \mathrm{E}\,(X_i \mid \ldots) \;=\; \frac{1}{m_i}\sum_{j\in\partial i} x_j, \qquad \mathrm{var}\,(X_i \mid \ldots) \;=\; \frac{\kappa}{m_i}.$$

Edge effects ?? Refinements ??

Compromising between Q and Q^- (Besag and Kooperberg, 1995)

Analogue of ordinary conditional autoregressions but autocovariances do not exist.

Suppose Q has rank $n-1$.

Define $\eta_{ij} = \mathrm{var}\,(X_i - X_j)$.

Specify which elements of Q can be non–zero …

and the η_{ij}'s for the corresponding i and j.

$\Rightarrow$ unique solution via modified Dempster's algorithm (if a solution exists).

In practice, required η_{ij}'s estimated from data.

Provides link with "geostatistical" approach to Gaussian formulations.

Applications: geographical epidemiology, agricultural field experiments.

Intrinsic autoregressions on regular arrays (Künsch, 1987)

Intrinsic autoregressions $\{X_{uv} : (u,v) \in \mathcal{Z}^2\}$

Defined via the class of generalized spectral density functions:

$$f(\omega_1, \omega_2) \;=\; \frac{\kappa}{4\pi^2}\,\{\,1 - \sum_k \sum_l (1 \;-\; \beta_{kl} \cos{(\omega_1 k + \omega_2 l)}\,\}^{-1},$$

(i) $\beta_{00} = 0$, (ii) $\beta_{-k,-l} = \beta_{kl}$, (iii) $\sum_k \sum_l \beta_{kl} \cos(\omega_1 k + \omega_2 l) \leq 1$, (iv) $\sum_k \sum_l \beta_{kl} = 1$.

Assuming existence of the *variogram* $\nu_{st} \;=\; \frac{1}{2}\,\mathrm{var}\;(X_{uv} - X_{u+s,v+t})$

$$\Rightarrow \qquad \nu_{st} \;=\; -\kappa\,\delta_{st} \,+\, \sum_k \sum_l \beta_{kl}\,\nu_{s-k,t-l}, \qquad\qquad (s,t) \in \mathcal{Z}^2,$$

$$\Rightarrow \qquad \mathrm{E}\,(X_{uv} \mid \ldots) \;=\; \sum_k \sum_l \beta_{kl}\,x_{u-k,v-l}, \qquad\qquad \mathrm{var}\,(X_{uv} | \ldots) \;=\; \kappa.$$

which agrees with the formulation on a finite set.

Also elegant maximum likelihood parameter estimation (cf. Yule–Walker equations).

Inversion formula for ν_{st}'s:

$$\nu_{st} \;=\; \kappa \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\{1 - \cos(\omega_1 s + \omega_2 t)\}\,d\omega_1 d\omega_2}{1 - \sum_k \sum_l \beta_{kl} \cos{(\omega_1 k + \omega_2 l)}}$$

but not very useful in practice (??) except for asymptotics.

Ex. First–order intrinsic autoregression $(\beta_{10},\, \beta_{01} > 0,\ \beta_{10} + \beta_{01} = \frac{1}{2})$

$$f(\omega_1, \omega_2) \ \propto\ \{\, 1 \,-\, 2\beta_{10} \cos\omega_1 \,-\, 2\beta_{01} \cos\omega_2 \,\}^{-1}$$

$$\mathrm{E}\,(X_{uv} \mid \ldots) \ =\ \beta_{10}(x_{u-1,v} + x_{u+1,v}) \ +\ \beta_{01}(x_{u,v-1} + x_{u,v+1})$$

$$\nu_{st} \ =\ -\kappa\delta_{st} \ +\ \beta_{10}(\nu_{s-1,t} + \nu_{s+1,t}) \ +\ \beta_{01}(\nu_{s,t-1} + \nu_{s,t+1})$$

Also, $\qquad (2t-1)\nu_{t-1,t-1} + (2t+1)\nu_{t+1,t+1} = 4t\nu_{tt}, \qquad t = 0, 1, \ldots\,,$

$\Rightarrow$

$$\nu_{10} \ =\ \frac{\kappa}{\pi\beta_{10}} \tan^{-1}\sqrt{\beta_{10}/\beta_{01}}, \qquad \nu_{01} \ =\ \frac{\kappa}{\pi\beta_{01}} \tan^{-1}\sqrt{\beta_{01}/\beta_{10}},$$

$$\nu_{t,t} \ =\ \nu_{-t,t} \ =\ \frac{\kappa}{\pi\sqrt{\beta_{10}\beta_{01}}}\,\Big(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2t-1}\Big)$$

But what about ν_{st} in general ???

Ex. First–order locally planar autoregression $(\beta_{10} = \beta_{01} = \frac{1}{4})$

All ν_{st} are determined by above results.

Suggests the isotropic approximation, excellent even for moderate $|s|$ and $|t|$,

$$\nu^*_{st} \ =\ \frac{1}{\pi}\,\{\ln(s^2 + t^2) + 2\gamma + \ln 8\}, \qquad \gamma \ =\ \text{Euler's constant} \ =\ 0.5772\ldots$$

References: McCrea and Whipple (1940), Spitzer (1976), Besag (1981).

Actual variogram $\nu_{s,t}$ for first–order symmetric intrinsic autoregression.

	t												
s	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.000	1.000	1.454	1.721	1.908	2.052	2.168	2.267	2.352	2.427	2.495	2.555	2.611
1	1.000	1.273	1.546	1.762	1.930	2.065	2.178	2.274	2.357	2.431	2.498	2.558	2.613
2	1.454	1.546	1.698	1.849	1.984	2.101	2.203	2.293	2.372	2.443	2.507	2.566	2.620
3	1.721	1.762	1.849	1.952	2.056	2.153	2.241	2.322	2.395	2.462	2.522	2.579	2.630
4	1.908	1.930	1.984	2.056	2.134	2.213	2.288	2.359	2.424	2.486	2.542	2.595	2.645
5	2.052	2.065	2.101	2.153	2.213	2.276	2.339	2.400	2.459	2.514	2.566	2.616	2.662
6	2.168	2.178	2.203	2.241	2.288	2.339	2.391	2.444	2.496	2.546	2.593	2.639	2.682
7	2.267	2.274	2.293	2.322	2.359	2.400	2.444	2.489	2.535	2.579	2.622	2.664	2.705
8	2.352	2.357	2.372	2.395	2.424	2.459	2.496	2.535	2.574	2.614	2.653	2.691	2.729
9	2.427	2.431	2.443	2.462	2.486	2.514	2.546	2.579	2.614	2.649	2.684	2.719	2.754
10	2.495	2.498	2.507	2.522	2.542	2.566	2.593	2.622	2.653	2.684	2.716	2.748	2.779
11	2.555	2.558	2.566	2.579	2.595	2.616	2.639	2.664	2.691	2.719	2.748	2.777	2.806
12	2.611	2.613	2.620	2.630	2.645	2.662	2.682	2.705	2.729	2.754	2.779	2.806	2.832

Isotropic approximation $\nu^*_{s,t}$ for first–order symmetric intrinsic autoregression.

	t												
s	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.000	1.029	1.471	1.729	1.912	2.054	2.170	2.268	2.353	2.428	2.495	2.556	2.611
1	1.029	1.250	1.542	1.762	1.931	2.066	2.179	2.275	2.358	2.432	2.498	2.559	2.614
2	1.471	1.542	1.691	1.846	1.983	2.101	2.204	2.293	2.372	2.444	2.508	2.566	2.620
3	1.729	1.762	1.846	1.949	2.054	2.152	2.241	2.322	2.395	2.462	2.523	2.579	2.631
4	1.912	1.931	1.983	2.054	2.133	2.211	2.287	2.358	2.424	2.486	2.542	2.595	2.645
5	2.054	2.066	2.101	2.152	2.211	2.275	2.338	2.399	2.458	2.514	2.566	2.616	2.662
6	2.170	2.179	2.204	2.241	2.287	2.338	2.391	2.444	2.495	2.545	2.593	2.639	2.682
7	2.268	2.275	2.293	2.322	2.358	2.399	2.444	2.489	2.534	2.579	2.622	2.664	2.705
8	2.353	2.358	2.372	2.395	2.424	2.458	2.495	2.534	2.574	2.614	2.653	2.691	2.728
9	2.428	2.432	2.444	2.462	2.486	2.514	2.545	2.579	2.614	2.649	2.684	2.719	2.753
10	2.495	2.498	2.508	2.523	2.542	2.566	2.593	2.622	2.653	2.684	2.716	2.748	2.779
11	2.556	2.559	2.566	2.579	2.595	2.616	2.639	2.664	2.691	2.719	2.748	2.777	2.805
12	2.611	2.614	2.620	2.631	2.645	2.662	2.682	2.705	2.728	2.753	2.779	2.805	2.832

Ex. Second–order locally quadratic autoregression

Least–squares 8–neighbour locally quadratic fit $\Rightarrow$

$$\begin{aligned} \mathrm{E}\,(X_{uv} \mid \dots) &= \tfrac{1}{2}(x_{u-1,v} + x_{u+1,v} + x_{u,v-1} + x_{u,v+1}) \\ &\quad + \tfrac{1}{4}(x_{u-1,v-1} + x_{u+1,v+1} + x_{u-1,v+1} + x_{u+1,v-1}) \end{aligned}$$

$$\mathrm{var}\,(X_{uv} \mid \dots) = \kappa$$

$$f(\omega_1, \omega_2) \propto \{(1 - \cos\,\omega_1)\,(1 - \cos\,\omega_2)\}^{-1}$$

Simple differences $X_{uv} - X_{u+s,v+t}$ no longer have proper distributions ...

and must consider genuine two–dimensional contrasts; in particular,

$$\{X_{uv} - X_{u+1,v} - X_{u,v+1} + X_{u+1,v+1}\}\,'\mathrm{s} \sim \text{ i.i.d. } \mathrm{N}\,(0,\, 4\kappa)$$

Invariant to addition of constants to any rows and to any columns.

Edge corrections: restriction to a finite array $\Rightarrow$ e.g.

$$\mathrm{E}\,(X_{11} \mid \dots) = x_{21} + x_{12} - x_{22}, \qquad \mathrm{var}\,(X_{11} \mid \dots) = 4\kappa,$$
$$\mathrm{E}\,(X_{u1} \mid \dots) = x_{u2} + \tfrac{1}{2}(x_{u-1,1} + x_{u+1,1} - x_{u-1,2} - x_{u+1,2}), \qquad \mathrm{var}\,(X_{u1} \mid \dots) = 2\kappa,$$

Limiting case of separable stationary autoregressions of Martin (1979, 1990) ...

and resembles implicit fertility model in Cullis and Gleeson (1991) but ...

independence of row and column effects $\Rightarrow$ unsuitable as prior distribution.

Ex. Third–order locally quadratic autoregression

Least–squares 12–neighbour locally quadratic fit $\Rightarrow$

$$\begin{aligned} \mathrm{E}\,(X_{uv} \mid \ldots) &= \tfrac{1}{4}(x_{u-1,v} + x_{u+1,v} + x_{u,v-1} + x_{u,v+1}) \\ &+ \tfrac{1}{8}(x_{u-1,v-1} + x_{u+1,v+1} + x_{u-1,v+1} + x_{u+1,v-1}) \\ &- \tfrac{1}{8}(x_{u-2,v} + x_{u+2,v} + x_{u,v-2} + x_{u,v+2}), \end{aligned}$$

$$f(\omega_1, \omega_2) \;\propto\; \{(1 - \cos\,\omega_1)\,(1 - \cos\,\omega_2) + (\cos\,\omega_1 - \cos\,\omega_2)^2\}^{-1}$$

Again, simple differences do not have proper distributions but e.g.

$$\left.\begin{array}{c} X_{uv} - X_{u+s,v} - X_{u,v+t} + X_{u+s,v+t} \\ X_{uv} - 2X_{u+s,v+t} + X_{u+2s,v+2t} \end{array}\right\} \qquad s \neq 0,\; t \neq 0 \quad \text{OK}$$

Invariant to addition of a plane but not to adding constants to rows and columns.

Useful in reconstruction of slowly varying surfaces ...

e.g. Higdon and Yamamoto (2000) on magnetoresistance microscopy.

Combining locally linear, planar and quadratic autoregressions

R.v.'s $\{X_{uv}\colon\ u=1,\ldots,p;\ v=1,\ldots,q\}$ with improper density $\propto\ \exp\{-\frac{1}{2}x^TQ\,x\}$,

where $$x^TQ\,x \;=\; \lambda_{10}\textstyle\sum_{\text{array}}(x_{uv}-x_{u+1,v})^2 \;+\; \lambda_{01}\sum_{\text{array}}(x_{uv}-x_{u,v+1})^2$$
$$+\,\lambda_{11}\textstyle\sum_{\text{array}}(x_{uv}-x_{u+1,v}-x_{u,v+1}+x_{u+1,v+1})^2$$

					rank
$\lambda_{10}>0$	$\lambda_{01}=0$	$\lambda_{11}=0$	$\Rightarrow$	locally linear in each column	$pq-1$
$\lambda_{10}=0$	$\lambda_{01}>0$	$\lambda_{11}=0$	$\Rightarrow$	locally linear in each row	$pq-1$
$\lambda_{10}=\lambda_{01}>0$		$\lambda_{11}=0$	$\Rightarrow$	locally planar	$pq-1$
$\lambda_{10}=0$	$\lambda_{01}=0$	$\lambda_{11}>0$	$\Rightarrow$	locally quadratic	$(p-1)\,(q-1)$
$\lambda_{10}>0$	$\lambda_{01}>0$	$\lambda_{11}>0$	$\Rightarrow$	combined formulation	$pq-1$

$$Q \;=\; (H_p\otimes H_q)^T\,(\lambda_{10}\,\Lambda_p\otimes I_q \;+\; \lambda_{01}I_p\otimes\Lambda_q \;+\; \lambda_{11}\,\Lambda_p\otimes\Lambda_q)\,(H_p\otimes H_q)$$

where H_p, H_q are known analytically, as are the diagonal matrices Λ_p, Λ_q.

$$Q^- \;=\; (H_p^{-1}\otimes H_q^{-1})\,(\lambda_{10}\,\Lambda_p\otimes I_q \;+\; \lambda_{01}I_p\otimes\Lambda_q \;+\; \lambda_{11}\,\Lambda_p\otimes\Lambda_q)^-\,(H_p^{-1}\otimes H_q^{-1})^T$$

where $(\ldots)^-$ is trivial. Also

$Q^{-\frac{1}{2}} \;=\; \ldots$ and eigenvalues of $Q \;=\; \ldots$

Extends immediately to $\lambda I_{pq}+Q$ but not immediately to $\lambda\Delta+Q$,

where Δ is a diagonal matrix with diagonal elements 0's and 1's.

Hierarchical–t intrinsic autoregressions (Besag and Higdon, 1999)

Write $w = \{w_{ij}\}$ and let $X|\,w$ have Gaussian full conditionals with

$$\mathrm{E}\,(X_i\,|\,x_{-i},\,w) \;=\; \sum_j \frac{w_{ij}}{w_{i+}}\,x_j, \qquad \mathrm{var}\,(X_i\,|\,x_{-i},\,w) \;=\; \frac{1}{\lambda^2 w_{i+}},$$

(i) $w_{ii} = 0$, (ii) $w_{ij} = w_{ji} \geq 0$, (iii) $w_{i+} > 0$, (iv) $\lambda > 0$

Define sites i and j to be *neighbours* and write $i \heartsuit j$ if $w_{ij} > 0$, so that

$$\pi(x\,|\,w) \;\propto\; \exp\{-\tfrac{1}{2}\lambda^2 \sum_{i \heartsuit j} w_{ij}(x_i - x_j)^2\},$$

and suppose the w_{ij}'s for $i \heartsuit j$ are a random sample from a $\Gamma(\frac{1}{2}f, \frac{1}{2}f)$ distribution.

Linear graph $\Rightarrow$ random walk with independent scaled t_f increments.

In general $\Rightarrow$ $\{\, X_i - X_j : \; i \heartsuit j \,\}$, i.i.d. scaled t_f, subject to logical constraints.

Pairwise difference Markov random fields (Besag, Green, Higdon and Mengerson, 1995)

$$\pi(x) \;\propto\; \exp\{-\sum_{i \heartsuit j} w_{ij}\psi(\lambda|x_i - x_j|)\}, \qquad x \in \mathcal{R}^n,$$

$$\Rightarrow \qquad \pi(x_i\,|\,\ldots) \;\propto\; \exp\{-\sum_{j \in \partial i} w_{ij}\psi(\lambda|x_i - x_j|)\}, \qquad x_i \in \mathcal{R},$$

e.g. $\psi(z) = |z|$: median–based prior (Besag, York and Mollié, 1991)

e.g. $\psi(z) = \delta(1+\delta)\ln\cosh(z/\delta)$: log–concave differentiable prior (Green, 1990)