En homenaje
al compañero

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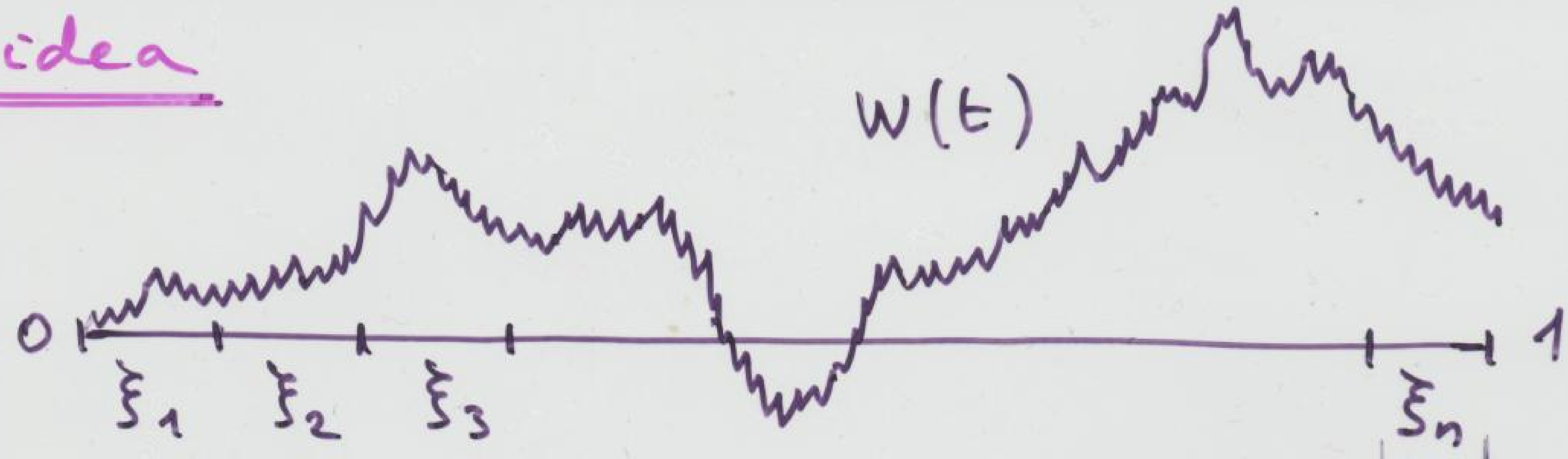
Que sont les processus à deux indices devenus ?

Paris 1 - 30 Mai 2003

1^{rst} idea



at $t = \frac{k}{n}$ $\rightsquigarrow$ $\xi_k = W\left(\frac{k+1}{n}\right) - W\left(\frac{k}{n}\right)$

$$Z \in \sigma(\xi_1, \dots, \xi_n) = f(\xi_1, \dots, \xi_n)$$

ordered in $k \downarrow$, then in ξ_k^r:

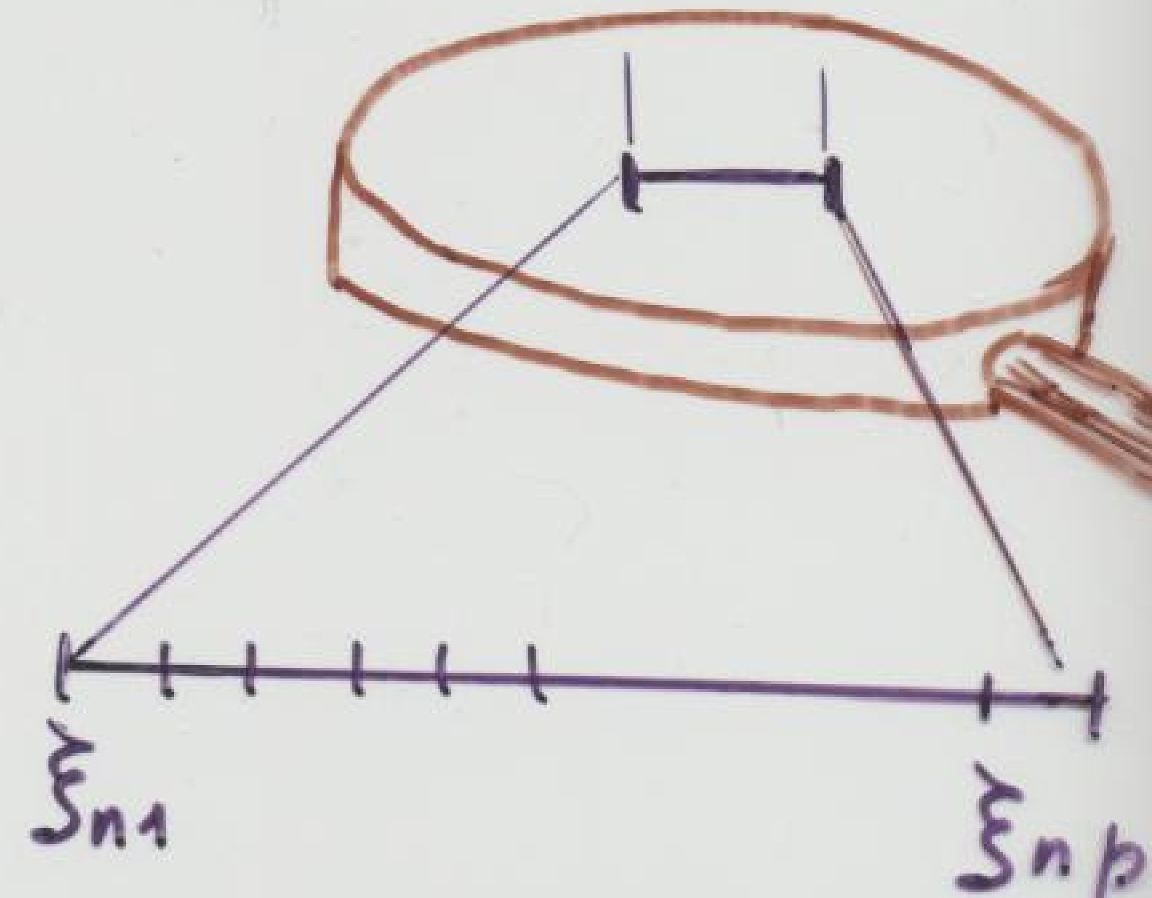
$$Z = a_0 + \sum_{k=n}^{1} \xi_k \, a_{1,k}(\xi_1, \dots, \xi_{k-1})$$
$$+ \sum_{k=n}^{1} \xi_k^2 \, a_{2,k}(\xi_1, \dots, \xi_{k-1})$$
$$+ \sum_{k=n}^{1} \sum_{r \geq 3} \xi_k^r \, a_{r,k}(\xi_1 \dots \xi_{k-1})$$

In a triangular array frame,

$$\xi_k = \sum_{j=1}^{p} \xi_{k,j}$$

var $\xi_{kj} = \frac{1}{p}$ var ξ_k "in" $\frac{1}{np}$

terms in ξ_k^r for $r \geq 3$ disappear



$$Z = a_0 + \int_0^1 a_1(t)\, W(dt) + \int_0^1 a_2(t)\, dt$$

$$Z = a_0 + \int_0^1 a_1(u)\, W(du) + \int_0^1 a_2(u)\, du$$

Moreover

$$\mathcal{F}_t = \sigma\left(W(u),\ u \leq t\right)$$

$$Z_t = E\left(Z \mid \mathcal{F}_t\right)$$

$$= a_0 + \int_0^t a_1(u)\, W(du) + \int_0^t a_2(u)\, du$$

also written

$$Z(dt) = a_0 + a_1(t)\, W(dt) + a_2(t)\, dt$$

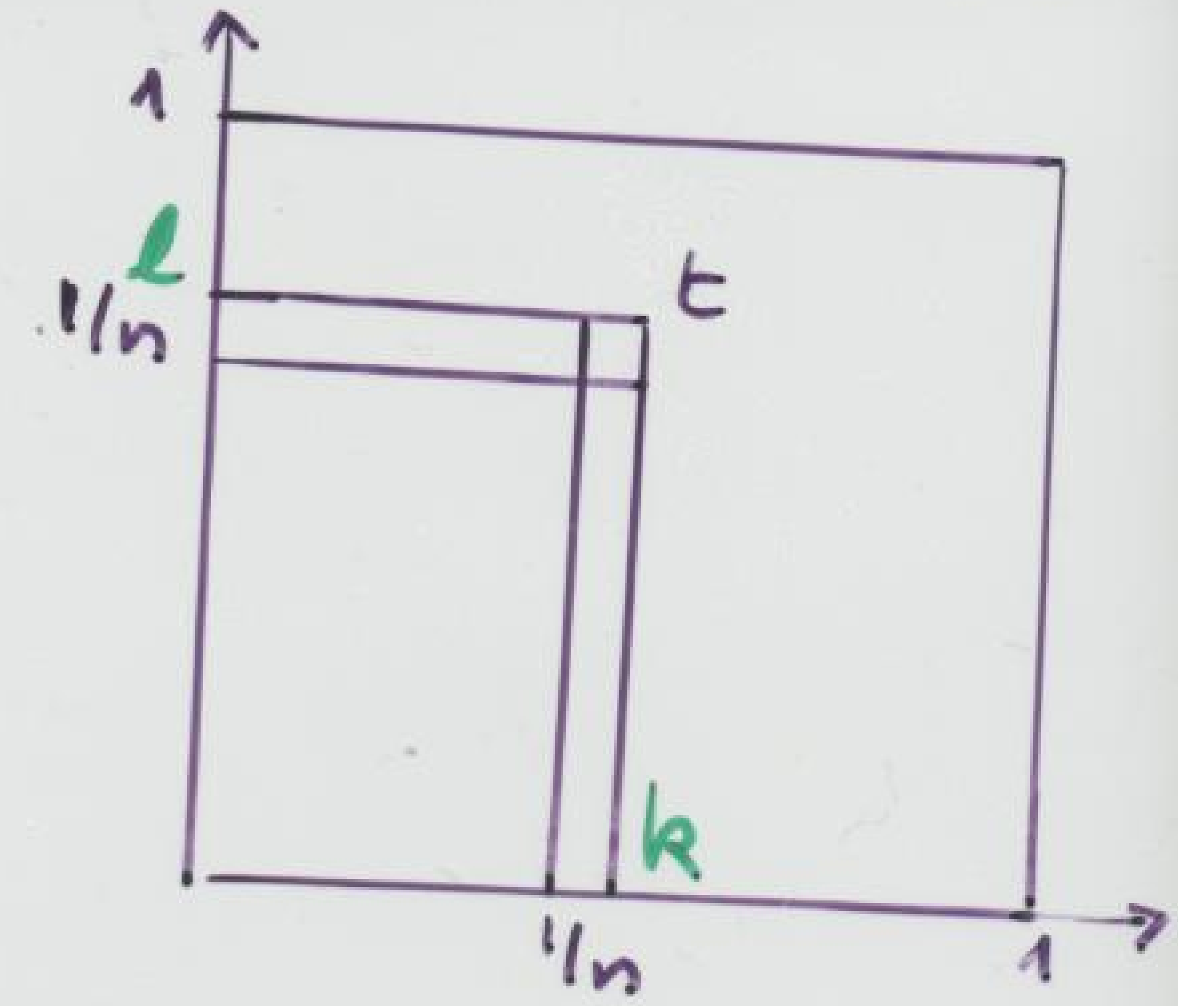
- diffusion
- semi-martingale

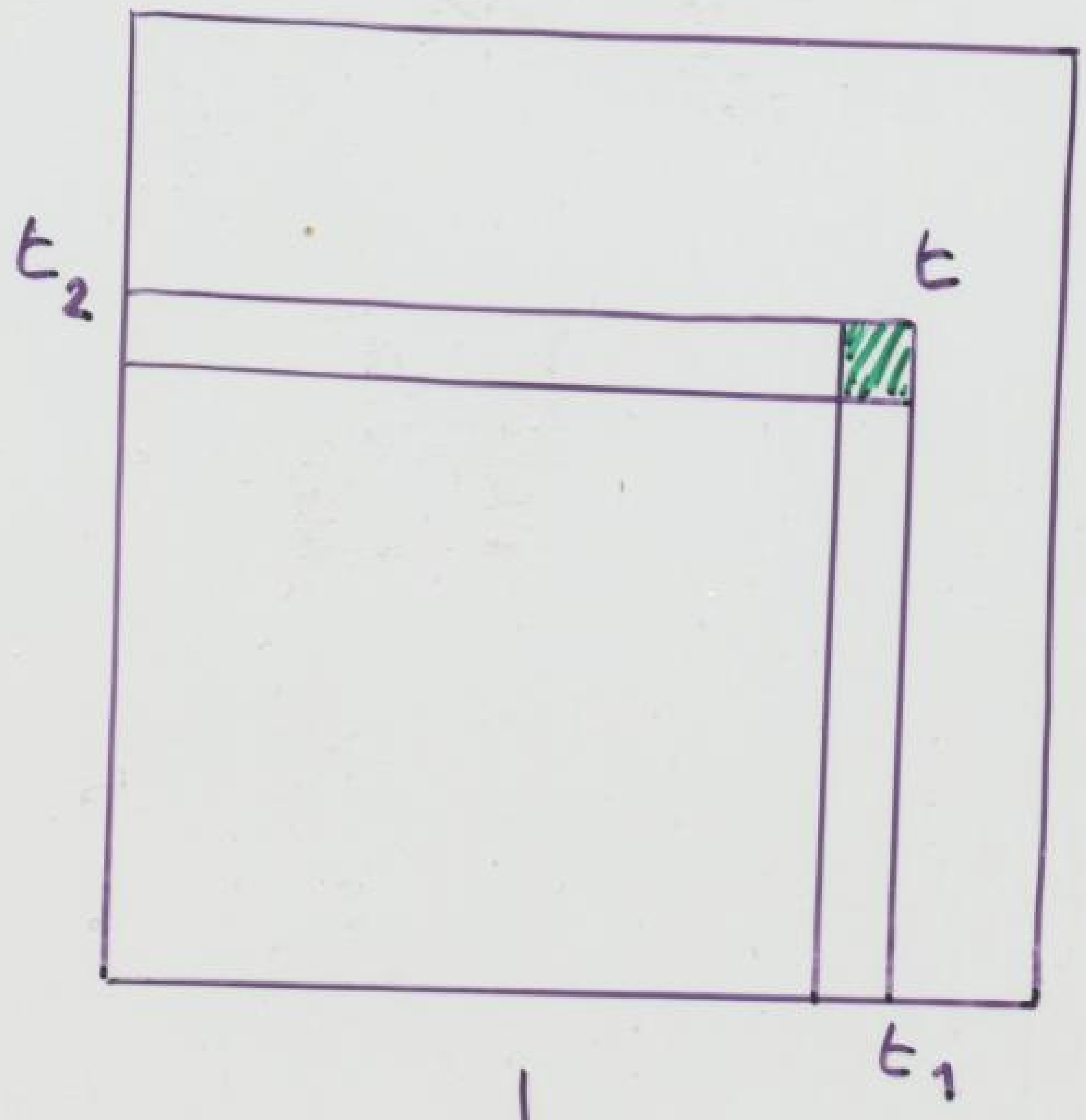
Martingale iff $a_2 \equiv 0$

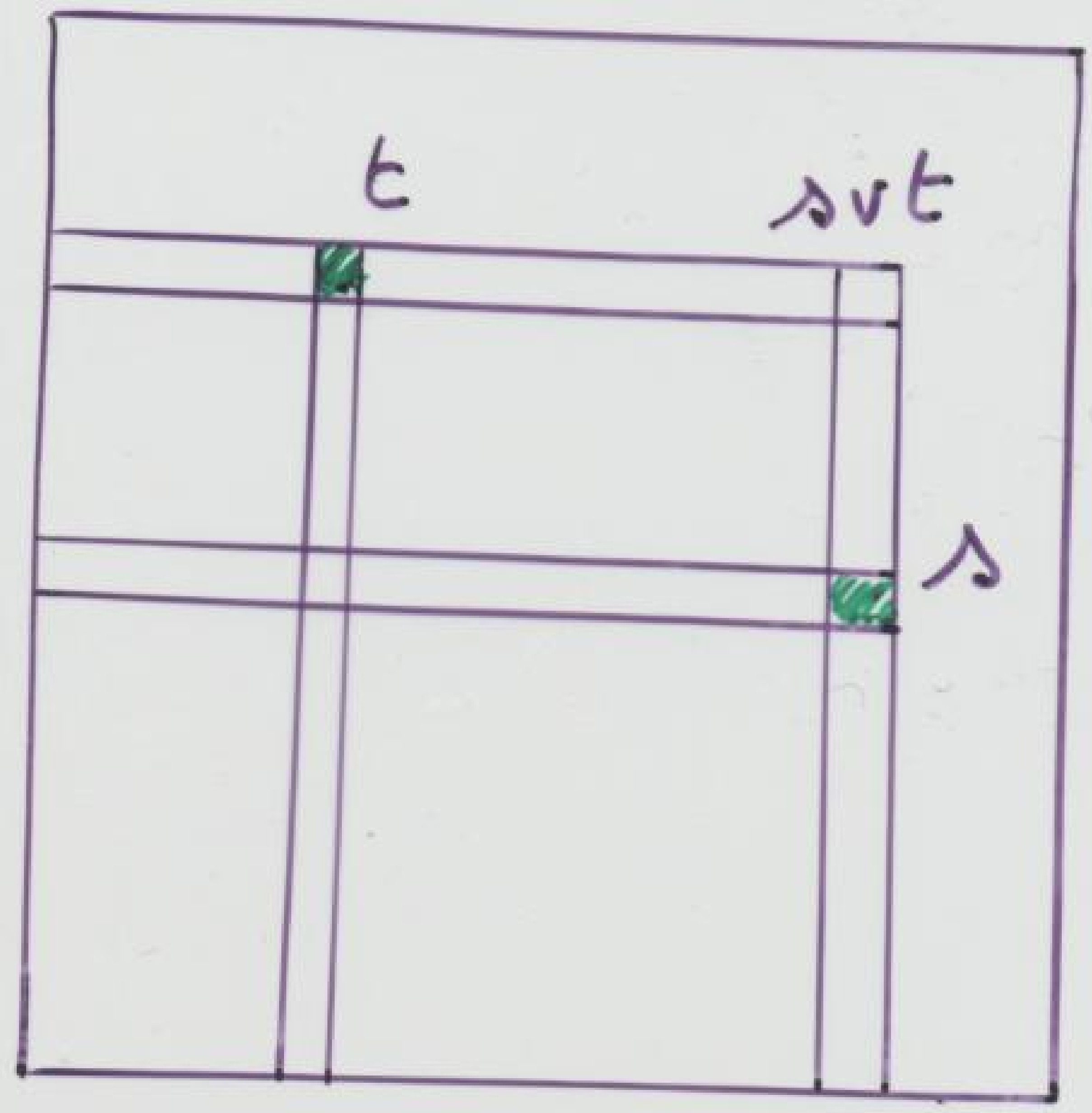
$$Z_t - Z_0 = \int_0^t a_1(u)\, W(du)$$

With two indices

$$\xi_{n,(k,\ell)} \sim \mathcal{N}\left(0;\frac{1}{n^2}\right)$$







$$\downarrow$$

$$\int \varphi(t_1,t_2)\, W(dt_1, dt_2)$$

double

and

$$\int b(t_1,t_2)\, dt_1\, dt_2$$

φ, b are $\mathcal{F}_t$-mesurable

$$\iiiint \psi((t_1,s_2),(t_2,s_1))\, W(dt_1, ds_2)\, W(ds_1, dt_2)$$

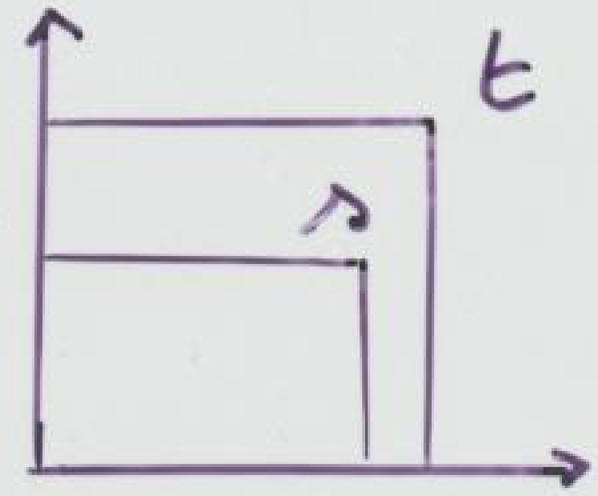
$$\iiiint \psi_1(\qquad)\, dt_1\, ds_2\, W(ds_1, dt_2)$$

$$\iiiint \psi_2(\qquad)\, W(dt_1, ds_2)\, dt_1\, dt_2$$

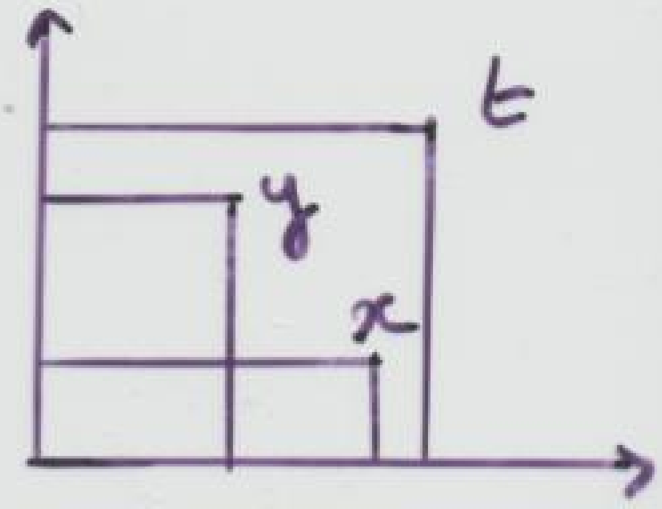
ψ, ψ_1, ψ_2 are $s \vee t$-mesurable

$$Z_t = \int \varphi(s)\, W(ds) + \int b(s)\, ds$$

indicates measurability

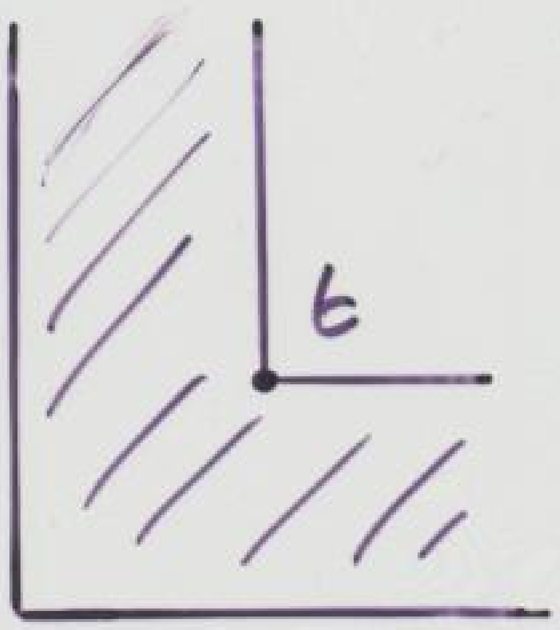


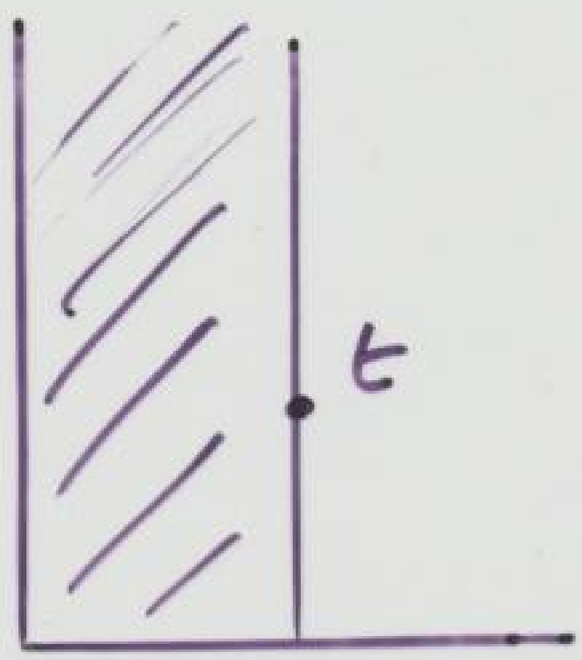
$$+ \iint \psi(x \vee y)\, W(dx)\, W(dy)$$

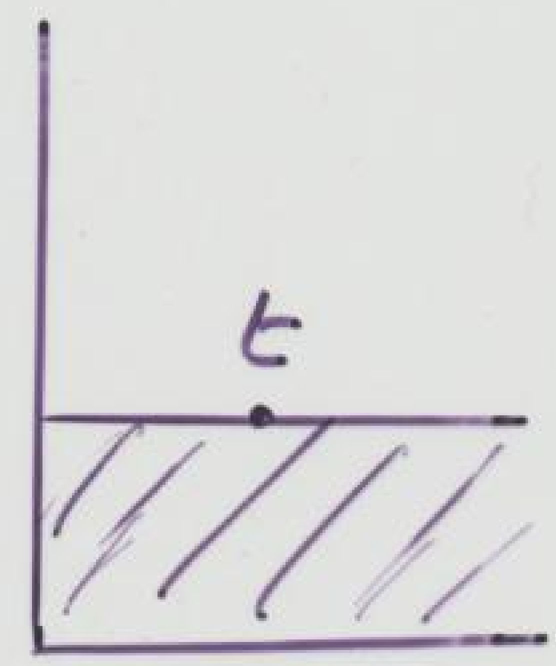


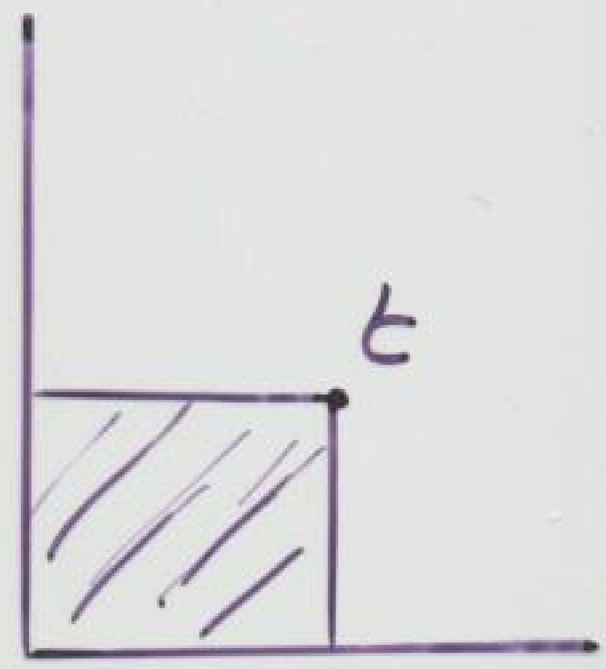
$$+ \iint \psi_1(x \vee y)\, W(dx)\, dy + \iint \psi_2(x \vee y)\, dx\, W(dy)$$

$\Rightarrow$ various notions of martingale:







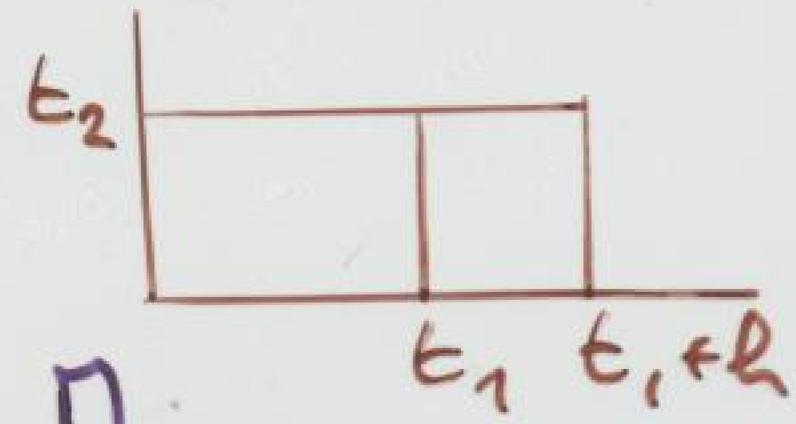


$\mathcal{G}_t$	$\mathcal{F}^1_t$	$\mathcal{F}^2_t$	$\mathcal{F}_t$
weak	1-mart	2 mart	strong
$b=0$	$b=0$	$b=0$	$b=0$
	$\psi_1=0$		$\psi_1=0$
		$\psi_2=0$	$\psi_2=0$
			$\psi=0$

Orthomartingales

New

$\mathbb{F}^1, \mathbb{F}^2$ two (one-parameter) filtrations



orthosubmartingale

measurability +

$$E\left(M_{t_1+h,\,t_2} \,\middle|\, \mathbb{F}^1_{t_1}\right) \geq M_{t_1 t_2}$$

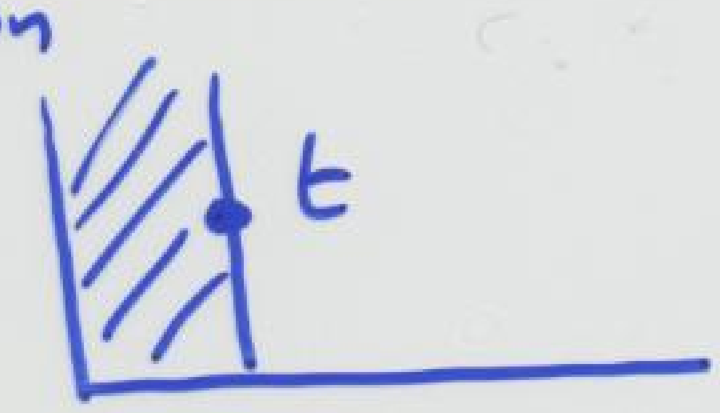
$$E\left(M_{t_1,\,t_2+\ell} \,\middle|\, \mathbb{F}^2_{t_2}\right) \geq M_{t_1 t_2}$$

orthosupermartingale $\leq$

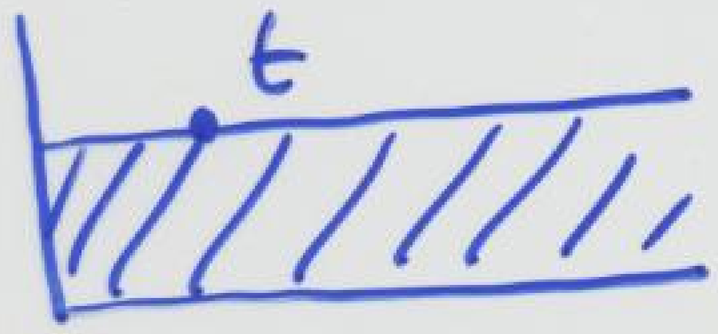
orthomartingale $=$

CP: $\mathbb{F}_t$ a two param. filtration

its marginal filtrations

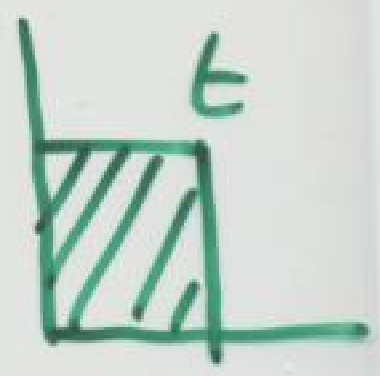


$$\mathbb{F}^1_{t_1} = \bigvee_{u_1=t_1} \mathbb{F}_u$$



$$\mathbb{F}^2_{t_2} = \bigvee_{u_2=t_2} \mathbb{F}_u$$

CPCP: orthohistories



$$\mathcal{H}_t = \sigma(M_u,\ u \leq t)$$

its marginals $\mathcal{H}^1_{t_1}$, $\mathcal{H}^2_{t_2}$

for orthomartingales ≥ 0 Doob inequality applies (Cairoli)

$$E\left[\max_{0<s<t} M_s^{\,p}\right] \leq \left(\frac{p}{p-1}\right)^{2p} E(M_t^{\,p})$$

$$\updownarrow$$

$$M_t \ln_+ M_t$$

also generalization for $p=1$

Martingales

$\mathbb{F}_t$ a (two-parameter) filtration

submartingale: $s < t$ $\quad E(M_t \mid \mathbb{F}_s) \geq M_s$

super martingale $\quad \leq$

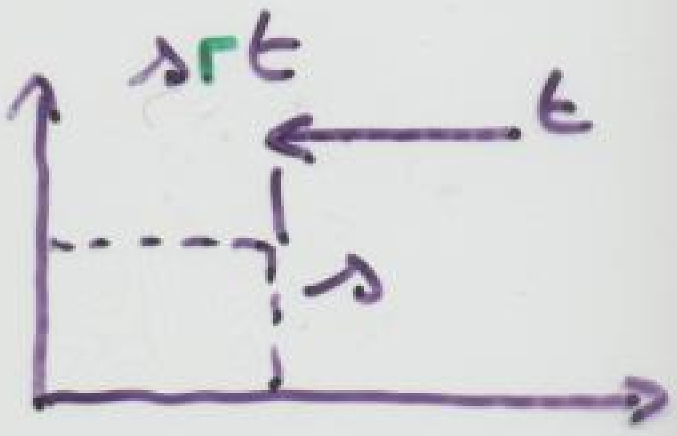
martingale $\quad =$

$$\boxed{\begin{array}{l}\textbf{Thm} \\ \left.\begin{array}{l}\text{let } M_t \text{ be a} \\ \mathbb{F}_t \text{ submartingale}\end{array}\right\} \Longrightarrow \begin{array}{l}M_t \text{ is an orthosubmartingale} \\ \text{for the marginal filtrations } \mathbb{F}^1, \mathbb{F}\end{array}\end{array}}$$

obvious: $s < t$:

$$E[M_t \mid \mathbb{F}_s] = E\left[E[M_t \mid \mathbb{F}_{s \Gamma t}] \mid \mathbb{F}_s\right]$$

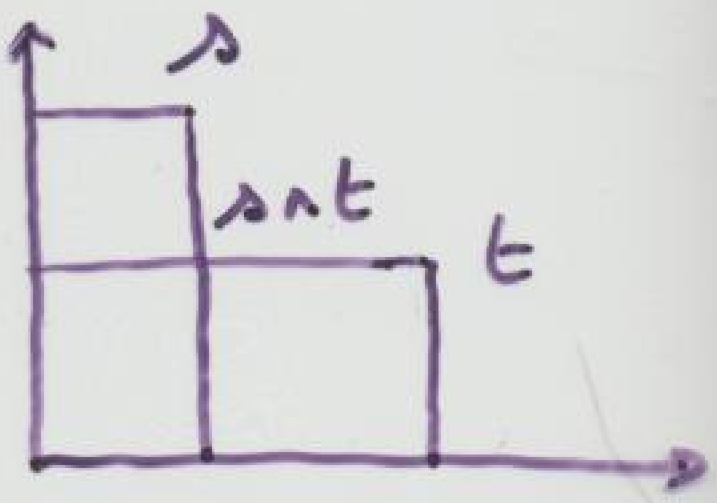
$$\geq E[\ M_{s \Gamma t} \mid \mathbb{F}_s] \geq M_s$$



Reciprocal: false in general

sufficient cd:

$$\boxed{\mathbf{F4:} \quad \mathbb{F}_s \underset{\mathbb{F}_{s \wedge t}}{\perp\!\!\!\perp} \mathbb{F}_t}$$



$$\Longleftrightarrow \quad E(Y_t \mid \mathbb{F}_s) = E(Y_t \mid \mathbb{F}_{s \wedge t})$$

$$\Longrightarrow \quad \geq E\left[E(Y_t \mid \mathbb{F}^1_{s_1}) \mid \mathbb{F}^2_{t_2}\right]$$

and

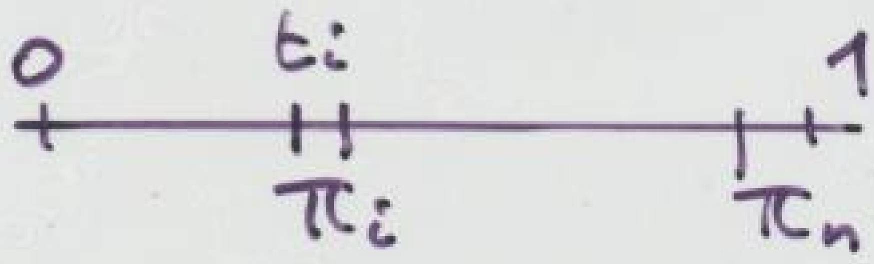
$$\geq E\left[E(Y_t \mid \mathbb{F}^2_{t_2}) \mid \mathbb{F}^1_{s_1}\right]$$

if $s < t$

$$E(Y_t \mid \mathbb{F}_s) \geq M_s$$

Quadratic variations and product variations

on $\mathbb{R}$

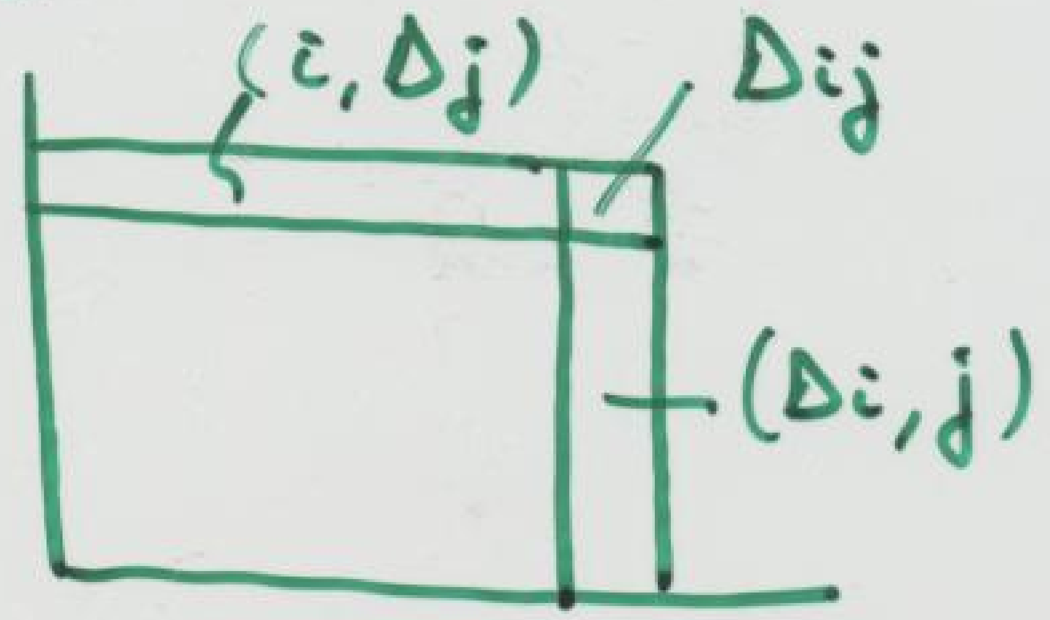


$X = smr = M + B$

$$\sum f_n(t_i)\, X(\Delta_i)^\alpha \longrightarrow \begin{cases} \int f(u)\, X(du) & \alpha = 1 \\ \int f(u) \langle M \rangle du & \alpha = 2 \end{cases}$$

$$\sum f_n(t_i)\, X(\Delta_i)\, Y(\Delta_i) \longrightarrow \int f(u) \langle M_X, M_Y \rangle du$$

on $\mathbb{R}^2$



$$\sum f_n(t_i)\, X(\Delta_{ij})^\alpha\, X(\Delta_i, j)^\beta\, X(i, \Delta_j)^\gamma$$

$\Rightarrow$ <u>huge</u> number of cases

With results like

sum $\longrightarrow$	integral w.r.t t
$X(\Delta_{ij})\, Y(i, \Delta_j)$	$\langle M_X^2(dx, \cdot)\, M_Y^2(x, \cdot) \rangle^{(2)} dy$
$X(\Delta_i, j)\, Y(i, \Delta_j)$	$J_{XY} := X(dx, y)\, Y(x, dy)$
$X(\Delta_{ij})\, Y(\Delta_i, j)\, Z(i, \Delta_j)$	$\langle M_X, J_{M_Z^2 M_Y^2} \rangle$

all with possible statistical application!

New

few: Peter Imkeller (LN 1308)

more dedicated to processes with jumps

Ito formula

on $\mathbb{R}$

$$X_t = M_t + B_t$$
$$Y_t = g(t, X_t)$$

$$Y_t = \int_0^t g'_t(u)\,du + \int_0^t g'_x(u)\,X(du) + \frac{1}{2}\int_0^t g''_{x^2}(u)\langle M\rangle(du)$$

$$= \underbrace{\int_0^t g'_x(u)\,M(du)}_{\text{mart.}} + \underbrace{\int_0^t g'_t(u)\,du + \int_0^t g'_x(u)\,B(du) + \frac{1}{2}\int g''_{x^2}(u)\langle M\rangle(du)}_{\text{bounded var.}}$$

$\Downarrow$ introduce

$$Dg = g'_t + g'_x \frac{d}{du} B + \frac{1}{2} g''_{x^2} \frac{d}{du}\langle M\rangle$$

ITO: $\boxed{Y_t = \int_0^t g'_x(u)\,M(du) + \int_0^t (Dg)(u)\,du}$

In $\mathbb{R}^2$ $\quad z = (x, y)$

$$X = M + P^1 + P^2 + B$$

D_1 = operator D with respect to 1^{rst} coordinate, x, taking into account the whole derivable part of X w.r.t x

$$Y_{xy} = f(x, y, X_{xy})$$

$$D_1 = f'_x + f'_X\left[\frac{\partial}{\partial x}(P^2 + B)\right] + \frac{1}{2} f''_{X^2}\left[\frac{\partial}{\partial x}(P^1 + M)\right]$$

idem D_2

2D Ito:

$$X = M + P^1 + P^2 + B$$

$$Y_{xy} = f(x, y, X_{xy}) = \mathcal{M}_{xy} + \mathcal{P}^1_{xy} + \mathcal{P}^2_{xy} + \mathcal{B}_{xy}$$

with

$$\mathcal{M} = \int f'_X(u,v)\, M(du,dv) + \int f''_{X^2}(u,v)\, M(du,v)\, M(u,dv)$$

$$\mathcal{P}^1 = \int f'_X(u,v)\, P^1(du,dv) + \int (D_2 f'_X)(M+P^1)(du,v)\, dv$$

$$+ \frac{1}{2} \int f''_{X^2}(u,v) \langle M(du,\cdot), (M+P^2)(u,\cdot) \rangle\, dv$$

$\mathcal{P}^2$: idem

$$\mathcal{B} = \int D_2(D_1 f)(u,v)\, du\, dv \qquad \left(= \int D_1(D_2 f) \right)$$

developped $\longrightarrow$ 22 terms

generalized for $f(z, X_1(z), X_2(z), \dots, X_k(z))$

new $\longrightarrow \emptyset$

What is new ?

(against) an "important" activity

exple: books <u>Multiparameter processes</u>
(Davar Khoshnevisan 2000)
<u>Spatial Stochastic Processes</u>
(LN 1802 to appear!)
Martina Franca. Ed.: Ely Merzbach
+ Capasso, Dalang, Dozzi,
Ivanoff, Mountford

<u>Axes</u>

- EDP stochastic: Robert Dalang, Annie Millet, Martha Sanz
  2 parameter $\mathbb{R}$ valued processes
  more than
  1 parameter process taking values in a functional space
- Processes indexed by sets: Ely Merzbach, Marco Dozzi, Gail Ivanoff
  «general theory of processes» in this frame,
  martingales, stochastic integrals, ...
- "Complete" analysis of some standard cases
  B. Bonfoussi, R Dalang, E Merzbach, P. Imkeller
  brownian sheet
  fractionary brownian sheet
  stable sheet

Envoi:

There remains a field of research concerning two (or more) parameter processes based on semi-martingales

- diffusion like
- cf 2D Ornstein Uhlenbeck

with very important statistical applications

- surfaces in metallurgy, swell, ..., agronomy!

and appropriate tools

- Ito formula
- Girsanov theorem
- Quadratic variations

To whom can we handover?

Qui prend le relais?

¿ Quien hace el relevo?