Université Paris I, Panthéon - Sorbonne

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Time Series Tutorial n^0 4 : Parameter estimation of ARMA and GARCH processes

The aims of this tutorial is the computation of QMLE estimators of ARMA and GARCH processes.

Yule-Walker and QMLE estimation of the parameters of an AR[1] process

In the sequel, we define and compute a natural estimator of the parameters of a AR[1] process, which is the Yule-Walker estimator. Hence, if $X_{t+1} = \theta\, X_t + \xi_t$ for $t \in \mathbf{Z}$, with $|\theta| < 1$, then $\theta = \mathrm{cov}(X_0, X_1)/\mathrm{var}(X_0)$. Hence, the Yule-Walker estimator of θ and σ_ξ^2 could be obtained by replacing "theoretical" covariances by empirical covariances:

$$\widehat{\theta} = \frac{\frac{1}{n}\sum_{i=1}^{n-1} X_i X_{i+1}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2}.$$

Use the previous program for exhibiting the asymptotic behavior of $\widehat{\theta}$. Consider the cases $n = 100$, $n = 500$ and $n = 1000$. How to check the $\sqrt{n}$ convergence rate of this estimator?

For estimating the variance σ_ξ^2 of ξ_t, a natural estimator is:

$$\widehat{\sigma}_\xi^2 = \frac{1}{n}\sum_{i=1}^{n-1}\big(X_{i+1} - \widehat{\theta}\, X_i\big)^2.$$

Explain this expression. Exhibit also the asymptotic behavior of this estimator. Which convergence rate could be expected?

Write the Gaussian conditional log-likelihood of (X_t) and deduce the QMLE-estimators. Which differences with the previous one?

QMLE estimation of the parameters of an ARCH[1] process

In GARCH framework, the Yule-Walker estimation could not be used since the auto-covariance of the process are vanished. As a consequence we use a QMLE estimator for estimating the parameters. For instance, write the following commands:

```
n=1000
spec = garchSpec(model = list(omega = 1, alpha = c(0.6), beta = 0))
X=garchSim(spec,n)
plot(X)
logLik=function(the){sum(log(the[1]+the[2]*X[1:(n-1)]^2)+X[2:n]^2/(the[1]+the[2]*X[1:(n-1)]^2))}
optim(c(0.1,0.1),logLik,method = "L-BFGS-B",lower = c(0.01,0.01), upper = c(100,0.999))
QMLE$par
```

Explain all the commands. By repeating these commands draw an histogram drawed by $\widehat{\omega}$ and $\widehat{a}_1$. Which distribution could be suspected? Do the same thing for $n = 100$ and $n = 10000$. What is the evolution of the standard error of these estimators with n?
Write a similar program for estimating the parameters of a GARCH[1,1] process, with $b_1 = 0.3$.

A numerically more accurate way for estimating these parameters can also be to use the following procedure:

```
QMLE2=garchFit(~ garch(1,0), data = X, trace = FALSE)
coef(QMLE2)
```

Do you find the same results? Note that this algorithm also provides an estimation of the standard deviation for each estimated parameter. Give it for all the parameters. By choosing different values of n remark how these standard deviations decrease with n.

Application to a financial time series

First, download the daily values of SP500 from November 28 2011 to last Friday (see Yahoo Finance website for instance or JMB webpage). A `.csv` file is downloaded. Then you can read it with R software from the following command (change the directory please!):

```
SP500=read.csv("C:/Users/.../TP/SP500.csv")
```

Then using the following commands, we consider the log-returns of the SP500 closing values and we estimate the parameters from a GARCH[1,1] model:

```
names(SP500)
X=SP500$Close
ts.plot(X)
n=length(X)
Y=log(X[2:n]/X[1:(n-1)])
ts.plot(Y)
QMLE3=garchFit(~ garch(1,1), data = Y, trace = FALSE)
coef(QMLE3)
```

Whhat is the result? But there is still a question: why use a GARCH[1,1] instead of other GARCH[p,q]? More generally, how to chose a model and test its goodness-of-fit?