Local correlation dimension of multidimensional stochastic process

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Abstract

The computation of the local correlation dimension is a way for estimating the Hausdorff dimension of the image of multidimensional stochastic processes. It can be obtained from the asymptotic behavior of the self-intersection occupation measure around zero. In this paper, we replace the usual indicator function of the occupation measure by a Gaussian kernel. Hence, we obtain the consistency of the local correlation dimension for multivariate fractional Brownian motion. On the other hand, we show that any used norms on $\mathbb{R}^d$ give the same asymptotic behavior of the occupation measure. The use of a numerical procedure based on $\log-\log$ least square estimator and Monte-Carlo experiments confirm the theoretical results and provide an efficient way of estimation of the Hausdorff dimension. In addition, we show that our proposed estimation method performs the univariate one on the estimation of the Hausdorff dimension.

Keywords: Occupation measure, Hausdorff dimension, local correlation dimension, Gaussian kernel correlation integral, semi-parametric estimation

1. Introduction

For a stochastic process X of $\mathbb{R}^d$, the Hausdorff dimension of its image $\{X_t, t \in [0,T]\}$ or its graph $\{(t, X_t), t \in [0,T]\}$ is an important measure to its roughness. On this point, the Hausdorff dimension D_H for different stochastic processes has been studied in literature. Taylor in [17] use the method of potential theory to determinate the Hausdorff dimension of the image of a d-dimensional Brownian motion and he proved that $D_H = \min\{d, 2\}$. The Lévy processes are studied by Pruit in [14], where he established the formula of the dimension of the image of a general Lévy process in terms of its potential measurement. The fractional Brownian motions are treated in [18] and their Gaussian or α-stable extensions are studied in [8].

To obtain or estimate the Hausdorff dimension of the image of an $\mathbb{R}^d$ stochastic process X, its occupation measure plays an important role using that it is the natural measure carried by its image. We remind that the occupation measure of the process X is defined, for all Borel set $A \subset \mathbb{R}^d$ and $T > 0$, by:

$$\mu_T(A) = \int_0^T \mathbb{I}_{(X_s \in A)}\, ds,$$

where $\mathbb{I}$ is the indicator function. The existence of the density of such occupation measure with respect to the Lebesgue measure (also called the local time) for a one-dimensional Brownian motion has been established by

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Lévy. For $d \geq 2$, this local time no more exists, and the appropriate tool to describe the local regularity is the asymptotic behavior of $\mu_T(A)$ when A is small. More precisely, and if it exists, define

$$D_P(x) = \lim_{r \to 0} \frac{\log(\mu_T(B(x,r)))}{\log(r)},$$

where $B(x,r)$ is the ball of centre x and radius r. Hence, Perkins and Taylor in [12] proved that $D_P(x) = D_H = 2$ for a 2-dimensional Brownian motion at any point of its support.

Bardet in [2] has considered another occupation measure, which he called bivariate occupation measure or self-intersection occupation measure of the process X. This measure is defined, for J a compact set of $\mathbb{R}^2$ and $I \subset \mathbb{R}^d$, by:

$$\mu_b(I, J) = \int_J \mathbb{I}_{(X_t - X_s \in I)}\, dsdt.$$

Bardet studied the asymptotic behavior of $\mu_b(B_d(0,r), [0,T]^2)$ when $r \to 0$ for continuous stochastic processes of $\mathbb{R}^d$. Interested to gaussian processes with stationary increments that are fractional Brownian motion and α-stable process with independent components. Using that

$$\mu_b(B_d(0,r), [0,T]^2) = \int_0^T \int_0^T \mathbb{I}_{(\|X_t - X_s\| \leq r)}\, dtds, \tag{1}$$

where $\|.\|$ is a norm on $\mathbb{R}^d$, Bardet studied the asymptotic behavior of $\mu_b(B_d(0,r), [0,T]^2)$ when $r \to 0$. Hence, if it exists, he defined the local correlation dimension ν by:

$$\nu = \lim_{r \to 0} \frac{\log(\mu_b(B_d(0,r), [0,T]^2))}{\log(r)}. \tag{2}$$

This dimension ν is proved to be less or equal to the Hausdorff dimension D_H, and for multivariate fractional Brownian motions or α-stable process, he proved that $\nu = D_H$.

In practice, Diks in [7] showed that there is a difference between a weight to be given to all distances $\|X_i - X_j\|$. In this way, the smallest distances between pairs (X_i, X_j) have the smallest weights. This problem is solved by using a Gaussian kernel function instead of the indicator function in the occupation measure. Hence he considered the Gaussian kernel bivariate occupation measure

$$\mu_G(B_d(0,r), [0,T]^2) = \int_0^T \int_0^T \exp\left(\frac{-\|X_t - X_s\|^2}{4r^2}\right) dtds,$$

as well as the Gaussian kernel correlation integral defined for $r > 0$ by:

$$C^G_{X,T}(r) = \frac{\mu_G(B_d(0,r), [0,T]^2)}{T^2} \tag{3}$$

and if it exists we define the local Gaussian kernel correlation dimension by:

$$\nu_G = \lim_{r \to 0} \frac{\log(C^G_{X,T}(r))}{\log(r)}. \tag{4}$$

In a practical framework, by considering $N+1$ vectors $(X_0, X_{T/N}, \ldots, X_T)$ from a trajectory $X(\omega) = \{X_t(\omega), t \in [0,T]\}$ of the process X, we consider $C^G_{X,N}(r)$ the empirical version of $C^G_{X,T}(r)$ defined by:

$$C^G_{X,N}(r) = \frac{2}{N(N-1)} \sum\sum_{0 \leq i < j \leq N} \exp\left(\frac{-\|X_{iT/N} - X_{jT/N}\|^2}{4r^2}\right). \tag{5}$$

Clearly, if X is an almost sure continuous process, we could expect for any $r > 0$,

$$\lim_{N\to\infty} C^G_{X,N}(r) = C^G_{X,T}(r).$$

In this paper, we study the asymptotic behavior of Gaussian kernel correlation integral given by (3) when r tends to zero for a $\mathbb{R}^d$-multidimensional fractional Brownian motion with independent components and with index $(\alpha_1, \alpha_2, \dots, \alpha_d)$, by convention $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_d$ (for an introduction to these processes, see chapter 8 of [1]). In this case, we partially show that the local Gaussian kernel correlation dimension of such process is given by:

$$\nu_G = D_H = \min\Big(\Big(\frac{1 + \sum_{k=1}^{i}(\alpha_i - \alpha_k)}{\alpha_i}\Big)_{1\le i\le d}, d\Big), \tag{6}$$

and therefore a $\log - \log$ regression of $C^G_{X,N}(r)$ onto r could provide an estimation of the Hausdorff dimension of the image of X. All theoretical results are detailed in section 2. The section 3 is devoted to Monte-Carlo's experiments on the estimation of Hausdorff dimension for $\mathbb{R}^d$-multidimensional fractional Brownian motion based on the computation of local Gaussian kernel correlation dimension ν_G defined in (4).

2. Asymptotic behavior of the Gaussian kernel correlation integral

In all this article we use the notation:

$$\overset{a.s.}{\underset{r\to 0}{\sim}} \ : \quad \text{almost sure asymptotic equivalence when } r \to 0 :$$

$$f(r) \overset{a.s.}{\underset{r\to 0}{\sim}} g(r) \iff \frac{f(r)}{g(r)} \overset{a.s.}{\underset{r\to 0}{\longrightarrow}} 1$$

$$\overset{a.s.}{\underset{p\to +\infty}{\sim}} \ : \quad \text{almost sure asymptotic equivalence when } p \to \infty :$$

$$f(p) \overset{a.s.}{\underset{p\to +\infty}{\sim}} g(p) \iff \frac{f(p)}{g(p)} \overset{a.s.}{\underset{p\to +\infty}{\longrightarrow}} 1.$$

Denote also

$$\begin{aligned} S(j) &= \alpha_1 + \alpha_2 + \cdots + \alpha_j, \ S(0) = 0, \quad \text{and} \\ V(j) &= \frac{1 + \sum_{k=1}^{j}(\alpha_j - \alpha_k)}{\alpha_j}. \end{aligned}$$

If $S(d) \ge 1$, let integer $i_0 \in \{2, \cdots, d\}$ be such that $S(i_0 - 1) < 1$ and $S(i_0) \ge 1$.

Assume that X is a multidimensional fractional Brownian motion in $\mathbb{R}^d$ with index $(\alpha_1, \alpha_2, \dots, \alpha_d)$, i.e. $X = \{X_t = (X^{(1)}_t, X^{(2)}_t, \dots, X^{(d)}_t)', t \in [0, T]\}$ satisfies the following assumptions:

- **(H1)** X is a d-dimensional Gaussian process, with zero-mean and stationary increments.
- **(H2)** For each $i = 1, \dots, d$, there exist $\alpha_i \in]0, 1[$ and $a_i > 0$ such that for all $t \in [0, T]$:

$$\mathbb{E}[(X^{(i)}_t - X^{(i)}_0)^2] = \sigma^2_i(t) = a_i |t|^{2\alpha_i}. \tag{7}$$

- **(H3)** For all $(s, t) \in [0, T]^2$, for each $(i, j) \in \{1, \dots, d\}^2$ such that $i \neq j$, $X^{(i)}_t$ is independent of $X^{(j)}_s$.

Then, the Gaussian kernel correlation integral of such a process verifies:

Proposition 2.1. *Let X verify assumptions* **(H1)**, **(H2)** *and* **(H3)**. *For $z = (z_1, \dots, z_d) \in \mathbb{R}^d$, denote $\|z\|^2 = \frac{z_1^2}{2a_1} + \cdots + \frac{z_d^2}{2a_d}$ and assume $0 < \alpha_1 \le \cdots \le \alpha_d < 1$. Then,*

1. *if $\alpha_1 + \cdots + \alpha_d = 1$,*
$$C^G_{X,T}(r) = \frac{2^d}{T\,\alpha_d}\, r^d \,\log(1/r)\big(1 + \eta_r\big) \qquad \textit{with} \qquad \mathbb{E}[\eta_r^2] \underset{r\to 0}{\longrightarrow} 0. \tag{8}$$
2. *if $\alpha_1 + \cdots + \alpha_d > 1$, then*
$$C^G_{X,T}(r) \underset{r\to 0}{\overset{a.s.}{\sim}} \begin{cases} \dfrac{2^{i_0+1}}{T}\left(\dfrac{\alpha_{i_0+1} - \alpha_{i_0}}{\alpha_{i_0+1}\alpha_{i_0}}\right) r^{i_0}\,\log(1/r) & \textit{if } \alpha_1 + \cdots + \alpha_{i_0} = 1 \textit{ and } \alpha_{i_0+1} \neq \alpha_{i_0}; \\ \dfrac{2^{\nu_G}}{T\alpha_{i_0}}\left(\dfrac{1}{1-S(j_0-1)} + \dfrac{1}{S(k_0)-1}\right) r^{\nu_G} & \textit{else,} \end{cases} \tag{9}$$
with
   - *$i_0 \in \{2, \dots, d\}$ be such that $\alpha_1 + \cdots + \alpha_{i_0-1} < 1$ and $\alpha_1 + \cdots + \alpha_{i_0} \ge 1$,*
   - *$j_0 = \min\{j \in \{1, \dots, i_0\}, \alpha_j = \alpha_{i_0}\}$ and $k_0 = \max\{k \in \{i_0, \cdots, d\}, \alpha_k = \alpha_{i_0}\}$.*

Proof of Proposition 2.1. In Bardet [2], the asymptotic behavior of the bivariate occupation measure $\mu_b(B_d(0,r), [0,T]^2)$ defined in (1) when $r \to 0$ is obtained from the asymptotic behavior of its Laplace transform $\widehat{\mu}(p)$ when $p \to \infty$ from a Tauberian theorem. But using usual its Laplace transform $\widehat{\mu}(p)$ satisfies:
$$\widehat{\mu}(p) = \int_0^T \int_0^T \exp\big(-p\,\|X_t - X_s\|^2\big) dtds.$$
Therefore, using its definition in (3), we have
$$C^G_{X,T}(r) = \frac{1}{T^2}\,\widehat{\mu}\big(1/4r^2\big).$$
Hence, we can use the asymptotic behavior of $\widehat{\mu}(p)$ when $p \to \infty$ established in [2] to obtain the one of $C^G_{X,T}(r)$ when $r \to 0$. As a consequence:

1. If $\alpha_1 + \cdots + \alpha_d = 1$, then it was proved that $\mathbb{E}\big[\widehat{\mu}(p)\big] \sim \frac{1}{T\,2\alpha_d}\, p^{-d/2}\,\log(p)$ when $p \to \infty$ and in Lemma 4.3 of [3], that
$$\mathbb{E}\Big[\Big|\frac{p^{d/2}}{\log(p)}\,\widehat{\mu}(p) - \frac{1}{T\,2\alpha_d}\Big|^2\Big] \le \frac{\log(\log(p))}{\log(p)},$$
inducing $\widehat{\mu}(p) = \frac{1}{T\,2\alpha_d}\, p^{-d/2}\,\log(p)\Big(1 + \eta_p\Big)$ with $\mathbb{E}\big[|\eta_p|^2\big] \underset{p\to+\infty}{\longrightarrow} 0$. Therefore, by replacing p with $1/4r^2$, we obtained the $\mathbb{L}^2$ consistency of $C^G_{X,T}(r)$ when $r \to 0$.

2. If $\alpha_1 + \cdots + \alpha_d > 1$, it was proved that
$$\widehat{\mu}(p) \underset{p\to+\infty}{\overset{a.s.}{\sim}} m_0(\alpha_1, \dots, \alpha_d, T) p^{-\nu_G/2}(\log(p))^{e_0}, \tag{10}$$
where $e_0 = 1$ if $S(i_0) = 1$ and $\alpha_{i_0} \neq \alpha_{i_0+1}$, else $e_0 = 0$, and
$$m_0(\alpha_1, \dots, \alpha_d, T) = \begin{cases} T\left(\dfrac{\alpha_{i_0+1} - \alpha_{i_0}}{2\alpha_{i_0+1}\alpha_{i_0}}\right) & \text{if } S(i_0) = 1 \text{ and } \alpha_{i_0+1} \neq \alpha_{i_0}, \\ T\left(\dfrac{1}{1-S(j_0-1)} + \dfrac{1}{S(k_0)-1}\right) & \text{else,} \end{cases}$$

where $j_0 = \min\{j \in \{1,\ldots,i_0\}, \alpha_j = \alpha_{i_0}\}$ and $k_0 = \max\{k \in \{i_0,\ldots,d\}, \alpha_k = \alpha_{i_0}\}$. Then, the asymptotic behavior of $C^G_{X,T}(r)$ when $r \to 0$ is obtained by replacing p with $1/4r^2$. $\square$

By this way, the Gaussian kernel correlation integral for planar and spatial Brownian motion admits this behavior:

Corollary 2.1. *If X is a $\mathbb{R}^d$-Brownian motion such that for all $(s,t) \in [0,T]^2$ and each $(i,j) \in \{1,\ldots,d\}$, $\mathbb{E}[(X_t^{(i)})^2] = |t|$ and $X_t^{(i)}$ and $X_s^{(j)}$ are independent when $i \neq j$. With $\|.\|_e$ the usual Euclidean norm on $\mathbb{R}^d$,*

1. *if $d = 2$ (X is a planar Brownian motion), $C^G_{X,T}(r) = \dfrac{8}{T}\, r^2 \log\big(1/r\big)\big(1+\eta_r\big)$ with $\mathbb{E}[\eta_r^2] \underset{r\to 0}{\longrightarrow} 0$.*
2. *if $d \geq 3$ (X is a $\mathbb{R}^d$-spatial Brownian motion), $C^G_{X,T}(r) \underset{r\to 0}{\overset{a.s.}{\sim}} \dfrac{8d}{T(d-2)}\, r^2$.*

Remark 2.1. Unfortunately, we did not succeed to obtain a consistency of $C^G_{X,T}(r)$ in the case where $\sum_{i=1}^{d} \alpha_i < 1$. However, in [2] it was proved that $\mu_b(B_d(0,r),[0,T]^2) \underset{r\to 0}{\overset{a.s.}{\sim}} Z\, r^d$ when $r \to 0$, with Z a random variable defined from the self-intersection local time. We could as well conjecture that $C^G_{X,T}(r) \underset{r\to 0}{\overset{a.s.}{\sim}} Z'\, r^d$ when $r \to 0$, with Z' a random variable.

Corollary 2.2. *If we consider the classical Euclidian norm $\|\cdot\|_e$ instead of $\|\cdot\|$ (or other norm on $\mathbb{R}^d$), using the usual property of equivalence of norms in $\mathbb{R}^d$, we have:*

$$c\,\|X_{iT/N} - X_{jT/N}\|^2 \leq \|X_{iT/N} - X_{jT/N}\|_e^2 \leq C\,\|X_{iT/N} - X_{jT/N}\|^2,$$

with $0 < c \leq C < \infty$. Then, if we denote $C^{G,e}_{X,T}(r)$ the random variable defined in (5) but computed with $\|\cdot\|_e$ instead of $\|\cdot\|$, then

$$C^G_{X,T}(r/\sqrt{C}) \leq C^{G,e}_{X,T}(r) \leq C^G_{X,T}(r/\sqrt{c}),$$

for any $r > 0$. Therefore, the asymptotic behavior of $\log(C^{G,e}_{X,T}(r))/\log(r)$ and $\log(C^G_{X,T}(r))/\log(r)$ are the same (and are the same for any norm on $\mathbb{R}^d$).

3. Numerical illustrations

3.1. A conjecture and numerical procedures

From the previous results, we could conjecture that:

$$\nu_G = D_H = \min\Big(\Big(\frac{1+\sum_{k=1}^{i}(\alpha_i - \alpha_k)}{\alpha_i}\Big)_{1\leq i\leq d},\, d\Big), \tag{11}$$

in the case of multivariate fractional Brownian motions. Monte-Carlo experiments will exhibit the validity of a such result.

For this, consider an observed path $(X_{t_0}, X_{t_1}, \ldots, X_{t_N})$ where $t_j = jT/N$ and (X_t) satisfies (H1), (H2) and (H3). Then, as in (5), define

$$C^G_{X,N}(r) = \frac{2}{N(N-1)} \sum\sum_{0 \le i < j \le N} \exp\left(\frac{-\|X_{iT/N} - X_{jT/N}\|^2}{4r^2}\right),$$

and for any $r > 0$, it is clear that $\lim_{N\to\infty} C^G_{X,N}(r) = C^G_{X,T}(r)$. Note that the choice of the norm is free from the result of Corollary 2.2. Hence we will consider the classical Euclidian norm $\|\cdot\|_e$ since it does not contain any parameter of the process (contrarily to $\|\cdot\|$). As a consequence, and as it was already done in several papers (see example [5] and [16]), a numerical procedure for estimating ν_G consists in two steps:

1. Select $M \in \mathbb{N}^*$ and $(r_1, \ldots, r_M)$ a family of positive real numbers. Theoretically $r_i \to 0$ and this means that the r_i should be chosen small enough numerically.
2. Compute $C^G_{X,N}(r_i)$ for $i = 1, \ldots, M$. Then the slope of the least squares regression of $\big(\log\big(C^G_{X,N}(r_i)\big)\big)_{1\le i\le M}$ onto $\big(\log(r_i)\big)_{1\le i\le M}$ provides an estimator $\widehat{\nu}_G$ of ν_G.

3.2. Estimation procedure of ν_G

The main drawback of the considered numerical procedure of computation of $\widehat{\nu}_G$ is the large sensitivity of this estimator to the choice of $(r_1, \ldots, r_M)$. Several papers provide empirical rules for choosing this family in this case (see for instance [5], [16]) as well as in the computation of the box-counting dimension (see for instance [10]).

Here, we will consider a data-driven procedure of choice $(r_1, \ldots, r_M)$. First, consider a very large choice of (r_i), *i.e.* $r_i = \exp(-10\, i/M)$ and $M = 300$, and therefore $-\log(r_i) \in [-10, 0]$. We can observe the curve drawn by the points $\big(-\log(r_i)\,,\log\big(C^G_{X,N}(r_i)\big)\big)$ on Figure 1. A linear region appears but for certain values of r_i (approximately for $\log(r_i)$ between -6 and -2 on this figure); r_i do not have to be chosen too small or too large for applying a least squares estimation. In the sequel, we develop a data-driven procedure for choosing $(r_1, \ldots, r_M)$. Note that the choice of M is not at all as crucial as the one of $(r_1, \ldots, r_M)$ and set $M = 30$, $M = 100$ or $M = 300$ does not significantly affect the estimation of ν_G.

Using the computation of $\mathbb{E}\big[C^G_{X,N}(r)\big]$ as it was done in [2], we deduce that the main term of its expansion is obtained for $(j-i)/N \in \big[r^{1/\alpha_{i_0-1}}, r^{1/\alpha_{i_0}}\big]$ where $i_0 = \min\big\{k \in \{1, \ldots, d\},\ \sum_{i=1}^k \alpha_i \ge 1\big\}$ when i_0 exists, and $(j-i)/N \in \big[r^{1/\alpha_d}, T\big]$ when i_0 does not exist, *i.e.* when $\sum_{i=1}^d \alpha_i < 1$. Therefore, this requires:

- when $\sum_{i=1}^d \alpha_i \ge 1$, $1/N < r^{1/\alpha_{i_0}}$ and thus $r > N^{-\alpha_{i_0}}$.
- when $\sum_{i=1}^d \alpha_i < 1$, no condition, except $r \to 0$.

But α_{i_0-1} and α_{i_0+1} are unknown. Then, we propose the following procedure for choosing (r_i):

1. Firstly, an estimation of $\alpha_1 = \min_{1\le j\le d}\{\alpha_j\}$ is however possible. Note that indeed, for any $1 \le i < j \le N$, we have $\mathbb{E}\big[\|X_{iT/N} - X_{jT/N}\|^2\big] = \frac{1}{2}\sum_{k=1}^d T^{\alpha_k}\left|\frac{j-i}{N}\right|^{2\alpha_k}$. It is clear that for $\frac{j-i}{N} \to 0$, the main term of the expansion of $\mathbb{E}\big[\|X_{iT/N} - X_{jT/N}\|^2\big]$ is given by $\frac{1}{2}T^{\alpha_1}\left|\frac{j-i}{N}\right|^{2\alpha_1}$. Therefore, an estimator of the Hurst

Fig1_Zouhaier-eps-converted-to.pdf

Figure 1: Plot of $\left(\log\left(C^G_{X,N}(r_i)\right)\right)_{1\le i\le M}$ onto $\left(\log(r_i)\right)_{1\le i\le M}$.

parameter based on quadratic variations (see for instance [9]) applied to $\|X_{iT/N} - X_{jT/N}\|^2$ will provide an estimation of α_1. Hence, define:

$$\widehat{\alpha}_1 = \frac{1}{2\,\log 2}\,\log\left(\frac{\sum_{k=1}^{N-2}\left\|X_{(k+2)T/N} - X_{kT/N}\right\|^2}{\sum_{k=1}^{N-1}\left\|X_{(k+1)T/N} - X_{kT/N}\right\|^2}\right). \tag{12}$$

Then, we know that $\alpha_{i_0} \geq \alpha_1$. We deduce the following first numerical conditions for choosing r_i:

- when $\sum_{i=1}^{2} \alpha_i \geq 1$ (typically for $d = 2$ and if $i_0 = 2$) and if $\widehat{\alpha}_1 < 0.5$ we deduce $\widehat{\alpha}_{i_0} = \widehat{\alpha}_2 \geq 1 - \widehat{\alpha}_1$, inducing

  $$r_i \geq r_{\min}(\widehat{\alpha}_1) = N^{-(1-\widehat{\alpha}_1)} \quad \text{for any } i = 1, \dots, M. \tag{13}$$

  But if $\widehat{\alpha}_1 \geq 0.5$, we could only restrict the condition to

  $$r_i \geq r_{\min}(\widehat{\alpha}_1) = N^{-\widehat{\alpha}_1}, \quad \text{for any } i = 1, \dots, M, \tag{14}$$

  since we know that $N^{-\alpha_1} \geq N^{-\alpha_{i_0}}$.

- when $\sum_{i=1}^{2} \alpha_i < 1$ and therefore $\widehat{\alpha}_1 < 0.5$, no condition is required except $r_i \to 0$ and therefore we could also chose the condition (13).

2. Secondly, it is required to choose $r_i \to 0$. Therefore, we chose

   $$r_i \leq r_{\max} = \frac{1}{2\,\log N} \quad \text{for any } i = 1, \dots, M.$$

   Note that the added constant 2 has been chosen to optimize first Monte-Carlo experiments.

3. Finally, we will only consider a family $(r_1, \dots, r_M)$ in the interval $[r_{\min}(\widehat{\alpha}_1), r_{\max}]$. We chose

   $$r_i = r_{\min}(\widehat{\alpha}_1)\left(\frac{r_{\max}}{r_{\min}(\widehat{\alpha}_1)}\right)^{i/M} \tag{15}$$

   for obtaining an uniform grid of $(\log(r_i))$.

3.3. Monte-Carlo experiments

We applied the numerical procedure of estimation of ν_G for 1000 independent replications of fractional Brownian motion trajectories generated with several combinations of parameters $(\alpha_1, \alpha_2) \in (0,1)^2$ and three values of trajectory length N. The results are detailed in Table 1. On the basis of these results, we can say that:

- The convergence from $\widehat{\nu}_G$ to ν_G seems to be occurring but is relatively slow;

- For values of ν_G close to 2, a small bias is present.

	N	$\overline{\mu}$	$\overline{\sigma}$
$\alpha_1 = 0.2,\ \alpha_2 = 0.9$	$N = 500$	1.8704	0.085
$\nu_G = 1.889$	$N = 1000$	1.8561	0.080
	$N = 2500$	1.8517	0.073
$\alpha_1 = 0.2,\ \alpha_2 = 0.4$	$N = 500$	1.9626	0.060
$\nu_G = 2$	$N = 1000$	1.9564	0.065
	$N = 2500$	1.9688	0.047
$\alpha_1 = 0.5,\ \alpha_2 = 0.5$	$N = 500$	1.9164	0.058
$\nu_G = 2$	$N = 1000$	1.9136	0.050
	$N = 2500$	1.9118	0.048
$\alpha_1 = 0.8,\ \alpha_2 = 0.6$	$N = 500$	1.6078	0.103
$\nu_G = 1.5$	$N = 1000$	1.5704	0.084
	$N = 2500$	1.4986	0.085
$\alpha_1 = 0.9,\ \alpha_2 = 0.9$	$N = 500$	1.1661	0.104
$\nu_G = 1.111$	$N = 1000$	1.1598	0.090
	$N = 2500$	1.1336	0.093

Table 1: Behavior of $\widehat{\nu}_G$ for multidimensional fractional Brownian motion process for $d = 2$ in several frameworks of parameters and trajectory length N.

	mean($\widetilde{\alpha}_i$)	sd($\widetilde{\alpha}_i$)	mean($\widetilde{\nu}_G$)
$\alpha_1 = 0.5$	0.5000	0.0160	1.9766
and $\alpha_2 = 0.9$	0.4988	0.0161	
$\nu_G = 1.5556$			
$\alpha_1 = 0.6$	0.6106	0.0161	1.6567
and $\alpha_2 = 0.8$	0.6000	0.0157	
$\nu_G = 1.5$			

Table 2: Estimation of ν_G using univariate method.

3.4. Isometry of the procedure

If we exactly know that $(X_t^{(j)})_j$ is a fractional Brownian motion with a parameter $\alpha_j \in (0,1)$, the estimation of ν_G using the Gaussian local correlation procedure is not really interesting. It is clearly preferable to use a method for estimating each α_i such as the one proposed in (12) and then we use the estimator given by:

$$\widetilde{\nu}_G = \min\Big(\Big(\frac{1+\sum_{k=1}^{i}(\widetilde{\alpha}_i - \widetilde{\alpha}_k)}{\widetilde{\alpha}_i}\Big)_{1\leq i\leq d}, \, d\Big)$$

$$\text{where} \quad \widetilde{\alpha}_i = \frac{1}{2\,\log 2}\,\log\left(\frac{\sum_{k=1}^{N-2}\big(X^{(i)}_{(k+2)T/N} - X^{(i)}_{kT/N}\big)^2}{\sum_{k=1}^{N-1}\big(X^{(i)}_{(k+1)T/N} - X^{(i)}_{kT/N}\big)^2}\right). \qquad (16)$$

But the general case is rather that of multidimensional processes that depend on each other in such a way that:

$$Y_t = Q\,X_t \quad \text{where} = (X_t)_t \text{ satisfies (H1), (H2) and (H3),}$$

where Q is an orthogonal matrix such as $Q\,Q' = I_d$. Note that Q is a constant matrix and is not supposed to depend on the time t. Simple computations show that (Y_t) does not satisfy (H2).
Now, we assume that $(Y_{T/N}, \dots, Y_T)$ is observed. Then, we have $\|Y_{iT/N} - Y_{jT/N}\|_e = \|X_{iT/N} - X_{jT/N}\|_e$ and therefore the Gaussian local correlation or Hausdorff dimensions of the images of $(Y_t)_{0\leq t\leq T}$ and $(X_t)_{0\leq t\leq T}$ are the same. Also, we deduce that the estimator $\widehat{\nu}_G$ is the same using X or Y.
Now, if we would like to use $\widetilde{\nu}_G$ applied to $(Y_{T/N}, \dots, Y_T)$ for estimating ν_G without knowing Q, then the estimators $\widehat{\alpha}_i$ are not at all consistent. To illustrate this, we consider a special case when $d = 2$, $N = 1000$ and $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then, after 1000 independent replications, we obtain the results shown in Table 2. It is clear that the estimator $\widetilde{\nu}_G$ is not at all consistent.

4. Conclusion

This article deals the roughness of multidimensional stochastic process by studying the asymptotic behavior of a modification of the most used occupation measure. By replacing the hard indicator function by Gaussian kernel one on the occupation measure, first, we determine, the theoretical value of local correlation dimension of multidimensional fractional Brownian motion with independent components. Second, we give a range of bandwidth values to obtain a consistent least square estimator of the local correlation dimension for such multivariate stochastic process. Finally, we confirms our theoretical results by a simulation study on the multidimensional fractional Brownian motion for different values of local correlation dimension and different length of trajectory.

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