EX 13 : 1ère méthode

$\forall x \geq 0$ $\int_x^\infty e^{-t^2} dt$ existe car $t \mapsto e^{-t^2}$ est intégrable sur $[0,+\infty[$

Notons $F : [0,+\infty[\to \mathbb{R}$ $\quad x \mapsto F(x) = \int_x^\infty e^{-t^2} dt$. F est positive, continue et dérivable avec $F'(x) = -e^{-x^2}$ (en effet si on note $G(x)$ une primitive de $u \mapsto e^{-u^2}$, $F(x) = \lim_{M \to \infty} G(M) - G(x)$ puisque $F(x) = \lim_{M\to\infty} \int_x^M e^{-t^2} dt = \lim_{M \to \infty} (G(M) - G(x))$ d'où $F'(x) = -G'(x) = -e^{-x^2}$). Soit $\alpha > 0$, par une intégration par partie on obtient :

$$\int_0^\alpha F(x)dx = [xF(x)]_0^\alpha - \int_0^\alpha xF'(x)dx = \alpha F(\alpha) + \int_0^\alpha x e^{-x^2} dx = \alpha F(\alpha) - \frac{1}{2}\int_0^\alpha \underbrace{-2xe^{-x^2}}_{f' \times f} dx$$

$$= \alpha F(\alpha) - \frac{1}{2}\left[e^{-x^2}\right]_0^\alpha = \alpha F(\alpha) - \frac{1}{2}(e^{-\alpha^2} - 1) = \alpha F(\alpha) + \frac{1}{2} - \frac{e^{-\alpha^2}}{2}$$

$$0 \leq F(\alpha) = \int_\alpha^\infty e^{-t^2} dt \leq \int_\alpha^\infty e^{-\alpha t} dt = \left[\frac{e^{-\alpha t}}{-\alpha}\right]_\alpha^\infty = \frac{e^{-\alpha^2}}{\alpha}.$$ Ce qui prouve que

$\begin{pmatrix} -t^2 \leq -\alpha t \\ \text{qd } t \geq \alpha \end{pmatrix}$ $\quad 0 \leq \alpha F(\alpha) \leq e^{-\alpha^2}$ et donc $\lim_{\alpha \to \infty} \alpha F(\alpha) = 0$

Finalement : $\int_0^\infty \left(\int_x^\infty e^{-t^2} dt\right) dx = \lim_{\alpha \to \infty} \int_0^\alpha \left(\int_x^\infty e^{-t^2} dt\right)dx = \lim_{\alpha\to\infty} \int_0^\alpha F(x)dx$

$$= \lim_{\alpha \to \infty} \alpha F(\alpha) + \frac{1}{2} - \frac{e^{-\alpha^2}}{2} = \frac{1}{2}$$

2ème méthode

Notons $\chi_A : t \mapsto \begin{cases} 0 \text{ si } t \notin A \\ 1 \text{ si } t \in A \end{cases}$.

$$\int_0^\infty F(x)dx = \int_0^\infty \left(\int_x^\infty e^{-t^2} dt\right)dx = \int_0^\infty \left(\int_0^\infty e^{-t^2} \chi_{[x,\infty[}(t) dt\right)dx$$

$$\circledast \quad = \int_0^\infty \left(\int_0^\infty e^{-t^2} \chi_{[0,t]}(x) dt\right)dx$$

$t \mapsto e^{t^2} \chi_{[0,t]}(x)$ positive + Fubini $\to$

$$= \int_0^\infty \left(\int_0^\infty e^{-t^2} \chi_{[0,t]}(x) dx\right)dt$$

$$= \int_0^\infty e^{-t^2} \left(\int_0^\infty \chi_{[0,t]}(x) dx\right)dt$$

$$= \int_0^\infty e^{-t^2} \times t \, dt = \int_0^\infty t e^{-t^2} dt$$

$$= -\frac{1}{2}\int_0^\infty -2te^{-t^2} dt$$

$$= -\frac{1}{2}\left[e^{-t^2}\right]_0^\infty = -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

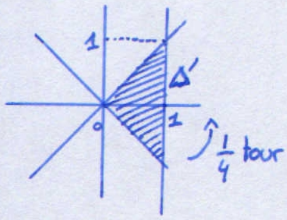
$\forall x \geq 0$

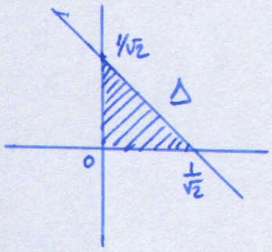
$\circledast$ $\chi_{[x,\infty[}(t) = \chi_{[0,t]}(x)$ car $t \in [x, \infty[\iff t \geq x \geq 0 \iff x \in [0,t]$

Ex 16 : $\Delta' = \{(u,v)\in\mathbb{R}^2 \,/\, u\leq 1 \text{ et } -u\leq v\leq u\} = \{(u,v)\,/\,0\leq u\leq 1 \text{ et } -u\leq v\leq u\} \cup \underbrace{\{(u,v)\,/\,u<0 \ -u\leq v\leq u\}}_{=\emptyset}$

car si $u<0$ on ne peut avoir $-u\leq u$!

a) $\Delta' = \{(u,v)\in\mathbb{R}^2 \,/\, 0\leq u\leq 1 \quad -u\leq v\leq u\}$





b) $$\iint_{\Delta'} u^2 e^{uv}\,du\,dv = \int_0^1 \left(\int_{-u}^{u} u^2 e^{uv}\,dv\right)du$$

$$= \int_0^1 u\left(\int_{-u}^{u} u e^{uv}\,dv\right)du$$

$$= \int_0^1 u \times \left[e^{uv}\right]_{-u}^{u}\,du$$

$$= \int_0^1 u\left(e^{u^2}-e^{-u^2}\right)du$$

$$= \int_0^1 2u\,\mathrm{sh}(u^2)\,du = \left[\mathrm{ch}(u^2)\right]_0^1 = \mathrm{ch}1 - \underbrace{\mathrm{ch}0}_{=1}$$

$$\iint_{\Delta'} u^2 e^{uv}\,dv = \mathrm{ch}(1) - 1$$

c) $$\psi(u,v) = \left(\overbrace{\frac{u-v}{\sqrt{2}}}^{x}, \overbrace{\frac{u+v}{\sqrt{2}}}^{y}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

ψ est une rotation vectorielle d'angle $\pi/4$. $\Delta = \left\{(u,v)\in\mathbb{R}^2 \,/\, \begin{array}{l} u\geq 0 \\ v\geq 0 \\ u+v\leq \frac{1}{\sqrt{2}} \end{array}\right\}$

d) $$\iint_{\Delta=\psi(\Delta')} (x+y)^2 e^{x^2-y^2}\,dx\,dy = \iint_{\Delta'} \left(\frac{2u}{\sqrt{2}}\right)^2 e^{\left(\frac{u-v}{\sqrt{2}}\right)^2 - \left(\frac{u+v}{\sqrt{2}}\right)^2} \times 1\,du\,dv$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\frac{\pi}{4} \end{pmatrix} = \cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4} = 1$$

$$= \iint_{\Delta'} 2u^2 e^{-2uv}\,du\,dv$$

$U = \sqrt{2}u$

$V = \sqrt{2}v$

$$= \iint_{\tilde{\Delta}'} U^2 e^{-UV} \frac{dU}{\sqrt{2}}\frac{dV}{\sqrt{2}} = \frac{1}{2}\iint U^2 e^{-UV}\,dU\,dV$$

si $0\leq u\leq 1$ alors $0\leq U\leq\sqrt{2}$

et si $-u\leq v\leq u$ alors $-U\leq V\leq U$

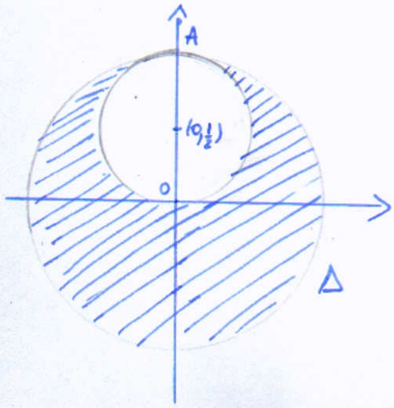
d'où $\Delta' = \{(U,V)\,/\,0\leq U\leq\sqrt{2} \text{ et } -U\leq V\leq U\}$

$$= \frac{1}{2}\left(\mathrm{ch}(\sqrt{2}) - 1\right) \quad \text{(même calcul)}.$$

Ex17 : a) $y \le x^2+y^2 \le 1 \iff x^2+y^2 \le 1$ et $0 \le x^2+y^2-y = x^2+(y-\frac{1}{2})^2$

$\iff x^2+y^2\le 1$ et $(\frac{1}{2})^2 \le x^2+(y-\frac{1}{2})^2$

$(x,y)\in\Delta \iff (x,y)\in \mathcal{D}(0,1)$ et $(x,y)\notin \mathcal{D}((0,\frac{1}{2});\frac{1}{2})$



b) $\displaystyle\iint_\Delta \frac{dxdy}{(1+x^2+y^2)} = \iint_{\Delta^+} \frac{dxdy}{1+x^2+y^2} + \iint_{\Delta'} \frac{dxdy}{(1+x^2+y^2)}$

$x = \rho\cos\theta$
$y = \rho\sin\theta$

où $\Delta^+ = \{(x,y)\in\Delta \,/\, y \geq 0\}$
$\Delta^- = \{(x,y)\in\Delta \,/\, y<0\}$

• $\displaystyle\iint_{\Delta^-}\frac{dxdy}{(1+x^2+y^2)^2} = \int_\pi^{2\pi}\int_0^1 \frac{\rho d\rho d\theta}{(1+\rho^2)^2} = \int_\pi^{2\pi}\left(\int_0^1 \frac{\rho}{(1+\rho^2)^2}d\rho\right)d\theta$

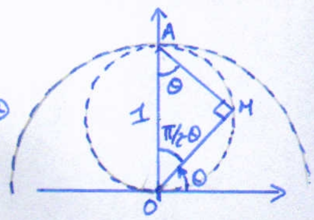
$\displaystyle = 2\pi\int_0^1 \frac{\rho}{(1+\rho^2)^2}d\rho = \pi\int_0^1\frac{2\rho}{(1+\rho^2)^2}d\rho = \pi\left[\frac{-1}{1+\rho^2}\right]_0^1$

$\displaystyle = \pi\left[\frac{1}{1+\rho^2}\right]_1^0 = \pi\left(1-\frac{1}{2}\right) = \frac{\pi}{2}$

• $\displaystyle\iint_{\Delta^+}\frac{dxdy}{(1+x^2+y^2)^2} = \int_0^\pi\left(\int_{OM}^1 \frac{\rho d\rho}{(1+\rho^2)^2}\right)d\theta$

Calculons OM :

$OM = 1\times\cos(\frac{\pi}{2}-\theta) = \sin\theta$



$\displaystyle = \int_0^\pi\left(\int_{\sin\theta}^1 \frac{\rho d\rho}{(1+\rho^2)^2}\right)d\theta$

$\displaystyle = \frac{1}{2}\int_0^\pi\left[\frac{-1}{1+\rho^2}\right]_{\sin\theta}^1 d\theta = \frac{1}{2}\int_0^\pi\left(\frac{1}{1+\sin^2\theta}-\frac{1}{2}\right)d\theta = \frac{1}{2}\int_0^\pi\frac{d\theta}{1+\sin^2\theta} - \frac{\pi}{4}$

Reste à calculer $\displaystyle\int_0^\pi\frac{d\theta}{1+\sin^2\theta} = 2\int_0^{\pi/2}\frac{d\theta}{1+\sin^2\theta} = 2\int_0^{+\infty}\frac{du}{1+2u^2} = \frac{2}{\sqrt{2}}\int_0^\infty\frac{dv}{1+v^2}$

$u = \tan\theta$ (règle de BIOCHE) $\qquad v = \sqrt{2}u$

$\displaystyle = \sqrt{2}\,[\text{Arctan}\, v]_0^{+\infty} = \sqrt{2}\times\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}$

Finalement : $\displaystyle\iint_{\Delta^+}\frac{dxdy}{(1+x^2+y^2)^2} = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{4}$

et $\displaystyle\iint_\Delta\frac{dxdy}{(1+x^2+y^2)^2} = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} = \frac{2+\sqrt{2}}{4\sqrt{2}}\pi$

Ex18 : Si A est définie positive et symétrique $A=\begin{pmatrix}a&b\\b&c\end{pmatrix}$ avec $ac-b^2>0$

car $(u\,v)A\,{}^t(u,v) = (u\,v)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = au^2+2buv+cv^2 = a\left(u+\frac{b}{a}v\right)^2+\frac{ac-b^2}{a}v^2$.

et $(u\,v)A\,{}^t(u\,v)>0$ ssi $ac-b^2\geq 0$

et $(u\,v)A\,{}^t(u\,v)=0$ ssi $(u,v)=(0,0)$ ssi $ac-b^2>0$

$$\iint_{\mathbb{R}^2} e^{-(u\,v)A\,{}^t(u,v)}\,du\,dv = \iint_{\mathbb{R}^2} e^{-(au^2+2buv+cv^2)}\,du\,dv$$

$$= \iint_{\mathbb{R}^2} e^{-a\left(u+\frac{b}{a}v\right)^2+\frac{ac-b^2}{a}v^2}\,du\,dv$$

On effectue le changement de variable : $\begin{cases} X=\sqrt{a}\left(u+\frac{b}{a}v\right)\\ Y=\sqrt{\frac{ac-b^2}{a}}\,v\end{cases}$ d'où $\begin{cases} u=\frac{X}{\sqrt{a}}-\frac{b}{a}\sqrt{\frac{a}{ac-b^2}}\,Y\\ v=\sqrt{\frac{a}{ac-b^2}}\,Y\end{cases}$

$$\mathrm{Jac}=\begin{pmatrix}\frac{1}{\sqrt{a}} & \frac{-b}{\sqrt{a}\sqrt{ac-b^2}}\\ 0 & \sqrt{\frac{a}{ac-b^2}}\end{pmatrix}=\frac{1}{\sqrt{ac-b^2}}$$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\frac{1}{\sqrt{ac-b^2}}\,dx\,dy = \frac{1}{\sqrt{ac-b^2}}\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \frac{4}{\sqrt{ac-b^2}}\iint_{[0,+\infty[^2} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \frac{4}{\sqrt{ac-b^2}}\iint_{[0,\frac{\pi}{2}]\times[0,+\infty[} e^{-\rho^2}\rho\,d\rho\,d\theta = \frac{-4}{\sqrt{\cdots}}\times\frac{1}{2}\int_0^{\pi/2}\left(\int_0^{+\infty}-2\rho\, e^{-\rho^2}d\rho\right)d\theta$$

$$= \frac{-2}{\sqrt{ac-b^2}}\int_0^{\pi/2}\left[e^{-\rho^2}\right]_0^{+\infty}d\theta = \frac{-2}{\sqrt{ac-b^2}}\int_0^{\pi/2}-1\,d\theta = \frac{2\pi/2}{\sqrt{ac-b^2}}$$

$$= \frac{\pi}{\sqrt{ac-b^2}}$$

$$= \frac{\pi}{\sqrt{\det A}}$$